

ALGEBRAIC SURFACES OF GENERAL TYPE WITH $c_1^2 = 3p_g - 7$

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Introduction. Let S be a minimal algebraic surface of general type defined over the complex number field C . Castelnuovo's second inequality states that if the canonical map of S is birational, then $c_1^2(S) \geq 3p_g(S) - 7$ (see [4], [10, II, §1], [1]).

In the present paper, we study minimal algebraic surfaces of general type with $c_1^2 = 3p_g - 7$. These surfaces are classified into two types according to the nature of their canonical map Φ_K :

Type I: Φ_K is a birational holomorphic map onto its image.

Type II: Φ_K birationally induces a double covering of a ruled surface.

Historically, surfaces of type I were already known to Castelnuovo [4]. He showed that the canonical image of a type I surface is always contained in a threefold of minimal degree and he determined its divisor class. For a modern treatment of his argument, see Harris [6]. On the other hand, Horikawa [9], [10, IV] has studied, among others, surfaces of types I and II in detail when $(p_g, c_1^2) = (4, 5), (5, 8)$. Especially he completely determined their deformation types. Surfaces of type I with $p_g = 7$ and $c_1^2 = 14$ were recently studied by Miranda [12].

The paper consists of two parts: §§1–4 and §§5–6. The former part is devoted to surfaces of type I. In §1, we show that surfaces with $c_1^2 = 3p_g - 7$ are divided into two types mentioned above and review Castelnuovo's argument to classify surfaces of type I according to the threefold W on which the canonical image lies. We remark that, in most cases, W is a rational normal scroll (see, [6] and [5]). We prove that the canonical image has only rational double points and that almost all type I surfaces have a pencil of nonhyperelliptic curves of genus three (Theorem 1.5). Proof of some Claims needed in §1, concerning the liftability of the canonical map to a nonsingular model of W , is postponed to §2. The technique employed here is essentially due to Horikawa [10]. In §3 and §4, we study deformations of type I surfaces and compute the number of moduli (Theorem 3.2 and Proposition 4.3). Though we try to determine their deformation types, many cases are left unsettled. In §4, we construct a family of surfaces in which the central fiber is of type II and a general fiber is of type I.

The latter part, §§5–6, is devoted to surfaces of type II. In view of the vanishing of irregularity of a type I surface (see, §1), we restrict ourselves to regular surfaces of type II. Our concerns here are pencils of hyperelliptic curves. From a remarkable result of Xiao [16], we know that a surface of type II has such a pencil of genus less than

five provided $p_g \geq 46$. In §5, we construct minimal surfaces with pencils of hyperelliptic curves of genus 3 whose invariants (p_g, c_1^2) cover a certain area in the zone of existence, which of course contains the line $c_1^2 = 3p_g - 7$ (Theorem 5.7). By the same method, we can show the existence of type II surfaces with pencils of hyperelliptic curves of genus 2, 3 or 4 (Proposition 6.3).

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1. Canonical map and surfaces of type I. Let S be a minimal algebraic surface of general type defined over the complex number field \mathbf{C} for which the geometric genus $p_g(S)$ and the Chern number $c_1^2(S)$ satisfy the conditions $c_1^2 = 3p_g - 7$ and $p_g \geq 3$. We let $\Phi_K: S \rightarrow \mathbf{P}^{p_g-1}$ denote the rational map defined by the canonical linear system $|K|$. We put $S' = \Phi_K(S)$ and call it the canonical image of S . We denote by $\phi_K: S \rightarrow S'$ the natural map induced by Φ_K .

LEMMA 1.1. *Let S be as above. Then we have the following two possibilities:*

- (1) *$|K|$ is free from base points and ϕ_K is a birational holomorphic map.*
- (2) *ϕ_K is a rational map of degree 2 and S' is birationally equivalent to a ruled surface.*

PROOF. We remark that ϕ_K is generically finite, since $|K|$ is not composite with a pencil by [1, Lemma 5.3]. Since S' is irreducible and nondegenerate (i.e., is not contained in any hyperplane in \mathbf{P}^{p_g-1}), we have the inequality

$$c_1^2 \geq (\deg \phi_K)(\deg S') \geq (\deg \phi_K)(p_g - 2).$$

Thus we have $\deg \phi_K \leq 2$. If $\deg \phi_K = 1$, then $|K|$ has no base points by [10, II, Lemma (1.1)] and [9, Lemma 2]. If $\deg \phi_K = 2$, then we get $\deg S' < 2p_g - 4$. Therefore it follows from [1, Lemma 1.4] that S' is birationally equivalent to a ruled surface. \square q.e.d.

We say that S is of type I or of type II according as whether the degree of ϕ_K is 1 or 2.

1.2. Surfaces of type I were essentially known to Castelnuovo [4]. Here we recall his argument. Our reference is [7] and [6].

We recall fundamental properties of the Hilbert function h_X defined for any projective variety $X \subset \mathbf{P}^r$ by

$$h_X(n) = \dim_{\mathbf{C}} \text{Im}\{\rho: H^0(\mathbf{P}^r, \mathcal{O}(n)) \rightarrow H^0(X, \mathcal{O}(n))\},$$

where ρ is the restriction map and n is a nonnegative integer. If Y is a general hyperplane section of X , then we have for any $n > 0$

$$(1) \quad \delta h_X(n) := h_X(n) - h_X(n-1) \geq h_Y(n).$$

We remark that X is projectively normal if $\delta h_X(n) = h_Y(n)$ holds for any n .

Now let S be a surface of type I and put $r=p_g-2$. Since $|K|$ has no base point, a general member $C \in |K|$ is irreducible nonsingular and has genus $g(C)=3r$. If we put $C'=\Phi_K(C)$, then it is an irreducible nondegenerate curve in $\mathbf{P}^r \subset \mathbf{P}^{r+1}$ and $\deg C'=K^2=3r-1$. We let Γ denote a general hyperplane section of C' . Since it is a nondegenerate set of $3r-1$ distinct points in uniform position, we have

$$(2) \quad h_\Gamma(n+1) \geq \min\{3r-1, h_\Gamma(n)+r-1\}.$$

Since $2K|_C$ is the canonical divisor of C and $h_C(1)=r+1$, it follows from (1) that

$$3r=h^0(C, \mathcal{O}(2K|_C)) \geq h_C(2) \geq r+1+h_\Gamma(2).$$

This and (2) show $h_\Gamma(2)=2r-1$ and $h_C(2)=3r$. By a similar calculation, one gets $h^0(C, \mathcal{O}(nK|_C))=h_C(n)$ and $\delta h_C(n)=h_\Gamma(n)$ for any $n>0$. This implies that C' is projectively normal.

We turn our attention to the canonical image S' . By the well-known formula for pluri-genera of minimal surfaces of general type combined with (1), we get

$$4r+2-q(S)=h^0(S, \mathcal{O}(2K)) \geq h_{S'}(2) \geq h_{S'}(1)+h_C(2)=4r+2.$$

From this, we have $q(S)=0$, $h^0(2K)=h_{S'}(2)$ and $\delta h_{S'}(2)=h_C(2)$. By a similar calculation, one can show $h_{S'}(n)=h^0(S, \mathcal{O}(nK))$, $\delta h_{S'}(n)=h_C(n)$ for any $n>0$. Therefore, S' is also projectively normal and the multiplication map $\text{Sym}^n H^0(S, \mathcal{O}(K)) \rightarrow H^0(S, \mathcal{O}(nK))$ is surjective for any $n \geq 0$. This implies that the canonical ring of S is generated in degree 1 and therefore S' is isomorphic to the canonical model of S . In particular, S' has only rational double points (RDP's, for short) as its singularity.

We show that S' is contained in an irreducible threefold W of minimal degree $r-1$ in \mathbf{P}^{r+1} , cut out by all quadrics through S' . Since $h_\Gamma(2)=2r-1$, Castelnuovo's Lemma (see, e.g., [7]) shows that Γ lies on a rational normal curve R of degree $r-1$ in \mathbf{P}^{r-1} cut out by all quadrics containing Γ . From this, we get $h^0(\mathbf{P}^{r-1}, \mathcal{I}_\Gamma(2))=h^0(\mathbf{P}^{r-1}, \mathcal{I}_R(2))=(r-1)(r-2)/2$, where \mathcal{I}_x is the ideal sheaf of x . On the other hand, we have $h^0(\mathbf{P}^{r+1}, \mathcal{I}_{S'}(2))=h^0(\mathbf{P}^{r+1}, \mathcal{O}(2))-h^0(S, \mathcal{O}(2K))=(r-1)(r-2)/2$. Therefore, the linear system $|\mathcal{I}_{S'}(2)|$ of quadrics through S' is restricted onto $|\mathcal{I}_\Gamma(2)|$ isomorphically, and its base locus W is an irreducible threefold of minimal degree.

1.3. To describe W , we introduce some notation. Let \mathcal{E} be a locally free sheaf of rank p on \mathbf{P}^q and let $\varpi: \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^q$ be the associated projective bundle. Then the Picard group of $\mathbf{P}(\mathcal{E})$ is generated by the tautological divisor T such that $\varpi_* \mathcal{O}(T)=\mathcal{E}$ and the pull-back F by ϖ of a hyperplane in \mathbf{P}^q . We note that the canonical bundle of $\mathbf{P}(\mathcal{E})$ is given by

$$(3) \quad K_{\mathbf{P}(\mathcal{E})}=\mathcal{O}(-pT+(\deg(\det \mathcal{E})-q-1)F).$$

According to the classification of irreducible nondegenerate threefolds of minimal degree in \mathbf{P}^{p_g-1} (cf. [5] or [6]), W is one of the following:

- (A) \mathbf{P}^3 ($p_g=4$).
- (B) a hyperquadric ($p_g=5$).
- (C) a cone over the Veronese surface, i.e., the image of the \mathbf{P}^1 -bundle $\tilde{W} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2))$ under the holomorphic map Φ_T induced by $|T|$ ($p_g=7$).
- (D) a rational normal scroll, i.e., the image of the \mathbf{P}^2 -bundle $\mathbf{P}_{a,b,c} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c))$ on \mathbf{P}^1 under the holomorphic map Φ_T induced by $|T|$ ($p_g \geq 6$), where a, b, c are integers satisfying

$$(4) \quad 0 \leq a \leq b \leq c, \quad a+b+c = p_g - 3.$$

1.4. We study S more closely in each of the above cases. Claims I-III below will be proved in the next section.

The first two may be clear:

Case (A): S' is a quintic surface in \mathbf{P}^3 .

Case (B): S' is a complete intersection of a quadric and a quartic.

These are extensively studied by Horikawa in [9], [10, IV].

Case (C): The map $\Phi: \tilde{W} \rightarrow W$ is the contraction of the divisor $T_\infty \sim T - 2F$, where the symbol \sim means the linear equivalence.

CLAIM I. *We have a holomorphic map $\mu: S \rightarrow \mathbf{P}^2$ of degree 3. Let $\phi: S \rightarrow W$ be the natural map induced by the canonical map. Then ϕ can be lifted to a holomorphic map $\psi: S \rightarrow \tilde{W}$ over μ such that $K = \psi^*T$. Further, $S'' = \psi(S)$ has only RDP's.*

We show that S'' is linearly equivalent to $3T + F$. Since μ is of degree 3, S'' is linearly equivalent to $3T + \alpha F$ for some integer α . Then, since $\deg S' = 14$, we have

$$14 = T^2(3T + \alpha F) = 12 + 2\alpha,$$

where we used the relation $T^2 = 2TF$ in the Chow ring of \tilde{W} . Therefore $S'' \sim 3T + F$. We note that the linear system $|3T + F|$ is free from base points and contains an irreducible nonsingular member.

We compute the invariants of S'' for the sake of completeness. Since \tilde{W} is rational, we have $H^q(\tilde{W}, \mathcal{O}(K_{\tilde{W}})) = 0$ for $q < 3$. By the cohomology long exact sequence for

$$0 \rightarrow \mathcal{O}(K_{\tilde{W}}) \rightarrow \mathcal{O}(K_{\tilde{W}} + S'') \rightarrow \omega_{S''} \rightarrow 0,$$

we get $H^q(S'', \omega_{S''}) \simeq H^q(\tilde{W}, \mathcal{O}(T)) \simeq H^q(\mathbf{P}^2, \mathcal{O} \oplus \mathcal{O}(2))$ for $q < 2$. This shows $p_g(S'') := h^0(\omega_{S''}) = 7$ and $h^1(\omega_{S''}) = h^1(\mathcal{O}_{S''}) = 0$. Further, since $\omega_{S''} = \mathcal{O}_{S''}(T)$, we get $\omega_{S''}^2 = 14 = 3p_g(S'') - 7$.

Case (D): This case is divided into three subcases

$$(D.1): a > 0, \quad (D.2): a = 0, b > 0, \quad (D.3): a = b = 0.$$

We remark that W is singular in the cases (D.2) and (D.3).

CLAIM II. *(D.3) cannot occur. If (D.2) is the case, then there is a lifting $\psi: S \rightarrow \mathbf{P}_{0,b,c}$*

of the natural map $S \rightarrow W$ such that $K = \psi^*T$. Further, $S'' = \psi(S)$ has only RDP's.

We let $\psi : S \rightarrow \mathbf{P}_{a,b,c}$ denote the map induced by Φ_K in Case (D.1) and the map in Claim II in Case (D.2). Put $S'' = \psi(S)$. It is nothing but S' in Case (D.1). We show that S'' is linearly equivalent to $4T - (p_g - 5)F$. For this purpose, put $S'' \sim \alpha T + \beta F$. Note that the fibers of $\varpi|_{S''}$ are plane curves of degree α . Since S'' is birational to the surface S of general type, we have $\alpha \geq 4$. Recall that we have $T^3 = (p_g - 3)T^2F$ in the Chow ring of $\mathbf{P}_{a,b,c}$. Since $\deg S' = 3p_g - 7$, we have

$$3p_g - 7 = T^2(\alpha T + \beta F) = (p_g - 3)\alpha + \beta.$$

On the other hand, it follows from (3) that $K_{\mathbf{P}_{a,b,c}} + S'' \sim (\alpha - 3)T + (p_g - 5 + \beta)F$. Since T and $K_{\mathbf{P}_{a,b,c}} + S''$ are equivalent on S'' , we get

$$0 = TS''(K_{\mathbf{P}_{a,b,c}} + S'' - T) = \alpha(\alpha - 4)T^3 + \beta(\alpha - 4)T^2F = (\alpha - 4)(\alpha T^3 + \beta).$$

From these, we get $S'' \sim 4T - (p_g - 5)F$. The numerical invariants can be computed similarly as in Case (C): for $q < 2$, we have $h^q(\omega_{S''}) = h^q(\mathbf{P}_{a,b,c}, \mathcal{O}(T)) = h^0(\mathbf{P}^1, \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c))$ and thus $p_g(S'') = a + b + c + 3 = p_g(S)$ by (4) and $h^1(\omega_{S''}) = 0$; since $\omega_{S''} = \mathcal{O}_{S''}(T)$, we get $\omega_{S''}^2 = 3p_g - 7$.

As to the linear system $|4T - (p_g - 5)F|$, we have the following:

CLAIM III. *The linear system $|4T - (p_g - 5)F|$ on $\mathbf{P}_{a,b,c}$ contains an irreducible member with only RDP's if and only if*

$$(5) \quad a + c \leq 3b + 2, \quad b \leq 2a + 2.$$

Now we get the following theorem essentially due to Castelnuovo [4]:

THEOREM 1.5. *If S is a surface of type I, then the irregularity $q(S)$ vanishes. Its canonical image S' is projectively normal and has only RDP's as its singularity. Furthermore, it is contained in an irreducible nondegenerate threefold of minimal degree. S' is either*

- (1) *a quintic surface in \mathbf{P}^3 ($p_g = 4$),*
- (2) *a complete intersection of a quadric and a quartic in \mathbf{P}^4 ($p_g = 5$),*
- (3) *the image in the cone over the Veronese surface of a member $S'' \in |3T + F|$ on $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(2))$ under the holomorphic map defined by $|T|$ ($p_g = 7$), or*
- (4) *the image in the rational normal scroll of a member $S'' \in |4T - (p_g - 5)F|$ on $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \oplus \mathcal{O}_{\mathbf{P}^1}(c))$ under the holomorphic map induced by $|T|$, where a, b, c are integers satisfying $0 \leq a \leq b \leq c$, $a + b + c = p_g - 3$, $a + c \leq 3b + 2$ and $b \leq 2a + 2$ ($p_g \geq 6$).*

2. Lifting of the canonical map. In this section, we prove Claims I, II and III which are assumed in 1.4. We make use of the standard fact that if a surface admits a map of degree less than three onto a ruled surface, then the canonical map cannot be birational.

Among others, we use the following notation. For any nonnegative integer e , we denote by $\Sigma_e = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(e))$ the Hirzebruch surface of degree e . We let C_0 and f denote the tautological divisor ($C_0^2 = e$) and a fiber, respectively.

2.1. In the cases (C) and (D.2), the threefold W is a cone over a nonsingular surface V . Let Λ_0 be the pull-back to S by Φ_K of the linear system of hyperplanes through the vertex of W . We can choose a basis $\{x_0, x_1, \dots, x_{p_g-1}\}$ of $H^0(S, \mathcal{O}(K))$ such that x_1, \dots, x_{p_g-1} span the module of Λ_0 . We let G denote the fixed part of Λ_0 and put $\Lambda_1 = \Lambda_0 - G$. Since $|K|$ is free from base points, we can assume that $\text{Supp}((x_0)) \cap \text{Supp}(G) = \emptyset$. In particular, we have $KG = 0$. When G is not 0, we denote by ζ the section of $\mathcal{O}([G])$ with $(\zeta) = G$.

2.2. PROOF OF CLAIM I. Since V is the Veronese surface, we have a net Λ such that $2H \in \Lambda_1$ for $H \in \Lambda$ and $K \sim 2H + G$. Since $K^2 = 14$ and $KG = 0$, we have $7 = KH = 2H^2 + HG$. Since $KH + H^2$ is even, we get $H^2 = 1$ or 3. Let $\mu: S \rightarrow \mathbf{P}^2$ denote the rational map induced by Λ . If $H^2 = 1$, then μ is birational. This contradicts the assumption that S is of general type. Therefore, we get $H^2 = 3$, $HG = 1$ and $G^2 = -2$. We claim that μ is holomorphic. Indeed, if μ is not holomorphic, then blow S up at any base point of Λ and let \tilde{H} be the proper transform of H . Then we have $\tilde{H}^2 < H^2 = 3$. This means that μ is of degree < 3 onto \mathbf{P}^2 , contradicting the fact that S is of type I. Therefore, μ is holomorphic and $\deg \mu = 3$. The pair (ζ, x_0) defines a homomorphism $\mathcal{O}_S \rightarrow \mathcal{O}_S(G) \oplus \mathcal{O}_S(K)$, which in turn gives a section $\sigma: S \rightarrow S \times_V \tilde{W}$ because $\text{Supp}((x_0)) \cap \text{Supp}(G) = \emptyset$. We get a holomorphic map $\psi: S \rightarrow \tilde{W}$ by setting $\psi = pr_2 \circ \sigma$, where pr_2 is the projection of $S \times_V \tilde{W}$ on the second factor. It is clear from the construction that $\psi^* T_\infty = G$. Therefore $K \sim 2H + G \sim 2\psi^* F + \psi^*(T - 2F) \sim \psi^* T$. Note that \tilde{W} is obtained by blowing up the vertex of W , and S'' is the proper transform of S' . Since S' has only RDP's, we see that S'' has only RDP's.

2.3. PROOF OF CLAIM II. We separately treat (D.2) and (D.3).

(D.2) $a=0, b>0, b+c \geq 3$: W is a cone over $V = \Sigma_{c-b}$ embedded into \mathbf{P}^{b+c+1} by $|C_0 + bf|$. Let X be the \mathbf{P}^1 -bundle $\mathbf{P}(\mathcal{O}_{\Sigma_{c-b}} \oplus \mathcal{O}_{\Sigma_{c-b}}(C_0 + bf))$ on Σ_{c-b} . We denote by π and L_0 the projection map and the tautological divisor, respectively. Then W is the image of X under the holomorphic map Φ_{L_0} defined by $|L_0|$. Let L_∞ be the divisor on X which is linearly equivalent to $L_0 - \pi^*(C_0 + bf)$. Then we have the holomorphic map $v: X \rightarrow \mathbf{P}_{0,b,c}$ which contracts L_∞ to a nonsingular rational curve Z and satisfies $\Phi_{L_0} = \Phi_T \circ v$, $v^* T = L_0$.

We first show that $\phi: S \rightarrow W$ can be lifted to a holomorphic map $\tilde{\phi}: S \rightarrow X$. Λ_1 induces a rational map $\mu: S \rightarrow \mathbf{P}^{b+c+1}$ whose image is V . We let $\rho: \tilde{S} \rightarrow S$ denote a composite of blowing-ups such that the proper transform Λ of Λ_1 is free from base points. We can assume that ρ is the shortest among those which enjoy the property mentioned above. Let E be the exceptional divisor of ρ . Then the canonical divisor \tilde{K} of \tilde{S} is linearly equivalent to $\rho^* K + E$. Further, we have $\rho^* K \sim \tilde{\mu}^*(C_0 + bf) + \tilde{E} + \rho^* G$,

where $\tilde{\mu}: \tilde{S} \rightarrow \Sigma_{c-b}$ is the holomorphic map induced by Λ and \tilde{E} is a sum of exceptional curves satisfying $\tilde{E} \geq E$. We put $L = \tilde{\mu}^*(C_0 + bf)$. Then

$$3(b+c) + 2 = (\rho^*K)^2 = L^2 + L(\tilde{E} + \rho^*G) \geq L^2 = (\deg \tilde{\mu})(b+c).$$

Since $\deg \tilde{\mu}$ is at least 3, we have $\deg \tilde{\mu} = 3$ and $L(\tilde{E} + \rho^*G) = 2$. We also remark that

$$0 = (\rho^*K)(\rho^*G) = L(\rho^*G) + G^2, \quad 0 = (\rho^*K)\tilde{E} = L\tilde{E} + \tilde{E}^2.$$

We have the following three possibilities:

- (1) $L\tilde{E} = 0, L(\rho^*G) = 2$.
- (2) $L\tilde{E} = 1, L(\rho^*G) = 1$.
- (3) $L\tilde{E} = 2, L(\rho^*G) = 0$.

If (1) is the case, then we have $L\tilde{E} = \tilde{E}^2 = 0$. By the Hodge index theorem, we get $\tilde{E} = 0$. This means that ρ is the identity map. Further we have $G^2 = -2$. If (2) is the case, then we get $G^2 = -1$ which contradicts the fact that $KG + G^2$ is even. If (3) is the case, then we have $G = 0$ and $\tilde{E}^2 = -2$. Since $\tilde{K}L + L^2 = 6(b+c) + 2 + LE$, we see that LE is even. Since ρ is the shortest, $\tilde{E} \neq 0$ implies the existence of a (-1) -curve E_0 with $LE_0 > 0$ which is contained in both \tilde{E} and E . Thus LE is positive. From this and $LE \leq L\tilde{E}$, we conclude $LE = 2$. We see that $\tilde{\mu}(\tilde{E} - E)$ cannot be a curve, because $L(\tilde{E} - E) = 0$ and L is the pull-back of the ample divisor $C_0 + bf$. This in particular implies $(\tilde{\mu}^*f)(\tilde{E} - E) = 0$. Then we get a contradiction, because $\tilde{K}(\tilde{\mu}^*f) + (\tilde{\mu}^*f)^2 = 3f(C_0 + bf) + (\tilde{E} + E)(\tilde{\mu}^*f) = 3 + 2E(\tilde{\mu}^*f)$ is odd.

In summary, ρ is the identity map and μ is holomorphic. Then, as in 2.2, we get a lifting $\tilde{\phi}: S \rightarrow X$ such that $\tilde{\phi}^*L_\infty = G$. We remark that $K \sim (\tilde{\phi} \circ \pi)^*(C_0 + bf) + \tilde{\phi}^*L_\infty \sim \tilde{\phi}^*L_0$. Thus we get the desired map ψ by putting $\psi = v \circ \tilde{\phi}$.

By the same reasoning as in the proof of Claim I, we see that $S^* = \tilde{\phi}(S)$ has only RDP's. Since $KG = 0$, G consists of (-2) -curves. Therefore, we obtain S'' from S^* by contracting some (-2) -curves. This implies that S'' has only RDP's.

(D.3) $a=b=0, c \geq 3$: W is a generalized cone over a rational normal curve of degree $c+1$ in P^{c+2} and the ridge of W is a line. We let Λ be the pull-back to S of the linear system of hyperplanes containing the ridge. Then it is composite with a pencil $|D|$ and we have $K \sim cD + G$, where G is the fixed part of Λ (see, [10, I, §1]). Since $3c+2 = K^2 = cKD + KG$, we get $KD = 1, 2$ or 3 . Since $KD + D^2$ is even and $KD = cD^2 + DG$, we have the following possibilities:

- (1) $KD = 2, D^2 = 0, DG = 2$.
- (2) $KD = 3, D^2 = 1, DG = 0$ (in this case $c = 3$).

If (1) is the case, then S has a pencil of curves of genus two, a contradiction. If (2) is the case, then we get $G^2 = 2$ by $11 = K^2 = 9D^2 + 6DG + G^2$. Since $DG = 0$, this contradicts the Hodge index theorem. Therefore the case (D.3) cannot occur.

2.4. PROOF OF CLAIM III. We choose sections X_0, X_1 and X_2 of $T - aF, T - bF$ and $T - cF$, respectively, in such a way that they form a system of homogeneous fiber

coordinates on each fiber of $\mathbf{P}_{a,b,c}$. Then any $\Psi \in H^0(\mathbf{P}_{a,b,c}, \mathcal{O}(4T - (p_g - 5)F)) \simeq H^0(\mathbf{P}^1, \text{Sym}^4(\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)) \otimes \mathcal{O}(-p_g + 5))$ can be written as

$$(6) \quad \Psi = \sum_{i,j \geq 0, i+j \leq 4} \psi_{ij} X_0^{4-i-j} X_1^i X_2^j,$$

where ψ_{ij} is a homogeneous form of degree $(4-i-j)a + ib + jc - (p_g - 5)$ on \mathbf{P}^1 . If $4b < p_g - 5$, then we can divide Ψ by X_2 and, therefore, the divisor (Ψ) is reducible. If $3a + c < p_g - 5$, then (Ψ) is singular along the curve Z defined by $X_1 = X_2 = 0$. Thus the condition (5) is necessary.

Conversely, assume that (5) holds. If $4a \geq p_g - 5$, then the linear system $|4T - (p_g - 5)F|$ has no base locus and contains an irreducible nonsingular member. So we assume $4a < p_g - 5$. Then (Ψ) contains Z , and $|4T - (p_g - 5)F|$ has no base locus outside it by (5). Thus it suffices to consider the singularity of (Ψ) in a neighborhood of Z . We shall identify Z with the base curve \mathbf{P}^1 of $\mathbf{P}_{a,b,c}$. If $3a + b \geq p_g - 5$, then we can assume that ψ_{10} and ψ_{01} have no common zero. Then (Ψ) is nonsingular in a neighborhood of Z . We next assume $3a + b < p_g - 5$. If $3a + c = p_g - 5$, then ψ_{01} is constant. Unless it is identically zero, (Ψ) is nonsingular along Z . If $3a + c > p_g - 5$, that is, ψ_{01} is of positive degree, then we can assume that it has only simple zeros. Then in a neighborhood of a zero P of ψ_{01} on Z , Ψ can be expressed locally as

$$\Psi = tx_2 + \psi_{20}(t)x_1^2 + \psi_{11}(t)x_1x_2 + \psi_{02}(t)x_2^2 + \dots,$$

where $x_i = X_i/X_0$ and t is a local parameter of Z at P . Thus (Ψ) is defined locally by

$$x_2(t + \psi_{11}(t)x_1 + \dots) + \psi_{20}(t)x_1^2 + \psi_{30}(t)x_1^3 + \psi_{40}(t)x_1^4 = 0.$$

This shows that P is an RDP if Ψ is general. Thus (5) is also sufficient.

We close this section with the following:

PROPOSITION 2.5. *Let S be a type I surface with $p_g = 4$ and S' its canonical image. S has a pencil of nonhyperelliptic curves of genus 3 if and only if S' contains a line.*

PROOF. Assume that S' contains a line l . We blow \mathbf{P}^3 up along l to get $\mathbf{P}_{0,0,1}$. Then the proper transform S'' of S' is linearly equivalent to $4T + F$ and has a pencil of nonhyperelliptic curves of genus 3 induced by the projection map of $\mathbf{P}_{0,0,1}$.

Conversely, assume that S has a pencil $|D|$ as in the statement. Then we have $KD = 4$, $D^2 = 0$. We choose a general $D \in |D|$ and consider the exact sequence

$$0 \rightarrow \mathcal{O}(K - (i+1)D) \rightarrow \mathcal{O}(K - iD) \rightarrow \mathcal{O}_D(K_D) \rightarrow 0,$$

for $i = 0, 1$. Since Φ_K is birational, $H^0(K) \rightarrow H^0(K_D)$ is surjective. Thus $h^0(K - D) = 1$. We show $H^0(K - 2D) = 0$. For this purpose, we take a general $C \in |K|$ and consider

$$0 \rightarrow \mathcal{O}(-2D) \rightarrow \mathcal{O}(K - 2D) \rightarrow \mathcal{O}_C(K - 2D) \rightarrow 0.$$

We have $H^0(-2D) = 0$. Further, since $C(K - 2D) = -3$, we have $H^0(C, \mathcal{O}_C(K - 2D)) = 0$.

Thus $H^0(K - 2D) = 0$. We can take $w_0 \in H^0(K - D)$ and $w_1, w_2 \in H^0(K)$ so that they span $H^0(K_D)$. Then, by using the triple (w_0, w_1, w_2) , we can lift the canonical map to $\psi : S \rightarrow \mathbf{P}_{0,0,1}$ and have $K = \psi^*T$. Then $S'' := \psi(S)$ is linearly equivalent to $4T + F$, since $\psi(D)$ is a plane curve of degree 4 (cf. §1). Φ_K is the composite of ψ and the map Φ_T induced by $|T|$. Since $H^0(\mathbf{P}_{0,0,1}, \mathcal{O}(T)) \simeq H^0(\mathbf{P}^1, \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$, we can take $\{X_0, X_1, z_0 X_2, z_1 X_2\}$ as a basis, where (X_0, X_1, X_2) is the same as that in 2.4 and (z_0, z_1) is a homogeneous coordinate system of \mathbf{P}^1 . Φ_T contracts the rational curve $X_1 = X_2 = 0$. If $(\zeta_0 : \zeta_1 : \zeta_2 : \zeta_3)$ is a homogeneous coordinate system on \mathbf{P}^3 and if Φ_T is given by

$$\zeta_0 = X_0, \quad \zeta_1 = X_1, \quad \zeta_2 = z_0 X_2, \quad \zeta_3 = z_1 X_2,$$

then, by substituting these to (6), we find that the equation of S' can be written as

$$\begin{aligned} \alpha_1 \zeta_0^4 + \alpha_2 \zeta_0^3 \zeta_1 + \alpha_3 \zeta_0^2 \zeta_1^2 + \alpha_4 \zeta_0 \zeta_1^3 + \alpha_5 \zeta_1^4 + \beta_1 \zeta_0^3 + \beta_2 \zeta_0^2 \zeta_1 + \beta_3 \zeta_0 \zeta_1^2 + \beta_4 \zeta_1^3 + \gamma_1 \zeta_0^2 \\ + \gamma_2 \zeta_0 \zeta_1 + \gamma_3 \zeta_1^2 + \delta_1 \zeta_0 + \delta_2 \zeta_1 + \varepsilon = 0, \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ and ε are homogeneous forms of respective degrees 1, 2, 3, 4 and 5 in ζ_2, ζ_3 . Therefore S' contains a line l defined by $\zeta_2 = \zeta_3 = 0$. q.e.d.

3. Number of moduli. In this and the next sections, we study deformations of surfaces of type I. Since we have Horikawa's works [9] and [10, IV] for $p_g \leq 5$, we assume $p_g \geq 6$ throughout. Further, we restrict ourselves to the case (D) in §1, because the case (C) can be found in [12]. Our main result here is Theorem 3.2 below. For a complex manifold M , we denote by Θ_M the tangent sheaf of M .

3.1. We say that S is a Castelnuovo surface of type (a, b, c) if W (or its nonsingular model) is $\mathbf{P}_{a,b,c}$, where the integers a, b, c satisfy the conditions (4) and (5). For the sake of simplicity, we put $W = \mathbf{P}_{a,b,c}$ even if $a=0$. We say S to be generic if it is the minimal resolution of a general member of $|4T - (p_g - 5)F|$.

THEOREM 3.2. *Let S be a generic Castelnuovo surface of type (a, b, c) with $c \leq 2a + 2$. Then*

$$h^1(S, \Theta_S) = \begin{cases} 5p_g + 18, & \text{if } a > 0, \\ 5p_g + 19, & \text{if } a = 0. \end{cases}$$

Further, the Kuranishi space is nonsingular of dimension $h^1(\Theta_S) = 5p_g + 18$ if $a > 0$.

For the proof, we need some lemmas.

LEMMA 3.3. *Let S be a Castelnuovo surface of type (a, b, c) and assume that $p_g(S) \geq 6$. Let $|D|$ be the pencil of curves of genus 3 on S induced by the projection map of $W = \mathbf{P}_{a,b,c}$.*

- (1) *If $a > 0$, then $h^0(2D) = 3$, $h^1(2D) = 0$ and $h^2(2D) = p_g - 6$.*
- (2) *If $a = 0$, then $h^0(2D) = 3$, $h^1(2D) = 1$ and $h^2(2D) = p_g - 5$.*

PROOF. Let S'' be the image of S in $\mathbf{P}_{a,b,c}$ described in §1. Since it has only RDP's,

we have $\psi_*\mathcal{O}_S \simeq \mathcal{O}_{S''}$ and $R^q\psi_*\mathcal{O}_S = 0$ for $q > 0$, where $\psi: S \rightarrow S''$ is the natural map. Thus $H^p(S, \mathcal{O}(2D)) \simeq H^p(S'', \mathcal{O}(2F|_{S''}))$ for any p . We consider the cohomology long exact sequence for

$$0 \rightarrow \mathcal{O}_W(2F - S'') \rightarrow \mathcal{O}_W(2F) \rightarrow \mathcal{O}_{S''}(2F) \rightarrow 0.$$

We have $H^p(W, \mathcal{O}_W(2F)) \simeq H^p(\mathbf{P}^1, \mathcal{O}(2))$ and $H^p(2F - S'') \simeq H^{3-p}(\mathcal{O}_W(K_W + S'' - 2F))^*$ by the Serre duality. Since $S'' \sim 4T - (p_g - 5)F$ and $a + b + c = p_g - 3$, we have $K_W + S'' - 2F \sim T - 2F$. Thus $H^{3-p}(\mathcal{O}_W(K_W + S'' - 2F)) \simeq H^{3-p}(\mathbf{P}^1, \mathcal{O}(a-2) \oplus \mathcal{O}(b-2) \oplus \mathcal{O}(c-2))$. From these, Lemma 3.3 follows. q.e.d.

LEMMA 3.4. *If $W = \mathbf{P}_{a,b,c}$, then*

$$h^q(W, \Theta_W) = \begin{cases} 2(c-a) + 8 + (a-b+1)^+ + (a-c+1)^+ + (b-c+1)^+, & (q=0), \\ (b-a-1)^+ + (c-a-1)^+ + (c-b-1)^+, & (q=1), \\ 0, & (q \geq 2), \end{cases}$$

where $m^+ = \max(m, 0)$.

PROOF. We recall the fundamental exact sequences

$$(7) \quad 0 \rightarrow \Theta_{W/\mathbf{P}^1} \rightarrow \Theta_W \rightarrow \varpi^*\Theta_{\mathbf{P}^1} \rightarrow 0$$

and

$$(8) \quad 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(T - aF) \oplus \mathcal{O}(T - bF) \oplus \mathcal{O}(T - cF) \rightarrow \Theta_{W/\mathbf{P}^1} \rightarrow 0,$$

where Θ_{W/\mathbf{P}^1} is the relative tangent sheaf. Since any automorphism of \mathbf{P}^1 preserves $\mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$, the natural map $\text{Aut}(W) \rightarrow \text{Aut}(\mathbf{P}^1)$ is surjective, hence so is the map $H^0(\Theta_W) \rightarrow H^0(\varpi^*\Theta_{\mathbf{P}^1})$. By (7) and the isomorphism $H^q(\varpi^*\Theta_{\mathbf{P}^1}) \simeq H^q(\Theta_{\mathbf{P}^1})$, we have $h^0(\Theta_W) = h^0(\Theta_{W/\mathbf{P}^1}) + 3$ and $h^q(\Theta_W) = h^q(\Theta_{W/\mathbf{P}^1})$ for $q > 0$. Then a calculation using (8) shows Lemma 3.4. q.e.d.

LEMMA 3.5. *Let S be as in Lemma 3.3 and consider the linear map $\psi_p^*: H^p(W, \Theta_W) \rightarrow H^p(S, \psi^*\Theta_W)$.*

- (1) *If $a > 0$, then ψ_p^* is bijective for $p \leq 1$ and $h^2(\psi^*\Theta_W) = p_g - 6$.*
- (2) *If $a = 0$, then ψ_p^* is bijective for $p = 0$ and is injective for $p = 1$. Furthermore, $h^1(\psi^*\Theta_W) = h^1(\Theta_W) + 1$, $h^2(\psi^*\Theta_W) = p_g - 5$.*

PROOF. We use the commutative diagram

$$\begin{array}{ccccc} H^p(\psi^*\Theta_{W/\mathbf{P}^1}) & \rightarrow & H^p(\psi^*\Theta_W) & \rightarrow & H^p(2D) \\ \uparrow & & \uparrow & & \uparrow \\ H^p(\Theta_{W/\mathbf{P}^1}) & \rightarrow & H^p(\Theta_W) & \rightarrow & H^p(2F), \end{array}$$

where the bottom row comes from the exact sequence (7). By Lemmas 3.3 and 3.4, it suffices to show that $H^p(\Theta_{W/\mathbf{P}^1}) \rightarrow H^p(\psi^*\Theta_{W/\mathbf{P}^1})$ is bijective for any p . Since we have

$H^p(\psi^*\Theta_{W/\mathbf{P}^1}) \simeq H^p(S'', \Theta_{W/\mathbf{P}^1})$, we only have to show $H^p(W, \Theta_{W/\mathbf{P}^1}(-S'')) = 0$ for any p in view of the exact sequence

$$0 \rightarrow \Theta_{W/\mathbf{P}^1}(-S'') \rightarrow \Theta_{W/\mathbf{P}^1} \rightarrow \Theta_{W/\mathbf{P}^1}|_{S''} \rightarrow 0.$$

If Ω_{W/\mathbf{P}^1} is the relative cotangent sheaf, we get $H^p(\Theta_{W/\mathbf{P}^1}(-S''))^* \simeq H^{3-p}(\Omega_{W/\mathbf{P}^1}(T))$ by the Serre duality. We recall that $H^q(\mathbf{P}^2, \Omega^1(1))$ vanishes for any q . Thus we get $R^q\pi_*\Omega_{W/\mathbf{P}^1}(T) = 0$ for any q . Then it follows from the Leray spectral sequence that $H^{3-p}(\Omega_{W/\mathbf{P}^1}(T)) = 0$ for any p . q.e.d.

LEMMA 3.6. *Let S be as in Theorem 3.2 and denote by $\mathcal{T}_{S/W}$ the cokernel of the natural map $\Theta_S \rightarrow \psi^*\Theta_W$. Then $H^2(S, \mathcal{T}_{S/W}) = 0$. Further, the composite $P \circ \psi_1^*$ of $\psi_1^*: H^1(W, \Theta_W) \rightarrow H^1(S, \psi^*\Theta_W)$ and $P: H^1(S, \psi^*\Theta_W) \rightarrow H^1(S, \mathcal{T}_{S/W})$ is surjective.*

PROOF. We first assume that $4a \geq p_g - 5$. As we have seen in 2.4, a general member of $|4T - (p_g - 5)F|$ is irreducible and nonsingular. Thus we can assume $S \in |4T - (p_g - 5)F|$. Then $\mathcal{T}_{S/W}$ is nothing but the normal sheaf $N_{S/W}$. Consider the cohomology long exact sequence for

$$0 \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_W(4T - (p_g - 5)F) \rightarrow N_{S/W} \rightarrow 0.$$

We see that $H^q(S, N_{S/W}) = 0$ for $q > 0$, because we have $H^q(W, \mathcal{O}(4T - (p_g - 5)F)) = 0$ for $q > 0$ by the assumption $4a \geq p_g - 5$.

We next consider the case $4a < p_g - 5$. As we have seen in 2.4, $S'' = \psi(S)$ contains a rational curve Z defined by $X_1 = X_2 = 0$. We denote by $v: X \rightarrow \mathbf{P}_{a,b,c}$ the blowing-up along Z . It is easy to see that X is the total space of the \mathbf{P}^1 -bundle $\pi: \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(C_0 + (b-a)f)) \rightarrow \Sigma_{c-b}$. We denote by L_0 the tautological divisor of X . If we let L_∞ be the unique divisor linearly equivalent to $L_0 - \pi^*(C_0 + (b-a)f)$, then $L_\infty = v^{-1}(Z)$. The proper transform of S'' is in $|3L_0 + \pi^*(C_0 + (2a-c+2)f)|$. Since $c \leq 2a+2$, this linear system has no base points. Thus we can assume $S \in |3L_0 + \pi^*(C_0 + (2a-c+2)f)|$.

By a simple calculation, we have $H^q(X, \mathcal{O}(S)) = 0$ for $q > 0$. This implies $H^q(S, N_{S/X}) = 0$ for $q > 0$. Then by the exact sequence

$$0 \rightarrow N_{S/X} \rightarrow \mathcal{T}_{S/W} \rightarrow \mathcal{T}_{X/W}|_S \rightarrow 0,$$

we have $H^q(S, \mathcal{T}_{S/W}) \simeq H^q(S, \mathcal{T}_{X/W}|_S)$ for $q > 0$. Since $\mathcal{T}_{X/W}|_S$ is supported on a curve, we have $h^2(\mathcal{T}_{S/W}) = h^2(\mathcal{T}_{X/W}|_S) = 0$. By [10, III, p. 235], the following sequence is exact:

$$0 \rightarrow N_{L_\infty/X} \rightarrow v^*N_{Z/W} \rightarrow \mathcal{T}_{X/W} \rightarrow 0.$$

We identify Z and L_∞ with \mathbf{P}^1 and Σ_{c-b} , respectively. Then we have $N_{Z/W} \simeq \mathcal{O}(a-b) \oplus \mathcal{O}(a-c)$ and $N_{L_\infty/X} \simeq \mathcal{O}(-C_0 - (b-a)f)$. Thus $\mathcal{T}_{X/W} \simeq v^*(\det N_{Z/W}) \otimes N_{L_\infty/X}^* \simeq \mathcal{O}(C_0 - (c-a)f)$.

To show the surjectivity of $P \circ \psi_1^*$, it suffices to show that the map $H^1(X, v^*\Theta_W) \rightarrow H^1(S, \mathcal{T}_{X/W}|_S)$ is surjective, since ψ_1^* is injective by Lemma 3.5. Note

that we have $H^2(X, \Theta_X) = 0$ by the exact sequence

$$0 \rightarrow \mathcal{O}(2L_0 - \pi^*(C_0 + (b-a)f)) \rightarrow \Theta_X \rightarrow \pi^*\Theta_{\mathcal{E}_{c-b}} \rightarrow 0.$$

Thus $H^1(X, v^*\Theta_W) \rightarrow H^1(X, \mathcal{T}_{X/W})$ is surjective. Consider the exact sequence

$$0 \rightarrow \mathcal{T}_{X/W}(-S) \rightarrow \mathcal{T}_{X/W} \rightarrow \mathcal{T}_{X/W}|_S \rightarrow 0.$$

Since $S|_{L_\infty} \sim C_0 + (2a - c + 2)f$, we have $\mathcal{T}_{X/W}(-S) \simeq \mathcal{O}(-(a+2)f)$. Then $H^2(X, \mathcal{T}_{X/W}(-S)) = 0$ and thus $H^1(X, \mathcal{T}_{X/W}) \rightarrow H^1(S, \mathcal{T}_{X/W}|_S)$ is surjective. q.e.d.

3.7. PROOF OF THEOREM 3.2. By Lemmas 3.5 and 3.6, we have

$$h^2(S, \Theta_S) = \begin{cases} p_g - 6, & \text{if } a > 0, \\ p_g - 5, & \text{if } a = 0. \end{cases}$$

Since S is of general type, we have $H^0(S, \Theta_S) = 0$. Thus the formula for $h^1(\Theta_S)$ follows from the Riemann-Roch theorem.

In order to show the second assertion, we use Horikawa's deformation theory of holomorphic maps [8]. By Lemma 3.3, we have $H^1(2D) = 0$. Thus it follows from [8, II, Theorem 4.4] that there is a family $p: \mathcal{S} \rightarrow M$ of deformations of $S = p^{-1}(o)$, $o \in M$, such that the characteristic map $\tau: T_o M \rightarrow D_{S/\mathcal{P}}$ is bijective. Further, we see from [8, II, Lemma 4.2] that the Kodaira-Spencer map $\rho: T_o M \rightarrow H^1(\Theta_S)$ is surjective. Note that the parameter space M is nonsingular. Thus we can choose a submanifold N of M passing through o such that the Kodaira-Spencer map $\rho: T_o N \rightarrow H^1(\Theta_S)$ is bijective. This completes the proof.

COROLLARY 3.8. *Let S be as in Theorem 3.2. Then the infinitesimal Torelli theorem holds for S .*

PROOF. By the criterion of Kii [11], we only have to show $h^0(\Omega_S^1(K)) \leq p_g - 2$. Since $h^0(\Omega_S^1(K)) = h^2(\Theta_S) \leq p_g - 5$, we are done. q.e.d.

4. A remark on deformations.

4.1. We construct a family of deformations of $P_{a,b,c}$ (for a geometric treatment of deformations of scrolls, see [6]). We denote by d the greatest integer not exceeding $(a+b+c)/3$. By Lemma 3.3, we can assume $(a, b, c) \neq (d, d, d), (d, d, d+1), (d, d+1, d+1)$, since in these cases $P_{a,b,c}$ is rigid.

Let U_1 and U_2 be two copies of $\mathbb{C} \times \mathbb{P}^2$. We denote by $(z; X_0, X_1, X_2)$ and $(\hat{z}; \hat{X}_0, \hat{X}_1, \hat{X}_2)$ the coordinates of U_1 and U_2 , respectively. Then we can construct $P_{a,b,c}$ from $U_1 \cup U_2$ by identifying $(z; X_0, X_1, X_2)$ with $(\hat{z}; \hat{X}_0, \hat{X}_1, \hat{X}_2)$ if and only if

$$(9) \quad X_0 = \hat{z}^a \hat{X}_0, X_1 = \hat{z}^b \hat{X}_1, X_2 = \hat{z}^c \hat{X}_2, z\hat{z} = 1.$$

We let t be a complex parameter and identify $(z; X_0, X_1, X_2)$ with $(\hat{z}; \hat{X}_0, \hat{X}_1, \hat{X}_2)$ if and only if

$$(10) \quad X_0 = \hat{z}^a \hat{X}_0, \quad X_1 = \hat{z}^b \hat{X}_1, \quad X_2 = \hat{z}^c \hat{X}_2 + t \hat{z}^{a+k} \hat{X}_0, \quad z\hat{z} = 1,$$

where k is an integer satisfying $0 \leq k \leq c-a$. Then we get a family $\{W_t\}$ of P^2 -bundles on P^1 . For $t \neq 0$, we put

$$\begin{aligned} Y_0 &= tX_2, & Y_1 &= X_1, & Y_2 &= z^k X_2 - tX_0, \\ \hat{Y}_0 &= t^2 \hat{X}_0 + t \hat{z}^{c-a-k} \hat{X}_2, & \hat{Y}_1 &= \hat{X}_1, & \hat{Y}_2 &= \hat{X}_2. \end{aligned}$$

Then we get $Y_0 = \hat{z}^{a+k} \hat{Y}_0$, $Y_1 = \hat{z}^b \hat{Y}_1$, $Y_2 = \hat{z}^{c-k} \hat{Y}_2$. Thus $W_t \simeq P_{a+k, b, c-k}$ if $t \neq 0$. Similarly, if we consider the families

$$(11) \quad X_0 = \hat{z}^a \hat{X}_0, \quad X_1 = \hat{z}^b \hat{X}_1 + t \hat{z}^{a+k} \hat{X}_0, \quad X_2 = \hat{z}^c \hat{X}_2, \quad z\hat{z} = 1$$

and

$$(12) \quad X_0 = \hat{z}^a \hat{X}_0, \quad X_1 = \hat{z}^b \hat{X}_1, \quad X_2 = \hat{z}^c \hat{X}_2 + t \hat{z}^{b+k} \hat{X}_1, \quad z\hat{z} = 1$$

for a suitable k , then we see that $P_{a,b,c}$ is a deformation of $P_{a+k, b-k, c}$ and $P_{a, b+k, c-k}$, respectively. Thus we get:

PROPOSITION 4.2. *The P^2 -bundle $P_{a,b,c}$ is a deformation of $P_{d,d,d}$, $P_{d,d,d+1}$ or $P_{d,d+1,d+1}$ according as $a+b+c$ is 0, 1 or 2 modulo 3.*

PROPOSITION 4.3. *Let S be a Castelnuovo surface of type (a, b, c) and assume that $c \leq 2a+3$. Then S is a deformation of a Castelnuovo surface of type (d, d, d) , $(d, d, d+1)$ or $(d, d+1, d+1)$.*

PROOF. We showed in §1 that S is a minimal resolution of a surface $S'' \sim 4T - (p_g - 5)F$ on $P_{a,b,c}$.

If $3a+1 \geq b+c$, then $H^1(P_{a,b,c}, \mathcal{O}(4T - (p_g - 5)F)) = 0$. Thus if $s \in H^0(4T - (p_g - 5)F)$ defines S'' , then it can be extended to any family of deformations of $P_{a,b,c}$. Thus we get a family $\{S'_t\}$ with $S'' = S'_0$ from the family (10) for example. Since S'' has only RDP's, so does S'_t provided that t is sufficiently small. We simultaneously resolve RDP's (cf. [2] and [3]) and get a family $\{S_t\}$ of deformations of $S = S_0$. This family shows that S is a specialization of a Castelnuovo surface of type $(a+k, b, c-k)$. Continuing this procedure using (10), (11) or (12), we get the desired result.

If $3a+1 < b+c$ but $c \leq 2a+3$, we consider the P^1 -bundle X in the proof of Lemma 3.6 instead of $P_{a,b,c}$. Since X is a monoidal transform of $P_{a,b,c}$, a sufficiently small deformation of the former is a monoidal transform of a deformation of the latter (see, [8, III]). Indeed, by blowing up the rational curve defined by $X_1 = X_2 = 0$ in the family (12) simultaneously, we get a family $\{X_t\}$ of deformations of $X = X_0$. We remark that $H^1(X, \mathcal{O}(3L_0 + \pi^*(C_0 + (2a-c+2)f))) = 0$ if $c \leq 2a+3$. Thus similar arguments also work. q.e.d.

REMARK 4.4. The moduli space of Castelnuovo surfaces has several components

in general. To see this, let $\mu = \dim |4T - (p_g - 5)F| - h^0(\mathbf{P}_{a,b,c}, \Theta)$ be the number of parameters on which Castelnuovo surfaces of type (a, b, c) depend. Then μ is sometimes strictly greater than the number $5p_g + 18$ of moduli of a generic Castelnuovo surface as in Theorem 3.2. For example, if $p_g = 12$, we have

$$(a, b, c; \mu) = (3, 3, 3; 78), \quad (2, 3, 4; 77), \quad (2, 2, 5; 74), \quad (1, 2, 6; 74), \\ (1, 3, 5; 75), \quad (1, 4, 4; 76), \quad (0, 2, 7; 79).$$

4.5. Here we give a family of surfaces with $c_1^2 = 3p_g - 7$ such that the central fiber is of type II while a general fiber is of type I.

We let W be the \mathbf{P}^2 -bundle $\mathbf{P}_{a,b,c}$, where a, b, c are integers satisfying (4) and (5). We assume that p_g is odd and put $2k = p_g - 5$. The linear system $|L|$, $L = 2T - kF$, is free from base points if $2a - k \geq 0$, i.e., $3a + 2 \geq b + c$. We assume this condition for the sake of simplicity. We choose $\eta \in H^0(W, \mathcal{O}(L))$ which defines an irreducible nonsingular divisor Y . Let \hat{W} be the \mathbf{P}^1 -bundle $\mathbf{P}(\mathcal{O}_W \oplus \mathcal{O}_W(L))$ on W . We put $\Delta_\varepsilon = \{t \in C; |t| < \varepsilon\}$, where ε is a sufficiently small positive number. Consider a family $\{S_t\}$, $t \in \Delta_\varepsilon$, of subvarieties of \hat{W} given by the equation

$$(13) \quad S_t : \begin{cases} Z_0^2 + \alpha_1 Z_0 Z_1 + \alpha_2 Z_1^2 = 0 \\ tZ_0 = \eta X_1 \end{cases}, \quad t \in \Delta_\varepsilon,$$

where (Z_0, Z_1) is a system of homogeneous fiber coordinates on \hat{W} and $\alpha_i \in H^0(W, \mathcal{O}(iL))$, $1 \leq i \leq 2$. We assume that α 's are general.

If $t \neq 0$, then S_t is biholomorphically equivalent to a surface in W defined by the equation $\eta^2 + t\alpha_1\eta + t^2\alpha_2 = 0$. Thus $S_t \in |4T - (p_g - 5)F|$ and it is of type I.

On the other hand, S_0 is a double covering of Y via the projection map of \hat{W} . Since Y is a conic bundle on \mathbf{P}^1 , S_0 has a pencil of hyperelliptic curves of genus three. Thus it is of type II.

5. Surfaces with hyperelliptic pencils of genus 3. We call a pencil on a surface a *hyperelliptic pencil of genus g* if its general member is a nonsingular hyperelliptic curve of genus g . In this section, we study the geography of surfaces with hyperelliptic linear pencils of genus 3.

5.1. Let V be a normal Gorenstein surface and denote by $\sigma: V' \rightarrow V$ the minimal resolution of an isolated singularity ξ of V . Then there exists an effective divisor Z_ξ on V' supported on $\sigma^{-1}(\xi)$ such that $\omega_{V'} \simeq \sigma^*\omega_V \otimes \mathcal{O}(-Z_\xi)$ (e.g., [14]). Then we have $\omega_{V'}^2 = \omega_V^2 + Z_\xi^2$. On the other hand, the spectral sequence $H^p(V, R^q\sigma_*\mathcal{O}_{V'}) \Rightarrow H^{p+q}(V', \mathcal{O}_{V'})$ implies that $\chi(\mathcal{O}_{V'}) = \chi(\mathcal{O}_V) - p_g(\xi)$, where $p_g(\xi) = h^0(V, R^1\sigma_*\mathcal{O}_{V'})$ is the geometric genus of ξ . We call $(p_g(\xi); -Z_\xi^2)$ the type of singularity ξ . If ξ is a double point, then its type can be easily calculated by Horikawa's canonical resolution ([9, §2] or [13, §1]).

Let $\{\xi_i\}_{1 \leq i \leq s}$ be a set of isolated singularities of V and assume that ξ_i is of type

(m_i, n_i) . If $V^* \rightarrow V$ is the minimal resolution of these singularities, then we have

$$(14) \quad \chi(\mathcal{O}_{V^*}) = \chi(\mathcal{O}_V) - \sum_{i=1}^s m_i, \quad \omega_{V^*}^2 = \omega_V^2 - \sum_{i=1}^s n_i.$$

We list the types of the singularities which we shall need later:

- (i) If ξ is an RDP, then $Z_\xi = 0$. Thus ξ is of type $(0 : 0)$.
- (ii) If ξ is a simple elliptic singularity of type \tilde{E}_8 or \tilde{E}_7 (see [15]), then $Z_\xi = E$, where E is the exceptional elliptic curve. Thus ξ is of type $(1 : 1)$ or $(1 : 2)$ according as whether it is \tilde{E}_8 or \tilde{E}_7 .

5.2. Let $\pi_0: Y = \Sigma_e \rightarrow \mathbf{P}^1$ be the Hirzebruch surface of degree e . Put $L := 4C_0 + \beta f$, where β is an integer satisfying

$$(15) \quad e + \beta \geq 2.$$

We consider the \mathbf{P}^1 -bundle

$$\pi: X = \mathbf{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L)) \rightarrow Y,$$

and set $L_0 = \mathcal{O}_X(1)$, $D_0 = \pi^* C_0$ and $F = \pi^* f$.

LEMMA 5.3. *Let V be an irreducible reduced divisor on X linearly equivalent to $2L_0$. Then,*

- (1) $\omega_V \simeq \pi^*(K_Y + L)|_V \simeq \pi^*(2C_0 + (\beta + e - 2)f)|_V$,
- (2) $\omega_V^2 = 16e + 8\beta - 16$,
- (3) $p_g(V) = 6e + 3\beta - 3$, $q(V) = 0$.

PROOF. Since $K_X \simeq -2L_0 + \pi^*(K_Y + L)$ and $\omega_V \simeq (K_X + V)|_V$, we get (1). Since $L_0 D_0^2 = e$, $L_0 D_0 F = 1$ and $L_0 F^2 = 0$, we get (2) by (1). Considering the cohomology long exact sequence for $0 \rightarrow \mathcal{O}(K_X) \rightarrow \mathcal{O}(K_X + V) \rightarrow \omega_V \rightarrow 0$, we easily obtain (3) by (15).

q.e.d.

5.4. Let V be as above. By a suitable system of homogeneous fiber coordinates $(X_0 : X_1)$ of $\pi: X \rightarrow Y$, the equation ϕ of V can be written as

$$(16) \quad \phi = X_0^2 + \phi_{2L} X_1^2,$$

where $\phi_{2L} \in H^0(Y, \mathcal{O}(2L))$. We note that V is a double covering of Y via $\pi|_V$ and its branch locus is $B_V = (\phi_{2L})$. We assume that B_V is reduced. Then V is normal. Set $\lambda = \pi_0 \circ \pi|_V: V \rightarrow \mathbf{P}^1$. Then, by the Hurwitz formula, a general fiber of λ is a hyperelliptic curve of genus 3. Assume further that

- (*) B_V has k infinitely close triple points (cf. [13, §1]) and l ordinary quadruple points, and the other singularities of B_V are at most double points.

The V is a normal Gorenstein surface with k singular points of type \tilde{E}_8 and l singular points of type \tilde{E}_7 . We remark that the other singularities are at most RDP's. Let $\sigma: V^* \rightarrow V$ be the minimal resolution of all singularities of V . Then by (14), (i) and

(ii) of 5.1 and Lemma 5.3, we have

$$(17) \quad \chi(\mathcal{O}_{V^*}) = 6e + 3\beta - 2 - k - l, \quad \omega_{V^*}^2 = 16e + 8\beta - 16 - k - 2l.$$

In order to construct B_V satisfying (*), we use the method essentially due to Persson [13]. For an integer $r \geq 0$, any $\psi \in H^0(Y, \mathcal{O}(2C_0 + rf))$ can be written as

$$\psi = \psi_r Y_0^2 + \psi_{r+e} Y_0 Y_1 + \psi_{r+2e} Y_1^2$$

where $(Y_0 : Y_1)$ is a system of homogeneous fiber coordinates of $\pi_0 : Y \rightarrow \mathbf{P}^1$ and $\psi_{r+ie} \in H^0(\mathbf{P}^1, \mathcal{O}(r+ie))$, $0 \leq i \leq 2$. We put $C_0 = (Y_0)$ and $C_\infty = (Y_1)$. We often identify them with the base curve \mathbf{P}^1 of Y .

We define a sublinear system of $|2C_0 + rf|$ by

$$|2C_0 + rf|_P = \{(\psi) : \psi = \psi_r Y_0^2 + \psi_{r+2e} Y_1^2\}$$

and call it *Persson's system*. As is easily seen, it has the following properties:

(a) Put $\psi \in |2C_0 + rf|_P$. If ψ_{r+2e} and ψ_r have simple zeros only and if they have no common zero, then (ψ) is nonsingular. We regard (ψ_{r+2e}) and (ψ_r) as reduced divisors on C_0 and C_∞ , respectively. Set $(\psi_{r+2e}) = \sum_{i=1}^{2e+r} P_i$ and $(\psi_r) = \sum_{i=2e+r+1}^{2e+2r} P_i$. Then the tangent line $T_{P_i}(\psi)$ of (ψ) at P_i is vertical for $1 \leq i \leq 2e+2r$, i.e., T_{P_i} coincides with the fiber of π_0 passing through P_i .

(b) Let k_0 , k_∞ and l be nonnegative integers satisfying $k_0 \leq 2e+r$, $k_\infty \leq r$ and $l \leq 2e+2r-k_0-k_\infty-1$. Let P_1, \dots, P_{k_0} and Q_1, \dots, Q_{k_∞} be mutually distinct points on C_0 and C_∞ , respectively, and let R_1, \dots, R_l be generic points on $Y \setminus (C_0 \cup C_\infty)$. Let Λ be the linear subsystem of $|2C_0 + rf|_P$ consisting of those elements passing through all the $k_0+k_\infty+l$ points P_i, Q_j, R_k . Then a general member of Λ is nonsingular, and we have $\dim \Lambda = 2e+2r-k_0-k_\infty-l \geq 1$. Moreover, the system Λ has no base point except P_i, Q_j, R_k .

By using these properties, we can show the following:

LEMMA 5.5. *Fix a nonnegative integer l and put*

$$k_{\max} = \max_{(l_1, l_2)} \{ [(2/3)(4e + \beta - 2l_1 - 2)] + [(2/3)(\beta - 2l_2)] \},$$

where l_1 and l_2 run through nonnegative integers satisfying $l_1 + l_2 = l$, $4e + \beta - 2l_1 - 2 \geq 0$ and $\beta - 2l_2 \geq 0$, and $[q]$ is the greatest integer not exceeding q . Then, for any integer k with $0 \leq k \leq k_{\max}$, there exists a reduced divisor B on Y such that

$$(1) \quad B \sim 8C_0 + 2\beta f,$$

(2) B has k infinitely close triple points and l ordinary quadruple points. The other singularities of B are at most double points.

PROOF. Let k_0, k_∞ be nonnegative integers satisfying

$$(18) \quad k = k_0 + k_\infty, \quad k_0 \leq [(2/3)(4e + \beta - 2l_1 - 2)], \quad k_\infty \leq [(2/3)(\beta - 2l_2)].$$

We choose mutually distinct points P_1, \dots, P_{k_0} on C_0 and Q_1, \dots, Q_{k_∞} on C_∞ . We also choose general points R_1, \dots, R_l on $Y \setminus (C_0 \cup C_\infty)$.

When β is even, we set $r_i = [\beta/2]$ for $1 \leq i \leq 4$. When β is odd, we set $r_1 = r_2 = [\beta/2] + 1$ and $r_3 = r_4 = [\beta/2]$. By (18), we can choose nonnegative integers $k_0(i)$ and $k_\infty(i)$ for $1 \leq i \leq 4$ such that

$$(i) \quad k_0(i) \leq 2e + r_i - l_1 - 1, \quad k_\infty(i) \leq r_i - l_2,$$

(ii) there are subsets $\{P_1^{(i)}, \dots, P_{k_0(i)}^{(i)}\}$ and $\{Q_1^{(i)}, \dots, Q_{k_\infty(i)}^{(i)}\}$ of $\{P_j; 1 \leq j \leq k_0\}$ and $\{Q_j; 1 \leq j \leq k_\infty\}$, respectively, such that $\sum_{i=1}^4 (P_1^{(i)} + \dots + P_{k_0(i)}^{(i)}) = 3 \sum_{j=1}^{k_0} P_j$ and $\sum_{i=1}^4 (Q_1^{(i)} + \dots + Q_{k_\infty(i)}^{(i)}) = 3 \sum_{j=1}^{k_\infty} Q_j$.

Let A_i be the subsystem of $|2C_0 + r_i f|_P$ consisting of elements passing through all the $k_0(i) + k_\infty(i) + l$ points $P_1^{(i)}, \dots, P_{k_0(i)}^{(i)}, Q_1^{(i)}, \dots, Q_{k_\infty(i)}^{(i)}$ and R_1, \dots, R_l . Let B_i be a general member of A_i and set $B = \sum_{i=1}^4 B_i$. Then B satisfies (1) and (2). q.e.d.

Take k, l and B as in Lemma 5.5 and let $\mu := \pi|_V: V \rightarrow Y$ be the double cover branched along B . V has k_0 (resp. k_∞) singular points of type \tilde{E}_8 on $\mu^{-1}(C_0)$ (resp. $\mu^{-1}(C_\infty)$) with $k = k_0 + k_\infty$ and l singular points of type \tilde{E}_7 on $\mu^{-1}(Y \setminus (C_0 \cup C_\infty))$. Moreover the other singularities of V are at most RDP's. We let $\sigma: V^* \rightarrow V$ be the minimal resolution.

PROPOSITION 5.6. *Assume that $k_0 \leq 3e + \beta - 2$, $k_\infty \leq e + \beta - 2$ and $k + l \leq 6e + 3\beta - 7$. Then, we have:*

$$(1) \quad p_g(V^*) = p_g(V) - k - l \text{ and } q(V^*) = 0.$$

(2) $|K_{V^*}|$ is free from fixed components, and has exactly k base points. Especially V^* is relatively minimal.

(3) The canonical map of V^* is generically 2:1 map onto its image.

PROOF. Set $\tilde{\mu} = \sigma \circ \mu$, $\{\eta_1, \dots, \eta_{k+l}\} = \{P_1, \dots, P_{k_0}, Q_1, \dots, Q_{k_\infty}, R_1, \dots, R_l\}$ and $\xi_i = \mu^{-1}(\eta_i)$ for the sake of brevity. By (i) and (ii) of 5.1 and Lemma 5.3, we have

$$(19) \quad K_{V^*} \simeq \tilde{\mu}^*(K_Y + L) \otimes \mathcal{O}_{V^*} \left(- \sum_{i=1}^{k+l} E_i \right),$$

where $E_i = \sigma^{-1}(\xi_i)$ is the exceptional elliptic curve. Hence from the exact sequence

$$0 \rightarrow \mathcal{O}(K_{V^*}) \rightarrow \mathcal{O}(\tilde{\mu}^*(K_Y + L)) \rightarrow \bigoplus_i E_i \rightarrow 0,$$

we get the exact sequence

$$(20) \quad 0 \rightarrow H^0(V^*, K_{V^*}) \rightarrow H^0(V^*, \tilde{\mu}^*(K_Y + L)) \xrightarrow{\rho} \bigoplus_i \mathbf{C}_{E_i},$$

where \mathbf{C}_{E_i} is the sheaf of constant functions on E_i .

Since V is normal, $\sigma_* \mathcal{O}_{V^*} \simeq \mathcal{O}_V$. Thus we have $\tilde{\mu}_* \tilde{\mu}^* \mathcal{O}_Y(K_Y + L) \simeq \mu_* (\mu^* \mathcal{O}_Y(K_Y + L) \otimes \sigma_* \mathcal{O}_{V^*}) \simeq \mathcal{O}_Y(K_Y + L) \otimes \mu_* \mathcal{O}_V \simeq \mathcal{O}_Y(K_Y + L) \otimes (\mathcal{O}_Y \oplus \mathcal{O}_Y(-L)) \simeq \mathcal{O}_Y(K_Y + L) \oplus \mathcal{O}_Y(K_Y)$. Hence $H^0(V^*, \tilde{\mu}^*(K_Y + L))$ is isomorphic to $H^0(Y, K_Y + L)$ and its dimension is $p_g(V)$.

We show that ρ in (20) is surjective. For any $\psi \in H^0(Y, K_Y + L) \simeq H^0(V^*, \tilde{\mu}^*(K_Y + L))$, the map ρ is given by

$$\rho(\psi) = (\psi(\eta_1), \dots, \psi(\eta_{k+l})) \in \bigoplus_i C_{E_i}$$

where $\psi(\eta_i)$ is the value of ψ at η_i .

Denote by M_i , $0 \leq i \leq k+l$, the linear subsystem of $|K_Y + L|$ consisting of elements passing through η_1, \dots, η_i . If η_{i+1} does not belong to the base locus of M_i , then the descending filtration

$$|K_Y + L| = M_0 \supset M_1 \supset \dots \supset M_{k+l}$$

satisfies $\dim M_{i+1} = \dim M_i - 1$ for any i .

On the other hand, for any $(\phi) \in |K_Y + L| = |2C_0 + (e+\beta-2)f|$, ϕ can be written as

$$\phi = \phi_{e+\beta-2} Y_0^2 + \phi_{2e+\beta-2} Y_0 Y_1 + \phi_{3e+\beta-2} Y_1^2,$$

where $\phi_{ie+\beta-2} \in H^0(\mathbf{P}^1, \mathcal{O}(ie+\beta-2))$. Thus from our construction and assumption, it is easy to see that, for each $0 \leq j \leq k+l$, M_j separates points on $Y \setminus \{\eta_1, \dots, \eta_j\}$, that is, for any points $P, Q \in Y \setminus \{\eta_1, \dots, \eta_j\}$, there exists $(\phi) \in M_j$ such that $\phi(P) = 0$ and $\phi(Q) \neq 0$. Especially the base locus of M_j coincides with $\{\eta_1, \dots, \eta_j\}$. Thus ρ is surjective. Then, by (20), we get (1). Moreover, since the rational map of Y associated with M_{k+l} is birational onto its image, (3) follows.

It remains to prove (2). By the above argument, the base locus of $|K_{V^*}|$ is contained in $\bigcup_{i=1}^{k+l} E_i$. Since a generic member of M_{k+l} passes through η_i ($1 \leq i \leq k+l$) smoothly, E_i is not a fixed component of $|K_{V^*}|$ by (19). Thus $|K_{V^*}|$ is free from fixed components.

Let η_i be an infinitely close triple point of B . Let C be a member of M_{k+l} and C^* the proper transform of $\mu^{-1}(C)$ by σ . When C varies in M_{k+l} , C^* passes through the unique point on E_i . (This is easily observed by means of the canonical resolution.) However, it is not the case when η is an ordinary quadruple point. Thus $|K_{V^*}|$ has exactly k base points. q.e.d.

THEOREM 5.7. *Let x, y be any pair of integers satisfying one of the following two conditions:*

- (a) $(8/3)x - 8 \leq y \leq 4x - 16$, $y \neq (1/3)(8x - i)$ ($i = 21, 23$).
- (b) $y = 4x - i$, $x \geq 6$ and x is equivalent modulo 5 to j , where $(i, j) = (8, 0), (9, 2), (10, 2), (10, 4), (11, 1), (11, 4), (12, 1), (12, 3), (12, 4), (13, 0), (13, 1), (13, 3), (14, 0), (14, 1), (14, 3), (14, 4), (15, 0), (15, 2), (15, 3), (15, 4)$.

Then there exists a minimal surface S such that

- (1) $p_g(S) = x$, $q(S) = 0$ and $c_1^2(S) = y$,
- (2) *there is a fibration $\lambda: S \rightarrow \mathbf{P}^1$ whose general fiber is a hyperelliptic curve of genus 3,*
- (3) *$|K_S|$ is free from fixed components and Φ_{K_S} is of degree 2 onto its image.*

PROOF. We set $0 \leq l \leq 4$, $e + \beta \geq 3$ and $(e, \beta, k) \neq (0, 3, 0)$. Then the assumptions in

Proposition 5.6 are satisfied. Under these conditions, we let k vary with $0 \leq k \leq k_{\max}$. By (17), Lemma 5.5 and Proposition 5.6, a calculation shows that the invariants of our surfaces cover the area (a) and (b). q.e.d.

REMARK 5.8. If a regular surface has a hyperelliptic pencil of genus 3, then it satisfies $c_1^2 \geq (8/3)p_g - 8$. See, [10, V] or [13].

6. Surfaces of type II. In this section, we give some remarks on surfaces of type II. Let S be a minimal surface of type II in the sense of §1. We assume that the irregularity of S vanishes. Then the canonical image S' is a rational ruled surface. Thus S has a hyperelliptic pencil induced by the canonical map and the ruling of S' .

For the hyperelliptic structure of S , we have the following theorem due to Xiao [16, §1]:

THEOREM 6.1 (Xiao). *Let S be a regular minimal surface of general type with a hyperelliptic pencil. Suppose that the invariants of S satisfy*

$$p_g(S) > (2g-1)(g+1)+1, \quad c_1^2(S) < (4g/(g+1))(p_g(S)-g-1)$$

for some integer $g \geq 2$. Then S has a hyperelliptic pencil of genus g . Moreover, the hyperelliptic pencil of genus less than $g+1$ is unique.

COROLLARY 6.2. *Assume that S is a regular surface of type II with $c_1^2 = 3p_g - 7$. If $p_g(S) \geq 46$, then it has a hyperelliptic linear pencil of genus less than 5.*

For the existence of surfaces of type II with hyperelliptic pencils of genus less than 5, we have the following:

PROPOSITION 6.3. *Let g be 2, 3 or 4. Then, for any pair of integers (x, y) satisfying $y = 3x - 7$ and $x \geq 4$, there exists a minimal surface S with a hyperelliptic linear pencil of genus g such that $p_g(S) = x$, $q(S) = 0$ and $c_1^2(S) = y$.*

PROOF. The case $g=2$ follows from a more general result of Persson [13, §3]. The case $g=3$ with $p_g \geq 6$ follows from Theorem 5.7. For $p_g = 4, 5$, consult [9] and [10, IV].

We consider the case $g=4$. Set $(k, l)=(1, 3)$ or $(2, 1)$. By an argument similar to that in §5, there exists a reduced divisor B on $Y = \Sigma_e$ such that

(i) $B \sim 10C_0 + 2\beta f$ ($\beta \geq 0$),

(ii) B has k infinitely close triple points and l ordinary quadruple points, and the other singularities are at most double points.

Let V be the double covering of Y branched along B . If S is the minimal resolution of V , then

$$p_g(S) = 10e + 4(\beta - 1) - k - l, \quad q(S) = 0,$$

$$c_1^2(S) = 30e + 12(\beta - 2) - k - 2l = 3p_g(S) - 7,$$

and S has the desired properties.

q.e.d.

REMARK 6.4. For a given $g \geq 2$, there exists a regular surface of type II with a hyperelliptic pencil of genus g . Indeed, we have constructed in the proof of Proposition 6.3 a surface S using the double covering V of $Y = \mathbf{P}^1 \times \mathbf{P}^1$ whose branch locus B is linearly equivalent to $10C_0 + 2\beta f$ for any $\beta \geq 3$. The second projection of Y induces on S another hyperelliptic pencil of genus $g' = \beta - 1$. So we cannot give an upper bound on the genus of hyperelliptic pencils.

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