Algebraic Techniques in Differential Cryptanalysis

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Outline

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The Blockcipher PRESENT

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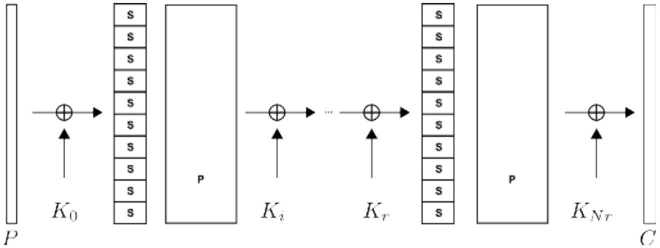
PRESENT [2] was proposed by Bogdanov et al. at CHES 2007.



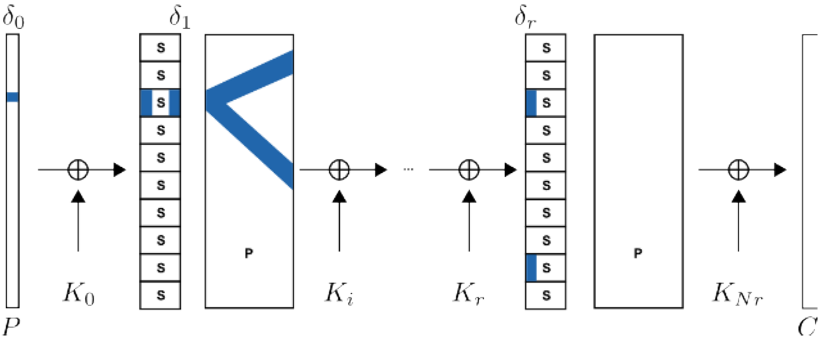
Where S is the 4-bit S-Box and P a permutation of bit positions.

We define reduced round variants and denote them by PRESENT-Ks-Nr.

Prior DC on Reduced Round Versions

Differential characteristics and two round filter function available in [3].

Differential Cryptanalysis I



$$Pr(\delta_i) = p_i \longrightarrow Pr(\Delta) = \prod p_i$$

Differential Cryptanalysis II

Key Recovery:

- **backward key guessing** to recover subkey bits of last rounds not covered by characteristic
- **right pairs** suggest correct and wrong key bits
- **wrong pairs** suggest random key bits
- **filter functions** used to remove wrong pairs
- **candidate key arrays** to count suggestions and observe peak

Differential Cryptanalysis of 16-round DES [1]

- distinguishes right pairs,
- uses outer round active S-Boxes to recover key bits and
- does not rely on candidate key arrays.

Algebraic Cryptanalysis

$$\longrightarrow$$

$$\begin{array}{l}
y_2x_3 + y_3x_3 + x_1x_3 + x_2x_3 + x_3, \\
y_0x_3 + y_3x_3 + x_1x_3 + x_2x_3 + \dots, \\
x_1x_2 + y_3 + x_0 + x_1 + x_3, \\
x_0x_2 + y_3x_3 + x_1x_3 + x_2x_3 + \dots \\
y_3x_2+y_3x_3+x_1x_3+y_0+y_1+y_3 \dots \\
y_0x_2 + y_1x_2 + y_1x_3 + y_3x_3 + \dots \\
x_0x_1 + y_3x_3 + x_1x_3 + x_2x_3 + \dots \\
y_3x_1 + y_3x_3 + x_2x_3 + \dots, \dots
\end{array}$$

We call $X_{i,j}$ and $Y_{i,j}$ the input resp. output variable for the j-th bit of the i-th S-Box application (i.e. round).

For example, for PRESENT-80-31 we have a system of 4172 variables in 13642 equations.

Multiple $P - C$ Pairs I

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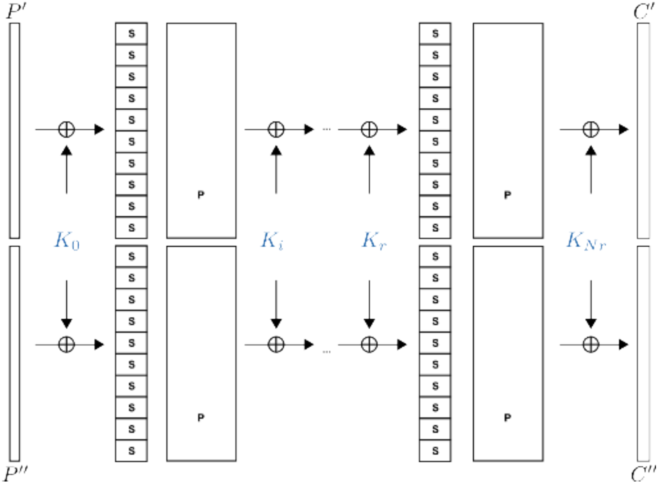
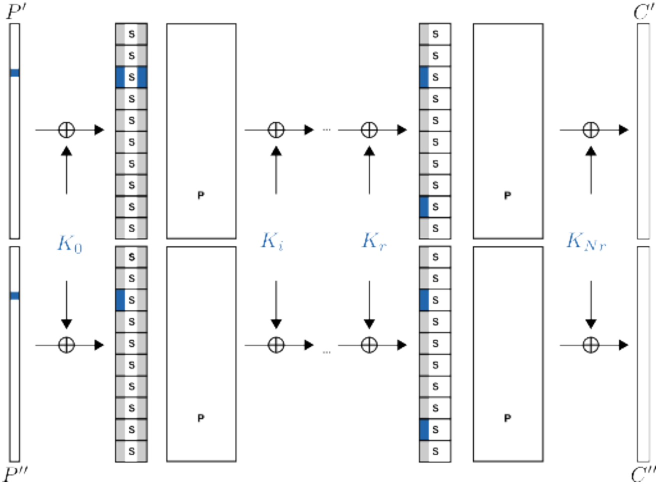
- Given two equation systems F' and $F"$ for two plaintext-ciphertext pairs (P', C') and $(P", C")$ under same encryption key K.
- We can combine these equation systems to form a system $F = F' \cup F"$.
- While F' and $F"$ do not share most of the state variables $X', X", Y', Y"$ but they share the key K and key schedule variables K_i.
- Thus by considering two plaintext–ciphertext pairs the cryptanalyst gathers twice as many equations, involving however many new variables.

Multiple $P - C$ Pairs II



Outline

Attack-A I



Attack-A II

- Each one-round difference gives rise to equations relating the input and output pairs for active S-Boxes.
- We have that the expressions

$$X'_{j,k} + X''_{j,k} = \Delta X_{j,k} \rightarrow \Delta Y_{j,k} = Y'_{j,k} + Y''_{j,k},$$

  where $\Delta X_{j,k}, \Delta Y_{j,k}$are known values predicted by the characteristic, are valid with some non-negligible probability $p_{j,k}$.
- For non-active S-Boxes we have the relations

$$X'_{j,k} + X''_{j,k} = 0 = Y'_{j,k} + Y''_{j,k}$$

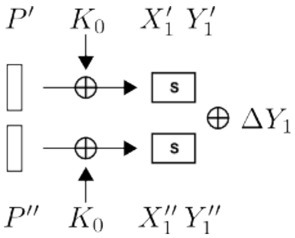
  also valid with a non-negligible probability.

These are $2n$ linear equations per round we can add to our equation system F. The resulting system $\overline{F}$ is expected to be easier to solve **but we need to solve $1/Pr(\Delta)$ such systems**.

Attack-B I

Restrict the first round bits to one active S-Box and assume we have a right pair. The S-Box can be written as a vectorial Boolean function

$$S(X_i) = \begin{array}{l} f_0(X_{i,0}, \ldots, X_{i,n-1}) \\ \ldots \\ f_{n-1}(X_{i,0}, \ldots, X_{i,n-1}) \end{array}.$$



If P', C' and $P"$, $C"$ is a right pair, we have

- $S(P' \oplus K_0) = S(X'_1) = Y'_1$
- $S(P" \oplus K_0) = S(X_1") = Y_1"$
- $Y'_1 \oplus Y_1" = \Delta Y_1$

$\rightarrow S(P'_1 \oplus K_0) \oplus S(P_1" \oplus K_0) = \Delta Y_1$

Attack-B II

We can use this small equation system F_s to recover bits of information about the subkey. Specifically:

Lemma

Given a differential characteristic Δ with a first round active S-Box with a difference that is true with probability 2^{-b}, then by considering F_s we can recover b bits of information about the key from this S-Box.

This is the algebraic equivalent of the well known subkey bit recovery from outer rounds in differential cryptanalysis.

In the case of PRESENT and Wang's differentials we can learn 4-bit of information per characteristic Δ.

Attack-B III

Experimental Observation

For some ciphers **Attack-A** can be used to distinguish **right pairs** and thus enables this attack.

Attack-B proceeds by measuring the time t it maximally takes to find that the system is inconsistent and assume we have a right pair if this time t elapsed without a contradiction.

Alternatively, we may measure other features of a Gröbner basis computation (degree reached, matrix dimensions, . . .).

Attack-B IV

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N_r	K_s	r	$Pr(\Delta)$	SINGULAR	POLYBORI
4	80	3	2^{-12}	106.55-118.15	6.18 – 7.10
4	80	2	2^{-8}	119.24-128.49	5.94 – 13.30
4	80	1	2^{-4}	137.84-144.37	11.83 – **33.47**
16	80	14	2^{-62}	N/A	43.42 – 64.11
16	128	14	2^{-62}	N/A	45.59 – 65.03
16	80	13	2^{-58}	N/A	80.35 – 262.73
16	128	13	2^{-58}	N/A	81.06 – 320.53
16	80	12	2^{-52}	N/A	$>$4 hours
17	80	14	2^{-62}	12,317.49-13,201.99	55.51 - **221.77**
17	128	14	2^{-62}	12,031.97-13,631.52	94.19 - 172.46
17	80	13	2^{-58}	N/A	$>$4 hours

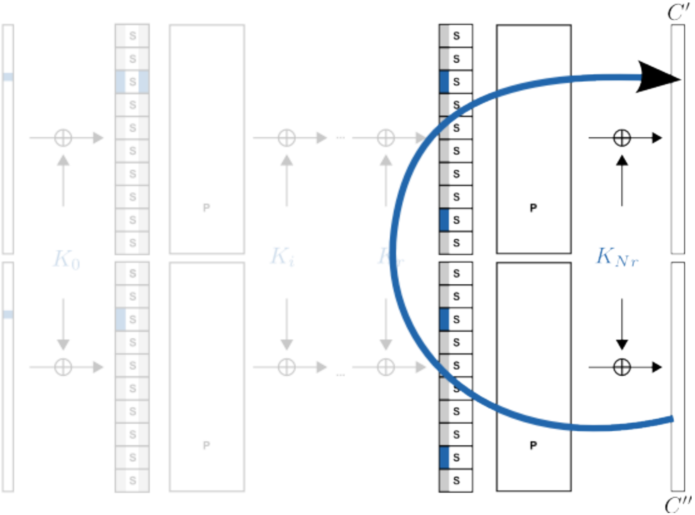
Table: Times in seconds for **Attack-B**

Times obtained on William Stein's `sage.math.washington.edu` computer purchased under NSF Grant No. 0555776.

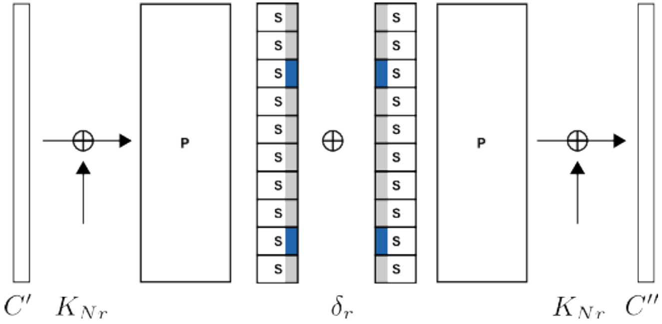
Why?

$$\frac{221.77\ s}{33.47\ s} \approx 6.626$$

Attack-C I



Attack-C II



The algebraic computation is essentially equivalent to solving a related cipher of $2(N_r - r)$ rounds (from C' to C" via the predicted difference δ_r) with a symmetric key schedule, using an algebraic meet-in-the-middle attack.

Attack-C III

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In a Nutshell

Attack-C is an algebraic filter.

Attack-C IV

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N	K_s	r	$Pr(\Delta)$	SINGULAR	POLYBORI	MINISAT2
4	80	4	2^{-16}	0.07 – 0.09	0.05 – 0.06	N/A
4	80	3	2^{-12}	6.69 – 6.79	0.88 – 1.00	0.14 – 0.18
4	80	2	2^{-8}	28.68 – 29.04	2.16 – 5.07	0.32 – 0.82
4	80	1	2^{-4}	70.95 – 76.08	8.10 – 18.30	1.21 – 286.40
16	80	14	2^{-62}	123.82 – 132.47	2.38 – 5.99	N/A
16	128	14	2^{-62}	N/A	2.38 – 5.15	N/A
16	80	13	2^{-58}	301.70 – 319.90	8.69 – 19.36	N/A
16	128	13	2^{-58}	N/A	9.58 – 18.64	N/A
16	80	12	2^{-52}	N/A	$>$ 4 hours	N/A
17	80	14	2^{-62}	318.53 – 341.84	9.03 – 16.93	0.70 – 58.96
17	128	14	2^{-62}	N/A	8.36 – 17.53	0.52 – 8.87
17	80	13	2^{-58}	N/A	$>$ 4 hours	$>$ 4 hours

Table: Times in seconds for **Attack-C**

Outline

PRESENT-80-6 and PRESENT-80-7

- We ran **Attack-C** against PRESENT-80-6 and PRESENT-80-7;
- the algorithm always suggested some key bits after the expected number of runs;
- the algorithm did return false positives (as expected);
- however, a simple majority vote over three experiments, always gave the correct answer.

PRESENT-80-16 I

4 bits:

- **Filter:** $\approx 2^{62}$ ciphertext checks
- **Algebraic Filter:** $\approx 2^{11.93} \cdot 6 \cdot 1.8 \cdot 10^9 \approx 2^{46}$ CPU cycles

Full Key Recovery:

- **Characteristics:** 6 characteristics from [4]
- **Filter:** $\approx 6 \cdot 2^{62}$ ciphertext checks
- **Algebraic Filter:** $\approx 6 \cdot 2^{46}$ CPU cycles
- **Guess:** $80 - 18 = 62$ bits

PRESENT-128-19

Consider the input difference for round 15 and iterate over all possible output differences. For the example difference we have 36 possible output differences for round 15 and $2^{13.93}$ possible output difference for round 16.

4 bits $\approx 2^{13.97} \cdot 1.8 \cdot 10^9 \cdot (18 \cdot 2^{62}) \approx 2^{111}$ CPU cycles.

full key $\approx 2^{13.97} \cdot 1.8 \cdot 10^9 \cdot (18 \cdot 2^{62} + 2 \cdot 6 \cdot 2^{64}) \approx 2^{116}$ CPU cycles.

Complexity Estimates

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Attack	N_r	K_s	r	#pairs	time	#bits
Wang	16	80	14	2^{63}	2^{65} MA	57
Attack-C	16	80	14	2^{62}	2^{62} MA	4
Attack-C	16	80	14	$6 \cdot 2^{62}$	2^{62} encr.	18
Attack-C	19	128	14	2^{62}	2^{111} cycles	4
Attack-C	19	128	14	$6 \cdot 2^{62}$	2^{116} cycles	128

Outline

Discussion

Properties:

- One right pair is sufficient to learn some information about the key.
- No requirement for candidate key counter.
- Silimar to DC attack on full DES [1] but **in theory** applicable to any block cipher.

Some open problems:

- Is this idea applicable to other ciphers?
- How long would it take to solve the small cipher system in Attack-C after a right pair has been identified?
- How about other techniques: linear cryptanalysis, saturation attacks, higher order differentials, ...
- Can we do PRESENT-128-20 with $r = 14$: "a situation without precedent" [2]?

Conclusion

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- We presented a new promising research direction: combining statistical and algebraic cryptanalysis instead of holding on to the "low data complexity dream" normally attached to algebraic cryptanalysis.
- In particular, we presented a new approach which uses algebraic techniques in differential cryptanalysis and showed how to invest more time in the last rounds not covered by a differential using algebraic techniques.
- To illustrate the viability of the attack we applied it against round reduced variants of PRESENT. Of course, this attack has no implication for the security of PRESENT!

Thank you!

Literature I

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  Differential Cryptanalysis of the Full 16-round DES.
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Literature II

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Differential Cryptanalysis of reduced-round PRESENT.
In Serge Vaudenay, editor, *Africacrypt 2008*, volume 5023 of *Lecture Notes in Computer Science*, pages 40–49. Springer Verlag, 2008.

Meiqin Wang.
Private communication: 24 differential characteristics for 14-round present we have found, 2008.