10. *Algebraic Threefolds with Ample Tangent Bundle*

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This is an announcement of our result proving the following conjecture of T. T. Frankel for $n=3$:

(F-n) *A compact Kaehler n-dimensional manifold with positive sectional* (*or more generally, positive holomorphic bisectional*) *curvature is biholomorphic to the complex projective space* $P^n(C)$.

We actually obtained the stronger result:

(G-3) *A non-singular irreducible* 3*-dimensional projective variety* M *with ample tangent bundle and the second Betti number* 1 *is* (*algebraically*) *isomorphic to* $P^3(C)$.

In order to prove (G-3), we first quote the following theorem of S. Kobayashi and T. Ochiai [3] which enables us to take a group-theoretic approach.

Theorem 1. *Let* M *be a non-singular irreducible* 3*-dimensional projective variety with ample tangent bundle. Then the group* Aut(M) *of algebraic transformations of* M *satisfies*:

(1) $\dim_C \mathrm{Aut}(M) \geqq 7$.

(2) M *can be embedded into* $P^{N-1}(C)$ *for some* N *in such a way that* Aut(M) *acts on* M *as a closed subgroup of* $PGL(N; C)$.

Secondly, note that a consideration of standard facts on linear algebraic groups gives us:

Theorem 2. *Any linear algebraic group of dimension* $\geqq 7$ *contains a closed subgroup which is isomorphic to one of the following four algebraic groups*:

(1) *The* 3*-dimensional algebraic torus* $(C^*)^3$.

(2) *A group which is isogenous to* $SL(3; C)$.

(3) *A group which is isogenous to* $SL(2; C) \times SL(2; C)$.

(4) *The* 3*-dimensional additive group* C^3.

The main point of these two theorems is that, for the proof of (G-3), we may assume one of the following four conditions on M:

(1) $(C^*)^3$ acts on M regularly and effectively.

(2) $SL(3; C)$ acts on M regularly and essentially effectively.

(3) $SL(2; C) \times SL(2; C)$ acts on M regularly and essentially effectively.

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(4) $\boldsymbol{C}^3$ acts on M regularly and effectively.

But then, we can prove (G-3) by combining Theorems 3, 4′, 5′ and 6.

Theorem 3 (T. Mabuchi [5]). *Let M be a non-singular irreducible n-dimensional projective variety with ample tangent bundle. Assume that the n-dimensional algebraic torus $(\boldsymbol{C}^*)^n$ acts on M regularly and effectively. Then M is isomorphic to $\boldsymbol{P}^n(\boldsymbol{C})$.*

This theorem is largely indebted to the systematic studies of torus embeddings made by several authors in recent years (cf. M. Demazure [1], D. Mumford *et al.* [10], K. Miyake and T. Oda [9]).

Theorem 4 (T. Mabuchi [6]). *Let M be a non-singular irreducible 3-dimensional complete variety on which the algebraic group $SL(3;\boldsymbol{C})$ acts regularly and essentially effectively. Then M is isomorphic to one of the following four types of varieties:*

(1) *The projective bundle* Proj $(T(\boldsymbol{P}^2(\boldsymbol{C})))$ *associated with the tangent bundle $T(\boldsymbol{P}^2(\boldsymbol{C}))$ of $\boldsymbol{P}^2(\boldsymbol{C})$. (This corresponds to the homogeneous $SL(3;\boldsymbol{C})$-action.)*

(2) $\boldsymbol{P}^3(\boldsymbol{C})$.

(3) Proj $(O_{\boldsymbol{P}^2}(m)\oplus O_{\boldsymbol{P}^2}(0))$, $m\in \boldsymbol{Z}_+$, *where $O_{\boldsymbol{P}^2}(m)$ denotes the m-fold tensor of the tautological line bundle over $\boldsymbol{P}^2(\boldsymbol{C})$.*

(4) $\boldsymbol{P}^2(\boldsymbol{C})\times C$, *where C is a complete non-singular curve. (In this case, the $SL(3;\boldsymbol{C})$-action on M factors to the product of a homogeneous action on $\boldsymbol{P}^2(\boldsymbol{C})$ and the trivial one on C.)*

Since a projective variety with ample tangent bundle can admit no non-trivial fibrations except for those which have finite fibres (cf. T. Ochiai [11]), the following is straightforward from Theorem 4:

Theorem 4′. *Let M be a non-singular irreducible* 3-*dimensional projective variety with ample tangent bundle. Assume that the algebraic group $SL(3;\boldsymbol{C})$ acts on M regularly and essentially effectively. Then M is isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$.*

A parallel argument goes through also in case of $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$-actions on M:

Theorem 5 (T. Mabuchi [7]). *Let M be a non-singular irreducible 3-dimensional complete variety on which the algebraic group $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ acts regularly and essentially effectively. Then M is isomorphic to one of the following five types of varieties:*

(1) $\boldsymbol{P}^1(\boldsymbol{C})\times \boldsymbol{P}^1(\boldsymbol{C})\times C$, *where C is a non-singular complete curve.*

(2) *The projective bundle* Proj $(\mathrm{pr}_1{*}(O_{\boldsymbol{P}^1}(\alpha))\oplus \mathrm{pr}_2{*}(O_{\boldsymbol{P}^1}(\beta)))$, $\alpha,\beta\in \boldsymbol{Z}$, *associated with the vector bundle* $\mathrm{pr}_1{*}(O_{\boldsymbol{P}^1}(\alpha))\oplus \mathrm{pr}_2{*}(O_{\boldsymbol{P}^1}(\beta))$ *over* $\boldsymbol{P}^1(\boldsymbol{C})\times \boldsymbol{P}^1(\boldsymbol{C})$, *where* $\mathrm{pr}_i : \boldsymbol{P}^1(\boldsymbol{C})\times \boldsymbol{P}^1(\boldsymbol{C})\to \boldsymbol{P}^1(\boldsymbol{C})$ *denotes the canonical projection to the i-th factor* $(i=1,2)$.

(3) *The hyperquadric* $\{(x:y:z:u:v)\in \boldsymbol{P}^4(\boldsymbol{C})\,;\, xu-yz=v^2\}$.

(4) $\boldsymbol{P}^3(\boldsymbol{C})$.

(5) Proj $(O_{\boldsymbol{P}^1}(m)\oplus O_{\boldsymbol{P}^1}(0)\oplus O_{\boldsymbol{P}^1}(0))$, $m\in \boldsymbol{Z}$.

Noting that any non-singular hyperquadric cannot have ample tangent bundle, by the same argument as in deriving Theorem 4′, we obtain:

Theorem 5′. *Let M be a non-singular irreducible 3-dimensional projective variety with ample tangent bundle. Assume that the algebraic group* $SL(2;\boldsymbol{C})\times SL(2;\boldsymbol{C})$ *acts on M regularly and essentially effectively. Then M is isomorphic to* $\boldsymbol{P}^3(\boldsymbol{C})$.

Finally, we need:

Theorem 6 (T. Mabuchi [8]). *Let M be a non-singular irreducible 3-dimensional projective variety with ample tangent bundle and the second Betti number* 1. *Assume that the 3-dimensional algebraic additive group* $\boldsymbol{C}^3$ *acts on M regularly and effectively (or more generally, the 3-dimensional complex Lie group* $\boldsymbol{C}^3$ *acts on M holomorphically and effectively). Then M is isomorphic to* $\boldsymbol{P}^3(\boldsymbol{C})$.

The proof of this theorem essentially depends on the following two facts.

Theorem A (T. Fujita [2], S. Kobayashi and T. Ochiai [4]). *Let M be a 3-dimensional irreducible non-singular projective variety with an ample tangent bundle. Assume that, in* $H^2(M)$ $(=H^2(M;\boldsymbol{Z})/\text{torsion classes})$, *the first Chern class* c_1 *of the tangent bundle is written in the form:*

$$c_1=r\cdot g \qquad \text{for some } 2\leqq r\in \boldsymbol{Z} \text{ and some } g\in H^2(M).$$

Then M is isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$.

Theorem B. *Let M be a non-singular irreducible 3-dimensional projective variety with ample tangent bundle and the second Betti number* 1. *Assume that there exists a section*

$$0\neq S\in H^0(M, T(M))$$

whose zero locus contains a (non-empty) 2-dimensional component. Then M is isomorphic to $\boldsymbol{P}^3(\boldsymbol{C})$.

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