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Algebraic trigonometric values at rational multipliers of π

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Abstract. The problem of finding all algebraic values of $\alpha \in [-1,1]$ when $\arccos\alpha, \arcsin\alpha$ and $\arctan\alpha$ are rational multiples of π is solved. The values of such α of degree less than five are explicitly determined.

1. Introduction

Rational and higher degree algebraic values of the cosine function have been of much interest for quite some time, cf. [4, 3, 8, 10]. As early as 1933, Lehmer [4] proved that if $k/n,\ n>2$, is an irreducible fraction, then $2\cos(2\pi k/n)$ and $2\sin(2\pi k/n)$ are algebraic integers of degree $\varphi(n)/2$, where $\varphi(n)$ is the Euler's totient function. Lehmer's proof makes use of cyclotomic polynomials. As a consequence, we have [7, Theorem 6.16, pp. 308–309]: let $\theta = r\pi$ be a rational multiple of π. Then $\cos\theta, \sin\theta$ and $\tan\theta$ are irrational numbers apart from the cases where $\tan\theta$ is undefined, and the exceptions

$$\cos\theta = 0, \pm 1/2, \pm 1;\ \sin\theta = 0, \pm 1/2, \pm 1;\ \tan\theta = 0, \pm 1.$$

Recently, Varona [10] proved that if $r \in \mathbb{Q}\cap[0,1]$ then $\arccos(\sqrt{r})$ is a rational multiple of π if and only if $r \in \{0, 1/4, 1/2, 3/4, 1\}$. His proof is elementary and is similar to the proof of [1, Theorem 4, p. 32].

In Section 2, we push further the result of Varona, using elementary trigonometric identities, to find all possible nonnegative rational and some quadratic (i.e., algebraic of degree 2) values of the cosine function at rational multiples of π. In Section 3, we adopt the approach of Lehmer in [4], i.e., using cyclotomic polynomials, to determine all other algebraic values. All algebraic values of degree less than 5 are explicitly worked out. In the last section, we consider the same problem for the sine and tangent functions.

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2. Nonnegative rational and some quadratic cosine values

From the well-known trigonometric identities $2\cos 2\theta = (2\cos\theta)^2 - 2$ and $2\cos(n+1)\theta = (2\cos\theta)(2\cos n\theta) - 2\cos(n-1)\theta$ ($n \in \mathbb{N}$), using an idea from [7, Theorem 6.16, pp. 308–309], it follows that there exists a monic $f_n(x) \in \mathbb{Z}[x]$ of degree n such that

$$2\cos n\theta = f_n(2\cos\theta). \tag{2.1}$$

Clearly,

$$f_1(x) = x,\ f_2(x) = x^2 - 2,\ f_{n+1}(x) = xf_n(x) - f_{n-1}(x) \quad (n \in \mathbb{N}). \tag{2.2}$$

The polynomials $f_n(x)$ are closely related to the Chebyshev polynomials of the first kind ([2, pp. 61–63]) defined by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)\ \ (n \geq 1),$$

i.e., $f_n(2x) = 2T_n(x)$.

Taking $\theta = 2k\pi/n$, $k \in \mathbb{N}$, in (2.1), we get

$$f_n(2\cos(2k\pi/n)) - 2 = 0. \tag{2.3}$$

Since $f_n \in \mathbb{Z}[x]$ is monic of degree n, we have thus proved

Theorem 2.1. *If $k, n \in \mathbb{N}$, then $2\cos(2k\pi/n)$ is an algebraic integer of degree $\leq n$.*

In [10], the author's main idea is to take a "proof from THE BOOK" for the case $\frac{1}{\pi}\arccos(1/\sqrt{n})$ ($n \in \mathbb{N}$, n odd, $n \geq 3$) and to do a simple variation of the proof to analyze the case $\frac{1}{\pi}\arccos(\sqrt{r})$ ($r \in \mathbb{Q}$) that is much more general. We now proceed analogously using Theorem 2.1 to give a simple proof of the result in [10].

Corollary 2.2. *Let $r \in \mathbb{Q}$ with $0 \leq r \leq 1$. Then, the number $\frac{1}{\pi}\arccos(\sqrt{r})$ is rational if and only if $r \in \{0, 1/4, 1/2, 3/4, 1\}$.*

Proof. Assume that $\frac{1}{\pi}\arccos(\sqrt{r}) = 2k/n$ is a rational number. Theorem 2.1 shows then that $2\sqrt{r} = 2\cos(2k\pi/n)$ is an algebraic integer. Thus, $4r$ is also an algebraic integer. Since $r \in \mathbb{Q} \cap [0, 1]$, we deduce that $4r$ is a rational integer in $[0, 4]$, which implies $r \in \{0, 1/4, 1/2, 3/4, 1\}$. The converse is trivial. □

An alternative simple derivation of Corollary 2.2 is to start from [7, Theorem 6.16] which asserts that for $r \in \mathbb{Q}$, the values $\cos(r\pi) \in \mathbb{Q}$ only for $\cos(r\pi) \in \{0, \pm 1, \pm 1/2\}$. By using $\cos^2(x) = (1 + \cos(2x))/2$, we get that $\cos^2(r\pi) \in \mathbb{Q}$ if and only if $\cos(2r\pi) \in \{0, \pm 1, \pm 1/2\}$. Consequently, $\cos^2(r\pi)$ can only take the rational values $(1+0)/2$, $(1 \pm 1)/2$ and $(1 \pm 1/2)/2$, so the possible rational values for $\cos^2(r\pi)$ correspond to the cases

$$\cos(r\pi) \in \left\{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}\right\}.$$

Recall from [9, Theorem 6.6] that a quadratic integer in $\mathbb{Q}(\sqrt{d})$, where d is a square-free rational integer, takes one of the following two forms:

1. all numbers of the form $a+b\sqrt{d}$, where a and b are rational integers,

and

2. if $d \equiv 1(\text{mod} 4)$, but not otherwise, all numbers of the form $(a+b\sqrt{d})/2$, where a and b are odd.

This result is proved by analyzing solutions

$$\frac{a+b\sqrt{d}}{c} \tag{2.4}$$

with $a, d, c, d \in \mathbb{N} \cup \{0\}$, $c, d \neq 0$, $\gcd(a, b, c) = 1$, of a quadratic equation $x^2 + sx + t = 0$ with $s, t \in \mathbb{Z}$.

Extending the proof of Corollary 2.2, we obtain the following generalization.

Theorem 2.3. *Let*

$$\alpha = \frac{a+b\sqrt{d}}{c} \in [0,1]$$

with $a, b, c, d \in \mathbb{N} \cup \{0\}$, $c, d \neq 0$, $\gcd(a, b, c) = 1$ *and* d *square free. Then, the number* $\frac{1}{\pi}\arccos(\alpha)$ *is a rational number if and only if* α *takes one of the following values*

$$0 = \cos\tfrac{\pi}{2},\ \tfrac{1}{2} = \cos\tfrac{\pi}{3},\ 1 = \cos 0, \tfrac{\sqrt{2}}{2} = \cos\tfrac{\pi}{4},\ \tfrac{\sqrt{3}}{2} = \cos\tfrac{\pi}{6},\ \tfrac{\sqrt{5}+1}{4} = \cos\tfrac{\pi}{5}. \tag{2.5}$$

Proof. The sufficiency part is clear from (2.5). We proceed now to prove the necessity part. If $\frac{1}{\pi}\arccos(\alpha) = 2k/n$ is a rational number, then, by Theorem 2.1,

$$2\cos\left(\frac{2k\pi}{n}\right) = 2\alpha = 2\left(\frac{a+b\sqrt{d}}{c}\right) \in [0,2]$$

is an algebraic integer. From the shapes of quadratic integers, given in (2.4), we must have $c = 2$ or 4, the latter possibility occuring only when $d \equiv 1$ mod 4 and a, b odd.

If $c = 2$, then $2\alpha = a + b\sqrt{d} \in [0,2]$. Since a, b are nonnegative, the only possible values of 2α are $0, 1, 2, \sqrt{2}, \sqrt{3}$.

If $c = 4, d \equiv 1 \bmod 4$, a and b odd, then $2\alpha = (a + b\sqrt{d})/2 \in [0,2]$ implying that the only possible value of 2α is $(1+\sqrt{5})/2$. □

To deal with the case where a, b are integers, we need more information about $\cos(2k\pi/n)$. We carry this out in the next section using Lehmer's results in [4].

3. General algebraic cosine values

Let $n \in \mathbb{N}, n > 2$ and $\zeta_n = e^{2\pi i/n}$. The nth cyclotomic polynomial (see, e.g., [6, pp. 33–36]) is defined by

$$\Phi_n(x) = \prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}} (x - \zeta_n^k). \tag{3.1}$$

Clearly, $\deg \Phi_n(x) = \varphi(n)$, where φ is the Euler totient function. It is well known that $\Phi_n(x) \in \mathbb{Z}[x]$. Since the roots of $\Phi_n(x)$ occur in reciprocal pairs, there is a monic polynomial $\psi_n(x) \in \mathbb{Z}[x]$ of degree $\varphi(n)/2$ such that

$$\psi_n(x + x^{-1}) = x^{-\varphi(n)/2}\, \Phi_n(x).$$

From [4, Theorem 1] and its proof, we have

Proposition 3.1. *Let $\alpha \in [-1, 1]$. If $\frac{1}{\pi} \arccos \alpha = 2k/n \in \mathbb{Q}$ with $k, n \in \mathbb{Z}$, $n > 2$ and $\gcd(k, n) = 1$, then $2\alpha = 2\cos(2k\pi/n)$ is an algebraic integer of degree $\varphi(n)/2$ whose minimal polynomial is $\psi_n(x)$.*

Note in addition that the cases where $n = 1$ and 2 are trivial for $\cos(2k\pi) = 1$, $\cos(k\pi) = (\pm 1)^k$. A special case of Theorem 3.1 is the following result, which is a slight extension of the main result in [3].

Corollary 3.2. *The value of $2\cos(a^\circ\, b'\, c'')$, where a, b and c are nonnegative integers, is an algebraic integer of degree $\le 2^8 3^3 5^2$.*

Moreover, $2\cos(a^\circ\, b'\, c'')$ is an algebraic integer of exact degree $2^8 3^3 5^2$ if and only if $\gcd(c, 30) = 1$.

Proof. The result follows at once from Proposition 3.1 by noting that

$$a^\circ b' c'' = \frac{(60^2 a + 60b + c)\pi}{2^6 3^4 5^3}.$$

□

Proposition 3.1 also enables us to answer the problem posed at the end of the last section, namely, to determine when $\frac{1}{\pi} \arccos\left(\frac{a+b\sqrt{d}}{c}\right)$ is rational for a, b being any integers, and much more.

Theorem 3.3. *Let $\alpha \in [-1, 1]$ Assume that $\frac{1}{\pi} \arccos \alpha = 2k/n$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, $\gcd(k, n) = 1$. Then*

(i) *the number $2\alpha = 2\cos(2\pi k/n)$ is an algebraic integer of degree 1 if and only if $n = 1, 2, 3, 4, 6$; in such cases, all the values taken by α are*

$$1 = \cos 0 = -\cos \pi,\ 0 = \cos\frac{2\pi}{4} = \cos\frac{6\pi}{4},$$
$$\frac{1}{2} = \cos\frac{2\pi}{6} = \cos\frac{10\pi}{6} = -\cos\frac{2\pi}{3} = -\cos\frac{4\pi}{3};$$

(ii) *the number* $2\alpha = 2\cos(2\pi k/n)$ *is an algebraic integer of degree* 2 *if and only if* $n = 5, 8, 10, 12$*; in such cases, all values of* α *are*

$$\begin{aligned}
\frac{\sqrt{5}-1}{4} &= \cos\frac{2\pi}{5} = \cos\frac{8\pi}{5} = -\cos\frac{6\pi}{10} = -\cos\frac{14\pi}{10},\\
\frac{\sqrt{5}+1}{4} &= -\cos\frac{4\pi}{5} = -\cos\frac{6\pi}{5} = \cos\frac{2\pi}{10} = \cos\frac{18\pi}{10},\\
\frac{\sqrt{2}}{2} &= \cos\frac{2\pi}{8} = \cos\frac{14\pi}{8} = -\cos\frac{6\pi}{8} = -\cos\frac{10\pi}{8},\\
\frac{\sqrt{3}}{2} &= \cos\frac{2\pi}{12} = \cos\frac{22\pi}{12} = -\cos\frac{10\pi}{12} = -\cos\frac{14\pi}{12};
\end{aligned}$$

(iii) *the number* $2\alpha = 2\cos(2\pi k/n)$ *is an algebraic integer of degree* 3 *if and only if* $n = 7, 9, 14, 18$*; in such cases all values of* α *are*

$$\begin{aligned}
\cos\frac{2\pi}{7} &= \cos\frac{12\pi}{7} = -\cos\frac{10\pi}{14} = -\cos\frac{18\pi}{14},\\
\cos\frac{6\pi}{14} &= \cos\frac{22\pi}{14} = -\cos\frac{4\pi}{7} = -\cos\frac{10\pi}{7},\\
\cos\frac{2\pi}{14} &= \cos\frac{26\pi}{14} = -\cos\frac{6\pi}{7} = -\cos\frac{8\pi}{7},\\
\cos\frac{2\pi}{9} &= \cos\frac{16\pi}{9} = -\cos\frac{14\pi}{18} = -\cos\frac{22\pi}{18},\\
\cos\frac{4\pi}{9} &= \cos\frac{14\pi}{9} = -\cos\frac{10\pi}{18} = -\cos\frac{26\pi}{18},\\
\cos\frac{2\pi}{18} &= \cos\frac{34\pi}{18} = -\cos\frac{8\pi}{9} = -\cos\frac{10\pi}{9};
\end{aligned}$$

(iv) *the number* $2\alpha = 2\cos(2\pi k/n)$ *is an algebraic integer of degree* 4 *if and only if* $n = 15, 16, 20, 24, 30$*; in such cases all values of* α *are*

$$\begin{aligned}
\frac{1}{8}\left(1+\sqrt{5}+\sqrt{6(5-\sqrt{5})}\right) &= \cos\frac{2\pi}{15} = \cos\frac{28\pi}{15} = -\cos\frac{26\pi}{30} = -\cos\frac{34\pi}{30},\\
\frac{1}{8}\left(1-\sqrt{5}+\sqrt{6(5+\sqrt{5})}\right) &= \cos\frac{4\pi}{15} = \cos\frac{26\pi}{15} = -\cos\frac{22\pi}{30} = -\cos\frac{38\pi}{30},\\
\frac{1}{8}\left(1+\sqrt{5}-\sqrt{6(5-\sqrt{5})}\right) &= \cos\frac{8\pi}{15} = \cos\frac{22\pi}{15} = -\cos\frac{14\pi}{30} = -\cos\frac{46\pi}{30},\\
\frac{1}{8}\left(1-\sqrt{5}-\sqrt{6(5+\sqrt{5})}\right) &= \cos\frac{14\pi}{15} = \cos\frac{16\pi}{15} = -\cos\frac{2\pi}{30} = -\cos\frac{58\pi}{30},\\
\frac{\sqrt{2+\sqrt{2}}}{2} &= \cos\frac{2\pi}{16} = \cos\frac{30\pi}{16} = -\cos\frac{14\pi}{16} = -\cos\frac{18\pi}{16},\\
\frac{\sqrt{2-\sqrt{2}}}{2} &= \cos\frac{6\pi}{16} = \cos\frac{26\pi}{16} = -\cos\frac{10\pi}{16} = -\cos\frac{22\pi}{16},\\
\frac{\sqrt{10+2\sqrt{5}}}{4} &= \cos\frac{2\pi}{20} = \cos\frac{38\pi}{20} = -\cos\frac{18\pi}{20} = -\cos\frac{22\pi}{20},\\
\frac{\sqrt{10-2\sqrt{5}}}{4} &= \cos\frac{6\pi}{20} = \cos\frac{34\pi}{20} = -\cos\frac{14\pi}{20} = -\cos\frac{26\pi}{20},\\
\frac{\sqrt{3}+1}{2\sqrt{2}} &= \cos\frac{2\pi}{24} = \cos\frac{46\pi}{24} = -\cos\frac{22\pi}{24} = -\cos\frac{26\pi}{24},\\
\frac{\sqrt{3}-1}{2\sqrt{2}} &= \cos\frac{10\pi}{24} = \cos\frac{38\pi}{24} = -\cos\frac{14\pi}{24} = -\cos\frac{34\pi}{24}.
\end{aligned}$$

Proof. The following table gives the values of n for the first four values of $\varphi(n)/2$. The determination of these values can be found in the paper [5] which gives an elementary approach to solve the equation $\varphi(x) = k$.

$\varphi(n)/2$	n
1	3, 4, 6
2	5, 8, 10, 12
3	7, 9, 14, 18
4	15, 16, 20, 24, 30

Using the values of n in the preceding table, the corresponding minimal polynomials are shown in the next table.

n	$\Phi_n(x)$	$\psi_n(t)$ $(t = x + x^{-1})$
3	$x^2 + x + 1$	$t + 1$
4	$x^2 + 1$	t
5	$x^4 + x^3 + x^2 + x + 1$	$t^2 + t - 1$
6	$x^2 - x + 1$	$t - 1$
7	$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$	$t^3 + t^2 - 2t - 1$
8	$x^4 + 1$	$t^2 - 2$
9	$x^6 + x^3 + 1$	$t^3 - 3t + 1$
10	$x^4 - x^3 + x^2 - x + 1$	$t^2 - t - 1$
12	$x^4 - x^2 + 1$	$t^2 - 3$
14	$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$	$t^3 - t^2 - 2t + 1$
15	$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$	$t^4 - t^3 - 4t^2 + 4t + 1$
16	$x^8 + 1$	$t^4 - 4t^2 + 2$
18	$x^6 - x^3 + 1$	$t^3 - 3t - 1$
20	$x^8 - x^6 + x^4 - x^2 + 1$	$t^4 - 5t^2 + 7$
24	$x^8 - x^4 + 1$	$t^4 - 4t^2 + 1$
30	$x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$	$t^4 + t^3 - 4t^2 - 4t + 1$

The stated values of α are computed by finding all the roots of $\psi_n(t)$ using Mathematica. □

Remark 3.4. The results in Proposition 3.1, Corollary 3.2, and Theorem 3.3 can be transformed to those of hyperbolic cosine function by noting that $\cosh(z) = \cos(zi)$.

4. Algebraic sine and tangent values

The following results are taken from [6, Theorems 3.9 and 3.11]. They were first considered by Lehmer [4]; however, the original Lehmer's result for the sine function was incomplete.

Proposition 4.1. *Let* $\alpha \in [-1, 1]$.

I. *If* $\frac{1}{\pi} \arcsin \alpha = 2k/n \in \mathbb{Q}$ *with* $k, n \in \mathbb{Z}$, $n > 2$, $n \neq 4$ *and* $\gcd(k, n) = 1$, *then* $2\alpha = 2\sin(2k\pi/n)$ *is an algebraic integer of degree* $\varphi(n)$, $\varphi(n)/4$ *or* $\varphi(n)/2$ *according as* $\gcd(n, 8) < 4$, $\gcd(n, 8) = 4$ *or* $\gcd(n, 8) > 4$.

II. *If* $\frac{1}{\pi}\arctan\alpha = 2k/n \in \mathbb{Q}$ *with* $k \in \mathbb{Z}$, $n > 2$, $n \neq 4$ *and* $\gcd(k,n) = 1$, *then* $2\alpha = 2\tan(2k\pi/n)$ *is an algebraic integer of degree* $\varphi(n)$, $\varphi(n)/2$ *or* $\varphi(n)/4$ *according as* $\gcd(n,8) < 4$, $\gcd(n,8) = 4$ *or* $\gcd(n,8) > 4$.

The cases when $n = 1$ and $n = 2$ are trivial for $0 = \sin(2k\pi) = \sin(k\pi) = \tan(2k\pi) = \tan(k\pi)$, and the case when $n = 4$ is also trivial for $\sin(k\pi/2) \in \{-1, 1\}$ and $\tan(k\pi/2) = 0$ or undefined.

Using Proposition 4.1 and Theorem 3.3, all algebric values of degrees < 5 of the sine and tangent functions are given in the following theorem, whose proof is omitted.

Theorem 4.2. *Let* $\alpha \in [-1, 1]$.

A. *Assume that* $\frac{1}{\pi}\arcsin\alpha = 2k/n$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, $\gcd(k,n) = 1$. *Then*

(A1) *the number* $2\alpha = 2\sin(2\pi k/n)$ *is an algebraic integer of degree* 1 *if and only if* $n = 1, 2, 4, 6, 12$; *in such cases, all the values taken by* α *are*

$$0 = \sin 0 = \sin\pi, \ -1 = \sin\frac{2\pi}{4} = \sin\frac{6\pi}{4},$$
$$-\frac{1}{2} = \sin\frac{14\pi}{12} = \sin\frac{22\pi}{12}, \ \frac{1}{2} = \sin\frac{10\pi}{12} = \sin\frac{26\pi}{12};$$

(A2) *the number* $2\alpha = 2\sin(2\pi k/n)$ *is an algebraic integer of degree* 2 *if and only if* $n = 3, 6, 8, 20$; *in such cases, all the values taken by* α *are*

$$\frac{\sqrt{3}}{2} = \sin\frac{2\pi}{3} = \sin\frac{14\pi}{6} = -\sin\frac{4\pi}{3} = -\sin\frac{10\pi}{6},$$
$$\frac{\sqrt{2}}{2} = \sin\frac{6\pi}{8} = \sin\frac{18\pi}{8} = -\sin\frac{10\pi}{8} = -\sin\frac{14\pi}{8},$$
$$\frac{\sqrt{5}-1}{4} = \sin\frac{18\pi}{20} = \sin\frac{42\pi}{20} = -\sin\frac{22\pi}{20} = -\sin\frac{38\pi}{20},$$
$$\frac{\sqrt{5}+1}{4} = \sin\frac{14\pi}{20} = \sin\frac{46\pi}{20} = -\sin\frac{26\pi}{20} = -\cos\frac{34\pi}{20};$$

(A3) *the number* $2\alpha = 2\sin(2\pi k/n)$ *is an algebraic integer of degree* 3 *if and only if* $n = 28, 36$; *in such cases all the values taken by* α *are*

$$\cos\frac{2\pi}{7} = \sin\frac{22\pi}{28} = \sin\frac{62\pi}{28} = -\sin\frac{30\pi}{28} = -\sin\frac{54\pi}{28},$$
$$\cos\frac{6\pi}{7} = \sin\frac{38\pi}{28} = \sin\frac{46\pi}{28} = -\sin\frac{18\pi}{28} = -\sin\frac{66\pi}{28},$$
$$\cos\frac{6\pi}{14} = \sin\frac{26\pi}{28} = \sin\frac{58\pi}{28} = -\sin\frac{34\pi}{28} = -\sin\frac{50\pi}{28},$$
$$\cos\frac{2\pi}{9} = \sin\frac{26\pi}{36} = \sin\frac{82\pi}{36} = -\sin\frac{34\pi}{36} = -\sin\frac{74\pi}{36},$$
$$\cos\frac{8\pi}{9} = \sin\frac{50\pi}{36} = \sin\frac{58\pi}{36} = -\sin\frac{22\pi}{36} = -\sin\frac{86\pi}{36},$$
$$\cos\frac{10\pi}{18} = \sin\frac{38\pi}{36} = \sin\frac{70\pi}{36} = -\sin\frac{46\pi}{36} = -\sin\frac{62\pi}{36};$$

(A4) *the number* $2\alpha = 2\sin(2\pi k/n)$ *is an algebraic integer of degree* 4 *if and only if* $n = 5, 10, 16, 24, 60$*; in such cases all values of* α *are*

$$\begin{aligned}
\frac{1}{8}(1+\sqrt{5}) + \frac{1}{4}\sqrt{\frac{3}{2}(5-\sqrt{5})} &= \cos\frac{2\pi}{15} = \sin\frac{38\pi}{60} = \sin\frac{142\pi}{60} \\
&= -\sin\frac{82\pi}{60} = -\sin\frac{98\pi}{60}, \\
\frac{1}{8}(1-\sqrt{5}) + \frac{1}{4}\sqrt{\frac{3}{2}(5+\sqrt{5})} &= \cos\frac{4\pi}{15} = \sin\frac{46\pi}{60} = \sin\frac{134\pi}{60} \\
&= -\sin\frac{74\pi}{60} = -\sin\frac{106\pi}{60}, \\
\frac{1}{8}(1+\sqrt{5}) - \frac{1}{4}\sqrt{\frac{3}{2}(5-\sqrt{5})} &= \cos\frac{8\pi}{15} = \sin\frac{62\pi}{60} = \sin\frac{118\pi}{60} \\
&= -\sin\frac{58\pi}{60} = -\sin\frac{122\pi}{60}, \\
\frac{1}{8}(1-\sqrt{5}) - \frac{1}{4}\sqrt{\frac{3}{2}(5+\sqrt{5})} &= \cos\frac{14\pi}{15} = \sin\frac{86\pi}{60} = \sin\frac{94\pi}{60} \\
&= -\sin\frac{34\pi}{60} = -\sin\frac{146\pi}{60}, \\
\frac{\sqrt{2+\sqrt{2}}}{2} &= \cos\frac{2\pi}{16} = \sin\frac{10\pi}{16} = \sin\frac{38\pi}{16} = -\sin\frac{22\pi}{16} = -\sin\frac{26\pi}{16}, \\
\frac{\sqrt{2-\sqrt{2}}}{2} &= \cos\frac{6\pi}{16} = \sin\frac{14\pi}{16} = \sin\frac{34\pi}{16} = -\sin\frac{18\pi}{16} = -\sin\frac{30\pi}{16}, \\
\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} &= \cos\frac{2\pi}{20} = \sin\frac{6\pi}{10} = \sin\frac{14\pi}{10} = -\sin\frac{14\pi}{10} = -\sin\frac{8\pi}{5}, \\
\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} &= \cos\frac{6\pi}{20} = \sin\frac{4\pi}{5} = \sin\frac{22\pi}{5} = -\sin\frac{6\pi}{5} = -\sin\frac{18\pi}{10}, \\
\frac{\sqrt{3}+1}{2\sqrt{2}} &= \cos\frac{2\pi}{24} = \sin\frac{14\pi}{24} = \sin\frac{58\pi}{24} = -\sin\frac{34\pi}{24} = -\sin\frac{38\pi}{24}, \\
\frac{\sqrt{3}-1}{2\sqrt{2}} &= \cos\frac{10\pi}{24} = \sin\frac{22\pi}{24} = \sin\frac{50\pi}{24} = -\sin\frac{26\pi}{24} = -\sin\frac{46\pi}{24}.
\end{aligned}$$

B. *Assume that* $\frac{1}{\pi}\arctan\alpha = 2k/n$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, $\gcd(k,n) = 1$. *Then*

(B1) *the number* $2\alpha = 2\tan(2\pi k/n)$ *is an algebraic integer of degree* 1 *if and only if* $n = 1, 2, 8$*; in such cases, all the values taken by* α *are*

$$\begin{aligned}
&0 = \tan 0 = \tan\pi, \\
&1 = \tan\frac{\pi}{4} = \tan\frac{5\pi}{4} = -\tan\frac{3\pi}{4} = -\tan\frac{7\pi}{4};
\end{aligned}$$

(B2) *the number* $2\alpha = 2\tan(2\pi k/n)$ *is an algebraic integer of degree* 2 *if and only if* $n = 3, 6, 12, 16, 24$*; in such cases, all the values of* α *are*

$$\begin{aligned}
\sqrt{3} &= \tan\frac{4\pi}{3} = \tan\frac{2\pi}{6} = -\tan\frac{2\pi}{3} = -\tan\frac{10\pi}{6},\\
\frac{1}{\sqrt{3}} &= \tan\frac{2\pi}{12} = \tan\frac{14\pi}{12} = -\tan\frac{10\pi}{12} = -\tan\frac{22\pi}{12},\\
1-\sqrt{2} &= \tan\frac{14\pi}{16} = \tan\frac{30\pi}{16} = -\tan\frac{2\pi}{16} = -\tan\frac{18\pi}{16},\\
1+\sqrt{2} &= \tan\frac{6\pi}{16} = \tan\frac{22\pi}{16} = -\tan\frac{10\pi}{16} = -\tan\frac{26\pi}{16},\\
2-\sqrt{3} &= \tan\frac{2\pi}{24} = \tan\frac{26\pi}{24} = -\tan\frac{22\pi}{24} = -\tan\frac{46\pi}{24},\\
2+\sqrt{3} &= \tan\frac{10\pi}{24} = \tan\frac{34\pi}{24} = -\tan\frac{14\pi}{24} = -\tan\frac{38\pi}{24};
\end{aligned}$$

(B3) *the number* $2\alpha = 2\tan(2\pi k/n)$ *is an algebraic integer of degree* 4 *if and only if* $n = 5, 10, 20, 32, 40, 48$*; in such cases all the values taken by* α *are*

$$\begin{aligned}
\sqrt{5+2\sqrt{5}} &= \tan\frac{2\pi}{5} = \tan\frac{14\pi}{10} = -\tan\frac{6\pi}{10} = -\tan\frac{8\pi}{5},\\
\sqrt{5-2\sqrt{5}} &= \tan\frac{6\pi}{5} = \tan\frac{2\pi}{10} = -\tan\frac{4\pi}{5} = -\tan\frac{18\pi}{10},\\
\sqrt{\frac{5-2\sqrt{5}}{5}} &= \tan\frac{2\pi}{20} = \tan\frac{22\pi}{20} = -\tan\frac{18\pi}{20} = -\tan\frac{38\pi}{20},\\
\sqrt{\frac{5+2\sqrt{5}}{5}} &= \tan\frac{6\pi}{20} = \tan\frac{26\pi}{20} = -\tan\frac{14\pi}{20} = -\tan\frac{34\pi}{20},\\
\sqrt{4+2\sqrt{2}}-\sqrt{2}-1 &= \tan\frac{2\pi}{32} = \tan\frac{34\pi}{32} = -\tan\frac{30\pi}{32} = -\tan\frac{62\pi}{32},\\
\sqrt{4-2\sqrt{2}}-\sqrt{2}+1 &= \tan\frac{6\pi}{32} = \tan\frac{38\pi}{32} = -\tan\frac{26\pi}{32} = -\tan\frac{58\pi}{32},\\
\sqrt{4-2\sqrt{2}}+\sqrt{2}-1 &= \tan\frac{10\pi}{32} = \tan\frac{42\pi}{32} = -\tan\frac{22\pi}{32} = -\tan\frac{54\pi}{32},\\
\sqrt{4+2\sqrt{2}}+\sqrt{2}+1 &= \tan\frac{14\pi}{32} = \tan\frac{46\pi}{32} = -\tan\frac{18\pi}{32} = -\tan\frac{50\pi}{32}.
\end{aligned}$$

5. Final remarks

1. As is easily checked from Proposition 4.1, II, there are no third degree algebraic values for the tangent function.

2. The result about all possible rational values of the sine, cosine and tangent functions derived in [8] is properly contained in Theorem 3.3, (i), and Theorem 4.2, (A1), (B1).

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