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ALGEBRAS ASSOCIATED WITH A FREE INVERSE MONOID

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Abstract

Let S be an ideal of the free inverse monoid on a set X, and let $\mathcal{B}$ denote the Banach algebra $l^1(S)$. It is shown that the following statements are equivalent: $\mathcal{B}$ is $*$-primitive; $\mathcal{B}$ is prime; X is infinite. A similar result holds if $\mathcal{B}$ is replaced by $\mathbb{C}[S]$, the complex semigroup algebra of S.

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By a $*$-*module* for an algebra A over the complex field $\mathbb{C}$, with involution *, we mean a (left) module V that admits an inner product $\langle \cdot \mid \cdot \rangle$ such that, for all $u, v \in V$ and all $a \in A$,

$$\langle au \mid v\rangle = \langle u \mid a^* v\rangle.$$

As in [2], we say that A is $*$-*primitive* if and only if it has a faithful irreducible $*$-module. Clearly, if A is $*$-primitive then it is primitive.

Let S be an inverse semigroup, and let $l^1(S)$ denote the Banach $*$-algebra consisting of all functions $a : S \to \mathbb{C}$ such that $\sum_{s \in S} |a(s)| < \infty$. Addition and scalar multiplication are defined pointwise, multiplication is convolution, and $\|a\| = \sum_{s \in S} |a(s)|$ for $a \in l^1(S)$. As is customary, we identify the elements of S with their characteristic functions, and write a typical element of $l^1(S)$ in the form $\sum_{x \in S} \alpha_x x$, where each $\alpha_x \in \mathbb{C}$ and $\sum_{x \in S} |\alpha_x| < \infty$. The involution on $l^1(S)$ is given by the rule that $(\sum \alpha_x x)^* = \sum \overline{\alpha_x} x^{-1}$, where $x \mapsto x^{-1}$ is inversion in S. Note that $\|a^*\| = \|a\|$ for all $a \in l^1(S)$.

The subalgebra of $l^1(S)$ consisting of all functions $S \to \mathbb{C}$ of finite support is the usual semigroup algebra of S over $\mathbb{C}$, and is denoted by $\mathbb{C}[S]$. The involution * on $l^1(S)$ restricts to an involution on $\mathbb{C}[S]$.

We shall not distinguish singleton sets from their elements. The cardinal of a set S is denoted by $|S|$.

The free inverse monoid on a nonempty set X can be constructed as follows. Let G_X denote the free group on X, the set of all reduced words in the formal alphabet $X \cup X^{-1}$ subject to the usual multiplication. For $w \in G_X$, denote by $\overline{w}$ the set of all prefixes of w in reduced form, including 1 and w. For $H \subseteq G_X$, define $\overline{H} := \bigcup_{h \in H} \overline{h}$. We say that H is *left-closed* if $\overline{H} = H$. Let $\mathcal{E}_X$ denote the set of all nonempty finite left-closed subsets of G_X. Note that $\overline{g} \in \mathcal{E}_X$ for all $g \in G_X$. Write

$$M_X := \{(A, g) \in \mathcal{E}_X \times G_X \mid g \in A\}.$$

It can be verified that if $(A, g), (B, h) \in M_X$ then $A \cup gB \in \mathcal{E}_X$, and so we can define a multiplication in M_X by the rule that

$$(A, g)(B, h) = (A \cup gB, gh).$$

With this definition, M_X is the free inverse monoid on X, where $(A, g)^{-1} = (g^{-1}A, g^{-1})$ for all $(A, g) \in M_X$ [4]. The element $(1, 1)$ is the identity of M_X and the ideal $M_X \setminus (1, 1)$ of M_X is the free inverse semigroup on X.

For $A \subseteq G_X$, define $\operatorname{con}(A)$, the *content* of A, by

$$\operatorname{con}(A) := \{x \in X \mid x \text{ or } x^{-1} \text{ occurs in the reduced form of some element of } A\}.$$

We require the following lemma, taken from [3]. For a proof, see [2] or [5].

LEMMA 1. *Let $\mathcal{A}$ be a Banach algebra, V a Banach space and $\circ$ a left action of $\mathcal{A}$ on V such that*

$$\|a \circ v\| \le \kappa \|a\| \|v\| \quad \textit{for all } a \in \mathcal{A},\ v \in V$$

where κ is a positive constant. Suppose that there exists a cyclic vector v_0 in V and that for all $v \in V \setminus 0$ there exists a sequence (a_n) in $\mathcal{A}$ such that $a_n \circ v \to v_0$. Then V is irreducible.

The next lemma provides the key step in the proof of the main result.

LEMMA 2. *Let X be an infinite set. Then $l^1(M_X)$ is $*$-primitive.*

PROOF. Write $\mathcal{A} := l^1(M_X)$. Since $|X| = |X \times \mathbb{N}|$, there exists a set $\mathcal{S}$ with cardinality $|X|$ whose elements are countably-infinite pairwise-disjoint subsets of X. Then $|\mathcal{S}| = |X| = |\mathcal{E}_X|$, and so there exists a bijection $\theta : \mathcal{E}_X \to \mathcal{S}$. For each $A \in \mathcal{E}_X$, write $\phi(A) := \theta(A) \setminus \operatorname{con}(A)$. Since $\operatorname{con}(A)$ is finite, each $\phi(A)$ is an infinite subset of X; further, if A and B are distinct elements of $\mathcal{E}_X$ then $\phi(A) \cap \phi(B) = \emptyset$. Define $H \subset G_X$ by

$$H := \{1\} \cup \left[\bigcup_{A \in \mathcal{E}_X} \bigcup_{x \in \phi(A)} xA \right].$$

Note that H is left-closed. Write $H^{-1} := \{h^{-1} \mid h \in H\}$, and define V to be the Banach space $l^1(H^{-1})$. Let V have the inner product $\langle \cdot \mid \cdot \rangle$, which has H^{-1} as an

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orthonormal set. We define an action $\circ$ of $l^1(M_X)$ on V as follows. For $(A, g) \in M_X$ and $v \in H^{-1}$, write

$$(A, g) \circ v := \begin{cases} gv & \text{if } A \subseteq gvH, \\ 0 & \text{otherwise.} \end{cases}$$

If here $A \subseteq gvH$, then $1 \in gvH$, and so $gv \in H^{-1}$. Let $(A, g), (B, h) \in M_X$ and $v \in H^{-1}$. Then

$$(B, h) \circ [(A, g) \circ v] = \begin{cases} hgv & \text{if } A \subseteq gvH \text{ and } B \subseteq hgvH, \\ 0 & \text{otherwise.} \end{cases}$$

Also,

$$\begin{aligned} [(B, h)(A, g)] \circ v &= (B \cup hA, hg) \circ v \\ &= \begin{cases} hgv & \text{if } B \cup hA \subseteq hgvH, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $B \cup hA \subseteq hgvH$ if and only if $A \subseteq gvH$ and $B \subseteq hgvH$, it follows that $[(B, h)(A, g)] \circ v = (B, h)[(A, g) \circ v]$. For all $(A, g) \in M_X$ and $v \in H^{-1}$, $\|(A, g) \circ v\| \leq 1 = \|v\|$. Hence, by linearity and continuity, we can extend $\circ$ to a left action of $\mathcal{A}$ on V with the property that

$$(\forall a \in \mathcal{A})(\forall v \in V) \quad \|a \circ v\| \leq \|a\| \|v\|.$$

We first show that the action is faithful. Let $a = \sum_0^\infty \alpha_k (A_k, g_k) \in \mathcal{A}$, with $\alpha_k \in \mathbb{C}$, $\alpha_0 \neq 0$, the (A_k, g_k) distinct in M_X, and $|A_0|$ minimal among the $|A_k|$. Choose $n \in \mathbb{N}$ such that $\sum_{k>n} |\alpha_k| < |\alpha_0|$, and then choose $z \in \phi(A_0) \setminus \bigcup_0^n \operatorname{con}(A_k)$. This is possible since $\phi(A_0)$ is infinite. Note that $z, zg_0 \in H$, since $zA_0 \subset H$. Consider $\langle a \circ g_0^{-1} z^{-1} \mid z^{-1} \rangle$. Contributions to this come only from terms $\alpha_k (A_k, g_k)$ with $g_k = g_0$ and $A_k \subseteq z^{-1} H$. Now,

$$z^{-1} H = z^{-1} \cup A_0 \cup \left[\bigcup_{A \in \mathcal{E}_X} \bigcup_{x \in \phi(A) \setminus z} z^{-1} x A \right].$$

In each set $z^{-1} x A$ here, x and z are distinct elements of X with $x \notin \operatorname{con}(A)$ (since $x \in \phi(A)$). Thus every element of $z^{-1} H \setminus A_0$ has z in its content. Hence if $k \leq n$ and $A_k \subseteq z^{-1} H$ then $A_k \subseteq A_0$, and so $A_k = A_0$; if also $g_k = g_0$, then $k = 0$. Since $A_0 \subseteq z^{-1} H$, $(A_0, g_0) \circ g_0^{-1} z^{-1} = z^{-1}$. Therefore

$$\langle a \circ g_0^{-1} z^{-1} \mid z^{-1} \rangle = \alpha_0 + \sum_{\text{some } k > n} \alpha_k \neq 0,$$

and so $a \circ V \neq 0$. This shows that the representation is faithful.

Now consider irreducibility. We first prove that the element $1 \in V$ is cyclic. For all $h \in H$, $\overline{h} \in \mathcal{E}_X$ and $\overline{h} \subseteq H$. Hence, $(h^{-1}\overline{h}, h^{-1}) = (\overline{h^{-1}}, h^{-1}) \in M_X$ and

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$(h^{-1}\overline{h}, h^{-1}) \circ 1 = h^{-1}$. Consider $v \in V$, $v = \sum_1^\infty \alpha_k h_k^{-1}$, with $\alpha_k \in \mathbb{C}$ and $h_k \in H$. Then $a := \sum_1^\infty \alpha_k (h_k^{-1}\overline{h_k}, h_k^{-1}) \in l^1(M_X)$ and $a \circ 1 = v$. Therefore $1 \in V$ is cyclic.

Let $v \in V \setminus 0$. Write $v = \sum_0^\infty \alpha_k g_k^{-1}$, with $\alpha_k \in \mathbb{C}$, $\alpha_0 \neq 0$, and the g_k distinct elements of H. Since $g_0 \in H$, $\overline{g_0} \subset H$, and so $(\overline{g_0}, g_0) \circ g_0^{-1} = 1$. For each k, $(\overline{g_0}, g_0) \circ g_k^{-1}$ is either $g_0 g_k^{-1}$ or 0. Therefore we can write $(\overline{g_0}, g_0) \circ v = \sum_0^\infty \beta_k h_k^{-1}$, where $h_k \in H$, $h_0 = 1$, $h_k \neq 1$ for all $k > 0$, $\beta_k \in \mathbb{C}$, and $\beta_0 = \alpha_0$. For each $k \in \mathbb{N}$, since $h_k \in H \setminus 1$, $h_k \in x_k A_k$ for some $A_k \in \mathcal{E}_X$ and $x_k \in \phi(A_k)$. For each $n \in \mathbb{N}$, choose $z_n \in \phi(1) \setminus \bigcup_1^n \operatorname{con}(A_k)$. Then $z_n = z_n 1 \in H$, and $\overline{z_n} \subset H$. Let $n \in \mathbb{N}$. For $k \in \{1, 2, \ldots, n\}$, $h_k z_n \in x_k A_k z_n$. Hence $h_k z_n \notin x_k A_k$, since $x_k, z_n \notin \operatorname{con}(A_k)$. Further, $h_k z_n$ has first letter x_k in reduced form, and the only elements of H with first letter x_k are those of $x_k A_k$. Thus $h_k z_n \notin H$. Hence $\overline{z_n} \not\subseteq h_k^{-1} H$, and so $(\overline{z_n}, 1) \circ h_k^{-1} = 0$. Also, $(\overline{z_n}, 1) \circ 1 = 1$. Therefore

$$[(\overline{z_n}, 1)(\overline{g_0}, g_0)] \circ v = \beta_0 1 + \sum_{\text{some } k > n} \beta_k h_k^{-1},$$

and so $a_n \circ v \to 1$ as $n \to \infty$, where $a_n = \beta_0^{-1}(\overline{z_n}, 1)(\overline{g_0}, g_0)$. By Lemma 1, the representation is irreducible.

Finally, we show that V is a $*$-module, by showing that

$$(\forall a \in \mathcal{A})(\forall v, w \in V) \quad \langle a \circ v \mid w \rangle = \langle v \mid a^* \circ w \rangle. \tag{1}$$

Consider first the case $a = (A, g) \in M_X$ and $v, w \in H^{-1}$. Then $a^* = (g^{-1}A, g^{-1})$, and

$$\begin{aligned} a \circ v = w &\Leftrightarrow A \subseteq gvH, \quad gv = w \\ &\Leftrightarrow g^{-1}A \subseteq g^{-1}wH, \quad g^{-1}w = v \\ &\Leftrightarrow a^* w = v. \end{aligned}$$

Hence both sides of (1) are equal to 1 if $a \circ v = w$ and are otherwise 0. Thus (1) is established in this case. Since, as is easily verified, $|\langle v \mid w \rangle| \leq \|v\| \|w\|$ for all $v, w \in V$, (1) follows in all cases by linearity and continuity. □

We also need the following standard result, showing that every nonzero ideal of a primitive algebra (over an arbitrary field) is primitive.

LEMMA 3. *let V be a faithful irreducible left module for an algebra $\mathcal{A}$, and let $\mathcal{B}$ be a nonzero ideal of $\mathcal{A}$. Then V is a faithful irreducible module for $\mathcal{B}$.*

The main result now follows.

THEOREM. *Let S be an ideal of M_X. The following are equivalent: (i) $l^1(S)$ is $*$-primitive; (ii) $l^1(S)$ is prime; (iii) X is infinite; (iv) $\mathbb{C}[S]$ is $*$-primitive; (v) $\mathbb{C}[S]$ is prime.*

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PROOF. By [1, Lemma 2], if X is finite then $\mathbb{C}[S]$ is not prime; and the proof shows also that $l^1(S)$ is not prime. This proves that (ii) implies (iii) and that (v) implies (iii). Since primitivity implies primeness, (i) implies (ii) and (iv) implies (v).

Now assume that X is infinite. By Lemma 3, the module V constructed in the proof of Lemma 2 is a faithful irreducible module for $l^1(S)$, and clearly also a $*$-module. This proves that (iii) implies (i). For $\mathbb{C}[M_X]$, we take as module $W := \text{lin}(H^{-1})$, with the action defined as before. Then W is a faithful irreducible $*$-module for $\mathbb{C}[M_X]$, and so for its ideal $\mathbb{C}[S]$. The proof is on the same lines as that for $l^1(M_X)$, but is simpler, not requiring Lemma 1, for instance. This shows that (iii) implies (iv). □

REMARKS. The argument of Lemma 2 also shows that $F[S]$ is primitive for any ideal S of M_X and any field F when X is infinite: a result previously obtained in [1]. The module is taken to be $\text{lin}(H^{-1})$. The proof is a simplified version, not requiring Lemma 1 and ignoring the $*$-condition.

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