ALGEBRAS, LATTICES, VARIETIES
VOLUME II

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This volume is dedicated to our teachers—
Robert P. Dilworth, J. Donald Monk, Alfred Tarski, and Garrett Birkhoff

Contents

CHAPTER TEN

The Classification of Varieties

10.1 Introduction

Although varieties have brought some order to the diversity which we saw among individual algebras, we can again see a great diversity in looking over varieties themselves. Fortunately there have arisen various classifications of varieties which help to bring some order to this further diversity. In this chapter we will review and study some of the better known and more important properties which concern congruence relations on the algebras of a given variety.

This section needs a rewrite. The concepts of Maltsev condition needs to be defined more precisely. Walter says he is leaning toward "Maltsev classes" so I think I will wait to see how he handles that in chapter 11. He also suggest that I say $\mathcal{V}$-term means a term is the language of $\mathcal{V}$.

The prototype for most of the properties treated in this chapter is that of congruence permutability, which we examined in §4.7 of Volume I. Recall that we proved in Theorem 4.141 that a variety $\mathcal{V}$ is congruence permutable (i.e., $\theta \circ \phi = \phi \circ \theta$ for all θ, $\phi \in \operatorname{Con} A$, $\mathbf{A} \in \mathcal{V}$) if and only if there exists a ternary term $p(x,y,z)$ such that $\mathcal{V}$ satisfies the equations

$$p(x,z,z) \approx x \approx p(z,z,x). \tag{10.1.1}$$

for:introEq1

In honor of this result of A. I. Maltsev 1954 varietal properties which can be defined in a similar manner, i.e., by the existence of terms satisfying a certain finite set Σ of equations, have come to be known as **strong Maltsev conditions**; we will also say that such a property is **strongly Maltsev definable**. Until §10.8, we will be examining properties P which are Maltsev definable; in most cases P will have a natural definition which is not of the Maltsev type, and our job will be to find a suitable set Σ of equations. In fact this Σ can be thought of as defining the 'most general' variety satisfying P, in that for P to hold, we require nothing more than the deducibility of the equations Σ. This can be made precise by using the concept of the interpretation of one variety into another as defined in §4.12. For example, Maltsev's result cited above, can be stated as follows: a variety $\mathcal{V}$ is congruence permutable if and only if the variety $\mathcal{M}$ is interpretable into $\mathcal{V}$, where $\mathcal{M}$ is the variety with a single ternary operation symbol p satisfying

(10.1.1). This is Theorem 4.141 of §4.12. The reader may want to review that section before reading this chapter.

Actually, most of the properties P that we study in this chapter fail to be strongly Maltsev definable, but satisfy the following more general situation instead. There exist strong Maltsev conditions $P_1, P_2, \ldots$ such that each P_i implies P_{i+1} and such that P is logically equivalent to the disjunction of all P_i. (Equivalently, there exist finite sets $\Sigma_1, \Sigma_2, \ldots$ of equations such that, if $\mathcal{V}_i$ is the variety defined by Σ_i, then $\mathcal{V}_{i+1}$ interprets into $\mathcal{V}_i$ and, for some i, $\mathcal{V}_i$ interprets into $\mathcal{V}$.) We emphasize that the similarity type of $\mathcal{V}$ need not be the same—nor even related in any way—to that of any of the Σ_i. We will call such a property **Maltsev definable**; sometimes also we refer to P, or to the sequence $\Sigma_1, \Sigma_2, \ldots$, as a **Maltsev condition**.

Many of the varietal properties of this chapter arise naturally if one observes the behavior of congruence relations in vector spaces and in lattices. For instance, if θ is a congruence on a vector space $\mathbf{V}$ (over any field), then any two congruence blocks a/θ and b/θ have the same number of elements; in our terminology, the congruences are **uniform**. In §10.5 we will examine this and some related properties, especially the somewhat weaker notion of congruence **regularity** (if $|a/\theta| = 1$, then $|b/\theta| = 1$). Vector spaces are also congruence permutable, and, as we saw in Theorem 4.67, permutability of the congruence lattice implies its modularity. In §10.4 we present some Maltsev conditions for congruence modularity. Except for modularity, these properties all fail for the variety $\mathcal{L}$ of lattices, but $\mathcal{L}$ is even congruence distributive, as we saw in Chapter 2.

There is a geometric way of thinking about these sorts of problems which some investigators have found helpful (see especially Wille 1970 and Gumm 1983). The traditional method (going back to Descartes) for constructing algebraic models of Euclidean geometry, i.e., analytic geometry, can be described as follows. Let $\mathbf{V}$ be a real vector space of dimension n. One then defines 'points' to be elements of $\mathbf{V}$ and 'lines' to be cosets of 1-dimensional subspaces of $\mathbf{V}$. In other words, lines are congruence blocks a/θ with $\mathbf{V}/\theta$ of dimension $n-1$ (and flats of dimension k are blocks a/θ with $\mathbf{V}/\theta$ of dimension $n-k$). If we extend this terminology from $\mathbf{V}$ to an arbitrary algebra $\mathbf{A}$, i.e., if we define all congruence blocks a/θ to be 'lines' or 'flats,' then some properties of this chapter have interesting geometric interpretations. For example congruence permutability simply asserts the existence of the fourth corner of a parallelogram (given $a\ \theta\ b\ \phi\ c$ there exists d such that $a\ \phi\ d\ \theta\ c$), as illustrated in Figure 10.1.

Congruence uniformity asserts that any two 'parallel' lines or flats have the same number of elements. (Here we are referring to a/θ and b/θ as 'parallel.') We refer the reader to the references above for a full treatment from this point of view; nevertheless various traces of these ideas will be seen in this chapter.

A lattice equation that holds identically in all the congruence lattices of the algebras in a variety $\mathcal{V}$ is called a **congruence identity** of $\mathcal{V}$. Theorem 1.144 of the first volume gives a Maltsev condition for a variety to be congruence distributive. In §10.4 we give a Maltsev condition for a variety to be congruece modular and in §10.6 we take up the subject of congruence identities in general.

Theorem 4.143 shows that arithmeticity of a variety is strongly Maltsev de-

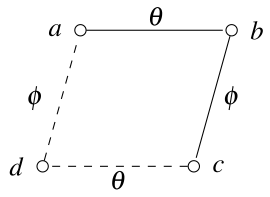

Figure 10.1:

fig:introFig

finable. Recall that $\mathcal{V}$ is *arithmetical* if it is both congruence permutable and congruence distributive. In the presence of permutability, distributivity is equivalent to the following 'geometric' condition: if $\{B_1,\ldots,B_k\}$ is a finite set of congruence blocks such that no two have empty intersection, then $B_1\cap\cdots\cap B_k$ is non-empty. (Compare with Helly's Theorem in geometry for convex sets.) This property is known as the **Chinese Remainder** property, since it generalizes the corresponding fact for the ring of integers, which is known as the Chinese Remainder Theorem (there the congruence blocks are simply arithmetic progressions).

rewrite this next paragraph

The main varietal property of this chapter which is not Maltsev definable is the Abelian property, which was defined and discussed in §4.13. This property asserts that the implications

$$t(x,\overline{u})\approx t(x,\overline{v})\rightarrow t(y,\overline{u})\approx t(y,\overline{v})$$

(one for each term t) hold over an entire variety. In combination with congruence modularity this property turns out to be very strong: every congruence modular Abelian variety is affine (i.e., derived from a ring and module in a simple manner made precise in §10.8).

The theory of varieties is essential for the results of this chapter: most of the results fail for algebras considered in isolation. For instance, any two element algebra obviously has permuting congruences, but there exist two element algebras with no Maltsev term p (such as we described at the beginning of this introduction). Nevertheless it is often the case that a stronger result than the result about varieties obtains. For example, by Theorem 10.46 if $\mathcal{V}$ is a regular variety then it is congruence modular. Although this result is not true for individual algebras, it is true that if every subalgebra of $\mathbf{A}^2$ is regular then **CON A** is modular; see Theorem 10.47. We loosely refer to these stronger results as *local* and call the results about varieties *global*. Another local result that will be proved in §10.6 is that if every subalgebra of $\mathbf{A}^2$ has a modular congruence lattice, **CON A** satisfies the arguesian equation.

10.2 Permutability of Congruences

First we recall some definitions and basic results from §4.7. If α and β are binary relations on a set A, and $\alpha \circ \beta$ denotes their relational product (as defined in the preliminaries of Volume I), $\alpha \circ^1 \beta = \alpha$, $\alpha \circ^{k+1} \beta = \alpha \circ (\beta \circ^k \alpha)$. We say that α and β k-**permute** if $\alpha \circ^k \beta = \beta \circ^k \alpha$. An algebra **A** is said to have k-**permutable conguences** if every pair of congruences of **A** k-permute. If **A** has k-permutible conguences then, by Lemma 4.66, $\alpha \vee \beta = \alpha \circ^k \beta$ for every pair of conguences of **A**. In this situation, the terminolgy **A** has **type** $k-1$ joins is used since there are $k-1$ $\circ$'s in the expression $\alpha \vee \beta = \alpha \circ^k \beta$. Thus type 1 is the same as 2-permutablity. In this case, we say that the algebra has permutable congruences. We saw in Theorem 4.67 that if an algebra **A** has 3-permuting congruences then **CON A** is modular. Later in this chapter we will construct a variety all of whose algebras have 4-permuting congruences, which is not congruence modular. Of course a variety is said to be k-permutable if each of its algebras is.

Permutability of congruences had a tremendous influence on the early development of universal algebra and its relation to lattice theory. The fact that permutability implies modularity goes back to R. Dedekind 1900. He derives the modularity of the lattice of normal subgroups of a group from the fact that any two normal subgroups permute with each other, i.e., $\mathbf{AB} = \mathbf{BA}$ holds for normal subgroups. Essentially the same proof shows that any permutable variety is modular. The fact that a lattice of permuting equivalence relations (and hence of permuting congruence relations) is modular is explicit in Ore 1942. Birkhoff's application of Ore's Theorem, Corollary 2.48, which yielded the results on unique factorization of Chapter 5, is a prime example of the importance of permutability. Moreover various generalizations of the Jordan-Hölder theorem to general algebra relied on congruence permutability, since they required congruence modularity.

Our first new Maltsev condition of this chapter is for k-permutability of congruences, and is due to J. Hagemann and A. Mitschke 1973.

perm1 **THEOREM 10.1.** *For a variety $\mathcal{V}$ the following conditions are equivalent:*

i. *$\mathcal{V}$ has k-permutable congruences;*

ii. $\mathbf{F}_{\mathcal{V}}(k+1)$ *has k-permutable congruences;*

iii. *there exist terms $p_1, \dots, p_{k-1}$ for $\mathcal{V}$ such that the following are identities of $\mathcal{V}$:*

$$\begin{aligned} x &\approx p_1(x,z,z) \\ p_1(x,x,z) &\approx p_2(x,z,z) \\ &\vdots \\ p_{k-2}(x,x,z) &\approx p_{k-1}(x,z,z) \\ p_{k-1}(x,x,z) &\approx z. \end{aligned} \tag{10.2.1}$$

for:permEq1

Proof. Clearly (i) implies (ii). Assume (iii) and let $\mathbf{A} \in \mathcal{V}$ and θ, $\phi \in \operatorname{Con} A$, and $a_0, \ldots, a_k \in A$ satisfy

$$a_0 \;\theta\; a_1 \;\phi\; a_2 \;\theta\; a_3 \;\cdots\; a_k.$$

Let $b_i = p_i(a_{i-1}, a_i, a_{i+1})$, for $1 \le i < k$. Then

$$\begin{aligned} a_0 &= p_1(a_0,a_1,a_1) \;\phi\; p_1(a_0,a_1,a_2) = b_1 \\ b_1 &= p_1(a_0,a_1,a_2) \;\theta\; p_1(a_1,a_1,a_2) = p_2(a_1,a_2,a_2) \;\theta\; p_2(a_1,a_2,a_3) = b_2. \end{aligned}$$

Continuing in this way, we obtain

$$a_0 \;\phi\; b_1 \;\theta\; b_2 \;\cdots\; a_k,$$

showing that (i) holds.

To see that (ii) implies (iii), we let λ_0 and λ_1 be the endomorphisms of $\mathbf{F}_{\mathcal{V}}(x_0,\ldots,x_k)$ such that $\lambda_0(x_i) = x_{2\lfloor i/2 \rfloor}$, $0 \le i \le k$, $\lambda_1(x_0) = x_0$, and $\lambda_1(x_i) = x_{2\lceil i/2 \rceil - 1}$, $1 \le i \le k$, where $\lceil n \rceil$ and $\lfloor n \rfloor$ denote the ceiling and floor functions.

A diagram here?

Thus λ_0 maps x_0 and x_1 to x_0, x_2 and x_3 to x_2, etc. Let θ_i be the kernel of λ_i, $i = 0, 1$. Clearly we have

$$x_0 \;\theta_0\; x_1 \;\theta_1\; x_2 \;\theta_0\; x_3 \;\cdots\; x_k.$$

Now by our assumption there are $q_1, \ldots, q_{k-1} \in \mathbf{F}_{\mathcal{V}}(x_0,\ldots,x_k)$ such that

$$x_0 \;\theta_1\; q_1 \;\theta_0\; q_2 \;\theta_1\; q_3 \;\cdots\; x_k.$$

Let r_i be terms such that $q_i = r_i(x_0,\ldots,x_k)$. Then

$$r_1(x_0,\ldots,x_k) \;\theta_1\; r_1(x_0,x_1,x_1,x_3,x_3,\ldots),$$

and hence $x_0 \;\theta_1\; r_1(x_0,x_1,x_1,x_3,x_3,\ldots)$. But λ_1 is the identity on the subalgebra of $\mathbf{F}_{\mathcal{V}}(x_0,\ldots,x_k)$ generated by x_0, x_1, $x_3, \ldots$. Thus

$$x_0 = r_1(x_0,x_1,x_1,x_3,x_3,\ldots)$$

in $\mathbf{F}_{\mathcal{V}}(x_0,\ldots,x_k)$, and consequently $\mathcal{V}$ satisfies the equation

$$x_0 \approx r_1(x_0,x_1,x_1,x_3,x_3,\ldots).$$

Similar arguments show that $\mathcal{V}$ satisfies the following equations.

$$\begin{aligned} r_1(x_0,x_0,x_2,x_2,\ldots) &\approx r_2(x_0,x_0,x_2,x_2,\ldots) \\ r_2(x_0,x_1,x_1,x_3,x_3,\ldots) &\approx r_3(x_0,x_1,x_1,x_3,x_3\ldots) \\ &\vdots \end{aligned}$$

For $1 \le i < k$, define

$$p_i(x_0,x_1,x_2) = r_i(\overbrace{x_0,\ldots,x_0}^{i \text{ times}},x_1,\overbrace{x_2,\ldots,x_2}^{k-i \text{ times}}).$$

It easy to verify that $\mathcal{V}$ satisfies the equations (10.2.1). ■

There are natural examples of varieties which are 3-permutable but not permutable. In the exercises we present some of these examples. The next example presents a variety constructed by S. V. Polin 1977 (see also Alan Day and Freese 1980) which is 4-permutable but not 3-permutable. In §10.6 we will see that this variety is not congruence modular but nevertheless its congruence lattices satisfy a nontrivial lattice equation.

polinexample

EXAMPLE 10.2. For $i=0,1$, let $\mathbf{A}_i = \langle \{0,1\}, \wedge, ', ^+ \rangle$ where $\wedge$ is the usual meet operation and $'$ and $^+$ are the unary operations given in the following tables.

$\mathbf{A}_0$	0	1
$'$	1	0
$+$	1	1

$\mathbf{A}_1$	0	1
$'$	0	1
$+$	1	0

Let $\mathbf{A} = \mathbf{A}_0 \times \mathbf{A}_1$ and let $\mathcal{P}$ denote the variety generated by $\mathbf{A}$. It is straighforward to verify that **CON A** is the lattice of Figure 10.2, where we have used juxtaposition to denote the elements of $\mathbf{A} = \mathbf{A}_0 \times \mathbf{A}_1$.

Possible use η_i notation instead.

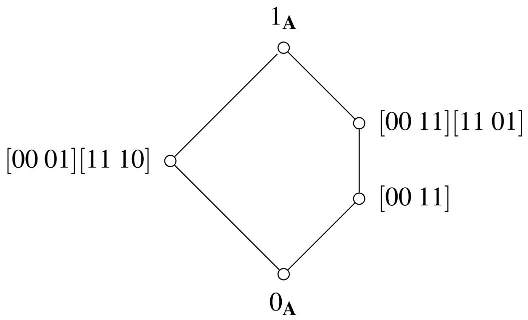

Figure 10.2: fig:permFig1

Thus $\mathcal{P}$ is not congruence modular and hence, by Theorem 4.67, not 3-permutable. To see that it is 4-permutable, we define three terms:

$$p_1(x,y,z) = x \wedge (y \wedge z^+)^+$$
$$p_2(x,y,z) = [(x \wedge y')' \wedge (z \wedge y')' \wedge (x \wedge z)']'$$
$$p_3(x,y,z) = z \wedge (y \wedge x^+)^+.$$

We need to verify that the equations (10.2.1) of Theorem 10.1 for $k=3$ are identities of $\mathcal{P}$. Since $p_1(x,y,z) = p_3(z,y,x)$ and $p_2(x,y,z) = p_2(z,y,x)$, we need only verify that $x \approx p_1(x,z,z)$ and $p_1(x,x,z) \approx p_2(x,z,z)$ are identities of $\mathbf{A}$. This is just a matter of checking on the algebras $\mathbf{A}_0$ and $\mathbf{A}_1$.

Since both $\mathbf{A}_0$ and $\mathbf{A}_1$ have the meet operation and each has a unary operation which is complementation, each is term equivalent (see Definition 4.139) to the two-element Boolean algebra. If $\mathcal{V}_i$ is the variety generated by $\mathbf{A}_i$ then

$$\mathcal{P} = \mathcal{V}_0 \vee \mathcal{V}_1.$$

Thus *the join of two distributive varieties need not be distributive, nor is it true that the direct product of two algebras with permuting congruences has permuting congruences.* Exercise 10.4.18 shows that $\mathcal{P}$ is residually large. Hence *the join of two residually finite varieites can be residually large.* No such counterexamples exist for congruence modular varieties. J. Hagemann and C. Herrmann 1979 have shown that *the join of two distributive varieties in a modular variety is distributive.* Moreover, *the join of two residually small subvarieties of a modular variety is residually small.* Furthermore, *the product of two algebras in a modular variety with permuting congruences has permuting congruences.* These results can be found in Freese and Ralph McKenzie 1987 as Exercise 8.2, Theorem 11.1, and Exercise 6.8, respectively.

For $k \geq 5$ there do not seem to exist any naturally occurring k-permutable varieties which are not $(k-1)$-permutable. Of course, the equations (10.2.1) define a variety $\mathcal{V}_k$ such that an arbitrary variety $\mathcal{V}$ is k-permutable if and only if $\mathcal{V}_k$ is interpretable (see §4.12) into $\mathcal{V}$.

It is not difficult to find an individual congruence lattice which is k-permutable but not $(k-1)$-permutable (see Exercise 1), but finding a k-permutable variety which is not $(k-1)$-permutable is more difficult. [It is possible to modify Polin's example above to produce a variety which is $2k$-permutible but not $2k-1$-permutable. The arguments are only slightly easier than the ones we will now present.]

Perhaps make this into an exercise.

The following example presents a variety which is k- but not $(k-1)$-permutable for all k. This fact was first established by E. T. Schmidt 1972. It is of course an immediate corollary of Schmidt's theorem that the variety $\mathcal{V}_k$ is k-permutable but not $(k-1)$-permutable.

kpermexample

EXAMPLE 10.3. Fix $k \geq 3$ and let $\mathbf{A}_i$, $i = 1, \ldots, k-1$, be algebras on the set $\{0, 1\}$ with ternary operation symbols p_i, $i = 1, \ldots, k-1$. The definition of $p_i^{\mathbf{A}_j}$ is given separately for $i < k/2$, $i = k/2$, and $i > k/2$. The second case only applies if k is even, of course. In these definitions, $x - y$ is an abreviation for $x \wedge y'$, where y' is the complement of y. First the case $i < k/2$:

$$p_i^{\mathbf{A}_j}(x,y,z) = \begin{cases} x \wedge z & \text{for } j < i \\ x - (y - z) & \text{for } j = i \\ x & \text{for } i < j < k - i \\ x \vee (z - y) & \text{for } j = k - i \\ x \vee z & \text{for } k - i < j. \end{cases}$$

When $i > k/2$ we have

$$p_i^{\mathbf{A}_j}(x,y,z) = \begin{cases} x \wedge z & \text{for } j < k-i \\ z - (y-x) & \text{for } j = k-i \\ z & \text{for } k-i < j < i \\ z \vee (x-y) & \text{for } j = i \\ x \vee z & \text{for } i < j. \end{cases}$$

Finally in the case $i = k/2$ we define

$$p_i^{\mathbf{A}_j}(x,y,z) = \begin{cases} x \wedge z & \text{for } j < i = k/2 \\ [(x \vee z) - y] \vee (x \wedge z) & \text{for } j = i = k/2 \\ x \vee z & \text{for } i = k/2 < j. \end{cases}$$

This definition is more clearly represented by a $k-1$ by $k-1$ table whose i, j^{th} entry is $p_i^{\mathbf{A}_j}$. In the case $k = 6$ this table is:

	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{A}_5$
p_1	$x-(y-z)$	x	x	x	$x \vee (z-y)$
p_2	$x \wedge z$	$x-(y-z)$	x	$x \vee (z-y)$	$x \vee z$
p_3	$x \wedge z$	$x \wedge z$	u	$x \vee z$	$x \vee z$
p_4	$x \wedge z$	$z-(y-x)$	z	$z \vee (x-y)$	$x \vee z$
p_5	$z-(y-x)$	z	z	z	$z \vee (x-y)$

where u, the element in the very center, is $[(x \vee z) - y] \vee (x \wedge z)$. In the general case, when k is even, the main diagonal is constant with value $x-(y-z)$ until the very middle element, which is u, and then it is constant with $z \vee (x-y)$. A similar situation holds for the sinister diagonal. What remains are four triangular wedges, each of which is constant. When k is odd, the pattern is the same except that the middle row and column are removed. Notice that if $j \le t$ then $p_i^{\mathbf{A}_j}(x,y,z) \le p_i^{\mathbf{A}_t}(x,y,z)$ as elements of the free Boolean algebra generated by $\{x,y,z\}$.

Let $\mathcal{P}_{k-1}$ be the variety generated by $\mathbf{A}_1, \ldots, \mathbf{A}_{k-1}$. It is evident from elementary set theory or Boolean algebra that in $\mathcal{P}_{k-1}$ the operations $p_1, \ldots, p_{k-1}$ obey the equations (10.2.1) of Theorem 10.1, and hence $\mathcal{P}_{k-1}$ has k-permutable congruences. For failure of $(k-1)$-permutability, it will be enough to establish that the vectors

$$\begin{aligned} \bar{a}_1 &= \langle 1, 1, \ldots, 1 \rangle \\ \bar{a}_2 &= \langle 0, 1, \ldots, 1 \rangle \\ &\vdots \\ \bar{a}_k &= \langle 0, 0, \ldots, 0 \rangle \end{aligned}$$

form a subuniverse of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_{k-1}$. For we then define congruences θ and ϕ on the corresponding subalgebra by $\bar{x}\ \theta\ \bar{y}$ if and only if $x_i = y_i$ for all even i, and $\bar{x}\ \phi\ \bar{y}$ if and only if $x_i = y_i$ for all odd i. It is easy to check that $\langle \bar{a}_1, \bar{a}_k \rangle$ is in $\theta \circ^k \phi$ but not in $\theta \circ^{k-1} \phi$.

Notice that $\{\bar{a}_1, \ldots, \bar{a}_k\}$ is the set of those function in $\mathbf{2}^{k-1}$ which are monotone. To see that these elements form a subalgebra we need to show that if $\bar{x}$, $\bar{y}$, and $\bar{z}$ are monotone, then $p_i(\bar{x}, \bar{y}, \bar{z})$ is also. That is, we must show

$$p_i^{\mathbf{A}_j}(x_j, y_j, z_j) \leq p_i^{\mathbf{A}_{j+1}}(x_{j+1}, y_{j+1}, z_{j+1})$$

for all j.

Now those $p_i^{\mathbf{A}_j}(x,y,z)$ which contain minus ($-$) fail to be monotone. Nevertheless one easily checks that there always exists a monotone $q_{i,j}(x,y,z)$ such that

$$p_i^{\mathbf{A}_j}(x,y,z) \leq q_{i,j}(x,y,z) \leq p_i^{\mathbf{A}_{j+1}}(x,y,z)$$

For example, $x - (y - z) \leq x \leq x \vee (z - y)$. Therefore, in every case

$$\begin{aligned} p_i^{\mathbf{A}_j}(x_j, y_j, z_j) &\leq q_{i,j}(x_j, y_j, z_j) \\ &\leq q_{i,j}(x_{j+1}, y_{j+1}, z_{j+1}) \\ &\leq p_i^{\mathbf{A}_{j+1}}(x_{j+1}, y_{j+1}, z_{j+1}), \end{aligned}$$

completing the argument.

ex10.2

Exercises 10.4

1. The variety of lattices is not congruence permutable. For every $k > 2$ there exists a lattice which is $(k-1)$-permutable but not k-permutable.

should we put Tschantz's example here?

2. Recall from Chapter 9

put a better reference here

that a quasi-primal algebra is a finite algebra whose clone of term operations contains the ternary discriminator operation:

$$t(a,b,c) = \begin{cases} c & \text{if } a = b \\ a & \text{if } a \neq b. \end{cases}$$

Prove that every quasiprimal algebra generates a congruence permutable variety.

3. If $\mathcal{V}$ has a 5-ary term q obeying the equations

$$x \approx q(x,y,y,z,z) \qquad q(x,x,y,y,z) \approx z$$

then $\mathcal{V}$ is congruence permutable.

4. The variety defined by the following equations is congruence permutable:

$$F(x,x,z) \approx z$$
$$H(u,u,x,y,w,z) \approx x$$
$$H(F(x,w,z),F(y,w,z),x,y,w,z) \approx y.$$

(This example will be used in the exercises of §10.5 below. As we will see in a later volume that it is in fact recursively undecidable whether a finite set of equations defines a congruence permutable variety.)

plicationAlgs

5. Implication algebras. (Mitschke 1971) The variety of implication algebras has a single binary operation symbol, $\to$, and is defined

this is the same as Exer 6 p.age 205 ov vol I, but I'll leave it.

by the equations

$$(x \to y) \to x \approx x$$
$$(x \to y) \to y \approx (y \to x) \to x$$
$$x \to (y \to z) \approx y \to (x \to z).$$

Prove that this variety is 3-permutable but not permutable. (Hint. Implication algebras can be interpreted in Boolean algebras by defining $x \to y$ as $y - x$; the reader should look for a nonrectangular subuniverse of $\langle 2, - \rangle^2$. To get 3-permutability, try something similar to the proof used in Example 10.3 with $k = 3$. In this case one column of the matrix will use only the Boolean operation $-$, and one can check that these same terms work for implication algebras.)

edSemigroups

6. Right-complemented semigroups. (Bosbach 1970). This variety has two

1969 ?? also check if all the equations are correct and necessary for 3-perm.

binary operations, $\cdot$ and $*$, and is defined by the following equations:

$$x \cdot (x * y) \approx y \cdot (y * x)$$
$$(x \cdot y) * z \approx y * (x * z)$$
$$x \cdot (y * y) \approx x$$

Prove that right-complemented semigroups are 3-permutable but not permutable. (Hint. This is similar to the previous exercise, except that now we need to notice that right-complemented semigroups can be interpreted in Boolean algebras by defining $x \cdot y$ to be $x \vee y$ and $x * y$ to be $y - x$. Now again look at Example 10.3.)

:HeytingAlgs

7. Heyting algebras. (Burris and Sankappanavar 1981) A Heyting algebra has operations $\vee$, $\wedge$, $\rightarrow$, 0, 1 and is defined

does this look like B–S invented them?

by the following equations, together with the equations of distributive lattice theory with 0 and 1:

$$\begin{aligned} x \rightarrow x &\approx 1 \\ (x \rightarrow y) \wedge y &\approx y \\ x \wedge (x \rightarrow y) &\approx x \wedge y \\ x \rightarrow (y \wedge z) &\approx (x \rightarrow y) \wedge (x \rightarrow z) \\ (x \vee y) \rightarrow z &\approx (x \rightarrow z) \wedge (y \rightarrow z) \end{aligned}$$

Prove that this variety is congruence permutable by verifying that

$$p(x,y,z) = (y \rightarrow x) \wedge (y \rightarrow z) \wedge (x \vee z)$$

is a Maltsev operation. (Hint: for a more conceptual approach, use the equations to verify that

$$(x \rightarrow y) \geq a \quad \text{if and only if} \quad y \geq a \wedge x,$$

i.e. $x \rightarrow y$ is the largest element a such that $a \wedge x \leq y$.)

kBooleanAlgs

8. k-Boolean algebras. In this exercise we present the variety that E. T. Schmidt used to show that k-permutability does not imply $(k-1)$-permutability. The variety used in Example 10.3 is a reduct of this variety. For fixed $k \geq 2$, let $\mathcal{B}_k$ be the variety with the operations of lattice theory, constants $c_1, \ldots, c_k$ and unary operations $f_1, \ldots, f_{k-1}$, which is defined by the identities of distributive lattices together with these equations:

$$\begin{aligned} x \vee c_1 &\approx c_1 \\ x \wedge c_k &\approx c_k \\ [(x \vee c_{i+1}) \wedge c_i] \vee f_i(x) &\approx c_i \\ [(x \vee c_{i+1}) \wedge c_i] \vee f_i(x) &\approx c_{i+1} \end{aligned}$$

for $1 \leq i \leq k-1$. In $\mathcal{B}_k$ we define $\mathbf{B}_{j,k}$, for $1 \leq j \leq k-1$ to be the algebra with universe $\{0,1\}$ and with

$$\begin{aligned} f_i(x) &= 1 \qquad \text{for } i \leq j \\ f_j(x) &= x' \\ f_i(x) &= 0 \qquad \text{for } i \geq j \end{aligned}$$

$$\begin{aligned} c_1 = \cdots = c_j &= 1 \\ c_{j+1} = \cdots = c_k &= 0 \end{aligned}$$

Show that $\mathcal{B}_k$ is k-permutable but not $(k-1)$-permutable. (Hint: The proof that $\mathcal{B}_k$ is not $(k-1)$-permutable is similar to that given in Example 10.3. The k-permutability can be proved by letting

$$p_i(x,y,z) = [x \wedge (f_i(y) \vee z)] \vee [z \wedge (f_{k-i}(y) \vee x)],$$

for $1 \leq i \leq k-1$, and verifying that the equations of (10.2.1) are identities of $\mathcal{B}_k$. It is much easier to appeal to the proof of Example 10.3 after first establishing two relatively easy facts: each $\mathbf{A}_j$ in that proof is the reduct of $\mathbf{B}_{j,k}$ to the operations $p_1, \ldots, p_{k-1}$ and the algebras $\mathbf{B}_{j,k}$ generate the variety $\mathcal{B}_k$ (in fact, they are its only subdirectly irreducible algebras).

9. Give an example of a variety $\mathcal{V}$ such that $\mathbf{F}_{\mathcal{V}}(k)$ has k-permutable congruences but $\mathcal{V}$ is not k-permutable.

10. (Wille 1970)

overlaps thm 4.68

Let $\theta \in \mathbf{CON\,A}$ and $f : \mathbf{A} \to \mathbf{B}$ is an onto homomorphism, with kernel ϕ. The relation

$$f(\theta) = \{\langle f(a), f(b)\rangle : \langle a,b\rangle \in \theta\}$$

may fail to be transitive, and thus to be a congruence. But if $\theta \circ \phi \circ \theta \subseteq \phi \circ \theta \circ \phi$ then $f(\theta)$ is a congruence. In fact, a variety $\mathcal{V}$ is congruence 3-permutable if and only if for every $\mathbf{A} \in \mathcal{V}$, every congruence $\theta \in \mathbf{CON\,A}$, and every onto homomorphism $f : \mathbf{A} \to \mathbf{B}$, $f(\theta)$ is a congruence. (Part of this exercise is worked out in Theorem 4.68.)

11. If $\theta \circ \phi \subseteq \phi \circ \theta$ for equivalence relations θ and ϕ on the same set, then $\theta \circ \phi = \phi \circ \theta = \theta \vee \phi$. But the corresponding assertion for triple products is false. Find an algebra $\mathbf{A}$ and congruences θ and ϕ on $\mathbf{A}$ such that $\theta \circ \phi \circ \theta < \phi \circ \theta \circ \phi$. (It still follows that $\theta \vee \phi = \phi \circ \theta \circ \phi$.) Is this situation possible inside a 3-permutable variety?

12. Give two Maltsev terms which differ on all nontrivial Boolean algebras.

13. (Hagemann and Mitschke 1973). For a variety $\mathcal{V}$, the following three conditions are equivalent:

i. $\mathcal{V}$ is congruence k-permutable;

ii. for every $\mathbf{A} \in \mathcal{V}$ and every reflexive subuniverse S of $A \times A$,

$$S^{-1} \subseteq \overbrace{S \circ \cdots \circ S}^{k-1 \text{ times}}$$

iii. for every $\mathbf{A} \in \mathcal{V}$ and every reflexive subuniverse S of $A \times A$,

$$\overbrace{S \circ \cdots \circ S}^{k \text{ times}} \subseteq \overbrace{S \circ \cdots \circ S}^{k-1 \text{ times}}$$

14. (Lakser 1982) If $\mathcal{V}$ is congruence k-permutable, $\mathbf{A} \in \mathcal{V}$, a, b, c, $d \in A$, and if $\langle c,d\rangle \in \mathrm{Cg}(a,b)$, then

$$\begin{aligned} c &= t_1(a,e_1,\ldots,e_m) \\ t_1(b,e_1,\ldots,e_m) &= t_2(a,e_1,\ldots,e_m) \\ &\vdots \\ t_{k-1}(b,e_1,\ldots,e_m) &= d \end{aligned}$$

for some $\mathcal{V}$-terms $t_1,\ldots,t_{k-1}$ and for some $e_1,\ldots,e_m \in A$. Conversely if all principal congruences in $\mathcal{V}$ can be expressed in this way, then $\mathcal{V}$ is congruence k-permutable. Compare this with Theorem 4.13 of §4.3. (Hint. Use the previous exercise.)

15. Suppose that every finite algebra in $\mathcal{V}$ has permutable congruences and that $\mathcal{V}$ is generated by its finite members. Must $\mathcal{V}$ be congruence permutable?

16. A subuniverse S of $\mathbf{A}\times\mathbf{B}$ is called **locally rectangular** if and only if whenever $\langle a,c\rangle$, $\langle b,c\rangle$, $\langle b,d\rangle \in S$, then $\langle a,d\rangle \in S$. Prove that $\mathcal{V}$ is congruence permutable if and only if every subuniverse of every product $\mathbf{A}\times\mathbf{B} \in \mathcal{V}$ is locally rectangular, and that this is equivalant to the property that every subuniverse of every direct square $\mathbf{A}\times\mathbf{A} \in \mathcal{V}$ is locally rectangular.

17. Show that if $\mathbf{A}$ is an algebra such that **CON A** contains a 0-1 sublattice isomorphic to $\mathbf{M}_3$ then $\mathbf{A}$ has at most one Maltsev term $p(x,y,z)$. The results of §4.13 are helpful.

PolinResLarge

18. **Polin's variety is residually large.** Let $\mathcal{P}$ be the variety of Example 10.2 and let $\mathbf{A}_0$ and $\mathbf{A}_1$ be the algebras given in that example. Let I be a set and let $\mathbf{C} = \mathbf{A}_1^I$

and let $\mathbf{B} = \mathbf{A}_0 \times \mathbf{C}$. Let $\overline{1} \in C$ be the element all of whose coordinates are 1. Let θ be the equivalence relation on B which identifies all pairs whose second coordinates are equal except that $\langle 1,\overline{1}\rangle$ and $\langle 0,\overline{1}\rangle$ are not θ-related. Show that θ is a completely meet irreducible congruence on $\mathbf{B}$ and that $|B/\theta| = |I|+1$. Thus Polin's variety $\mathcal{P}$ is residually large.

I added this 10/26/87. Check that references aren't screwed up.

We now present five exercises in which congruence permutability is applied to develop the theory of topological algebras. All these results may be found in Taylor 1977b. In all these exercises, $\mathbf{A} = \langle A, F_0, F_1, \ldots\rangle$ is an

do we need a reference for RL? Walter has 2 1977 papers; check if this is the right one.

algebra, and $\mathcal{T}$ a topology on A such that each F_i is continuous as a function $A^{n_i} \to A$, where n_i is the arity of F_i. This algebra is assumed to lie in a congruence permutable variety $\mathcal{V}$. We also assume that F_0 is a Maltsev operation: $F_0(x,x,z) \approx z \approx F_0(z,x,x)$ are identities of $\mathcal{V}$. Thus, these exercises form a generalization of the theory of topological groups.

19. If θ is a congruence relation on $\mathbf{A}$, then the closure $\overline{\theta}$ (in the space $A \times A$) is again a congruence on $\mathbf{A}$.

20. If $U \subseteq A$ is open, and θ is any congruence on $\mathbf{A}$, then $\{v \in A : \langle u,v\rangle \in \theta$, for some $u \in U\}$ is also open.

21. For $\theta \in \mathbf{CON\ A}$ there exists a unique topology on $\mathbf{A}/\theta$ so that all operations of $\mathbf{A}/\theta$ are continuous and $A \to A/\theta$ is an open continuous map.

22. If A is T_0, then A is Hausdorff.

23. In any case, $A/\overline{0}_{\mathbf{A}}$ is Hausdorff.

24. The last five exercises are false for topological algebras in general.

10.3 Congruence Semidistributive Varieties

We need to prove the following thms.

THEOREM 10.5. *The following are equivalent for a variety $\mathcal{V}$:*

(1) *$\mathcal{V}$ is congruence meet semidistributive.*

(2) *$[\alpha,\beta] = \alpha \wedge \beta$ for α, $\beta \in$ **CON A** and for all $\mathbf{A} \in \mathcal{V}$.*

(3) *$\mathcal{V}$ has no nontrivial abelian algebra.*

(4) *$\mathcal{V}$ has a set of Willard terms. (EXPAND THIS)*

Proof. NOTES: The equivalence of (2) and (3) is easy and should be in the commutator chapter. (2) implies (1) is also easy. (Make sure something like KK Theorem 2.19 is in the commutator chapter.) ■

MORE NOTES: p 8 of KK states that it is shown in Kearnes Szendrei that if $\mathcal{V}$ has a Taylor term, then Abelian algebras are affine. Using this, if A is Abelian then A x A has an M_3 (and so fails SD-meet), giving (1) implies (2). So we need to show that (2) implies a Taylor term. This is done in Theorem 3.13 of KK, which actually shows that no strongly abelian tolerance is equivalent to a Taylor term. The proof is involved. But maybe a direct proof that no abelian congruence implies Taylor term is easier. Lemma 3.5 may help.

thm: 3.1 **THEOREM 10.6.** *The following are equivalent for a variety $\mathcal{V}$:*

(1) *$\mathcal{V}$ is congruence semidistributive.*

(2) *$\mathcal{V}$ is congruence join semidistributive.*

(3) *$\mathcal{V}$ satisfies*

$$\gamma \cap (\alpha \circ \beta) \subseteq (\alpha \wedge \beta) \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma)$$

for congruences.

(4) *For some k, $\mathcal{V}$ has terms $d_0(x,y,z),\ldots,d_k(x,y,z)$ satisfying*

$$\begin{aligned}
&d_0(x,y,z) \approx x;\\
&d_i(x,y,y) \approx d_{i+1}(x,y,y) \quad \text{if } i \equiv 0 \text{ or } 1 \pmod 3;\\
&d_i(x,y,x) \approx d_{i+1}(x,y,x) \quad \text{if } i \equiv 0 \text{ or } 2 \pmod 3;\\
&d_i(x,x,y) \approx d_{i+1}(x,x,y) \quad \text{if } i \equiv 1 \text{ or } 2 \pmod 3;\\
&d_k(x,y,z) \approx z.
\end{aligned}$$

(5) *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in the variety of semilattices and in every non-trivial variety of modules.*

Proof. Clearly (1) implies (2). That (2) implies (3) is proved by considering the congruences $\gamma = \mathrm{Cg}(x,z)$ $\alpha = \mathrm{Cg}(x,y)$ and $\beta = \mathrm{Cg}(y,z)$ on $\mathbf{F}_{\mathcal{V}}(x,y,z)$. We assume that $\mathcal{V}$ satisfies SD($\vee$). Since, $\alpha \vee \beta = \alpha \vee \gamma = \beta \vee \gamma$ it follows that $\alpha \vee (\beta \wedge \gamma) = \beta \vee (\alpha \wedge \gamma) = \alpha \vee \beta$. Then

$$\alpha \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma) = \alpha \vee \beta\,;$$

$$\beta \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma) = \alpha \vee \beta\,;$$

and finally applying SD($\vee$) again we get

$$(\alpha \wedge \beta) \vee (\beta \wedge \gamma) \vee (\alpha \wedge \gamma) = \alpha \vee \beta\,,$$

and it is easy to see (3) follows from this.

Next, (4) is an easy consequence of (3)—or rather of the corollary that (x,z) belongs to the join of the three binary meets—using the usual characterization of the relations $f(x,y,z) \equiv g(x,y,z)$ mod α, mod β, and mod γ.

maybe refer to something in the previous section.

The Maltsev condition of (4) is idempotent and fails in the variety of semilattices and in every nontrivial variety of modules, as the reader can show; see Exercise XXXX.

[Put in a proof that (5) implies (2) and that (2) implies (1).] ■

10.4 Congruence Modularity

The early importance of modularity of the congruence lattice is evident in various generalizations of the Jordan-Hölder theorem (see pp. xxx-yyy of section 2.4), and the Birkhoff-Ore theory of factorization in modular lattices, later refined by B. Jónsson to a theory of factorization of finite algebras with modular congruence lattice (see Chapter 5). The varietal theory of congruence-modularity came into its own with the advent of the **commutator** in modular varieties; we will devote an entire chapter to this topic in a later volume.

Here our purpose is again the modest one of describing congruence-modularity, especially by writing a Maltsev condition for it. It turns out that there are two significant Maltsev conditions for modularity.

The modular law, which is a specialization of the distributive law, is:

$$(\alpha \vee \beta) \wedge \gamma \leq (\alpha \wedge \gamma) \vee \beta \qquad \text{if } \beta \leq \gamma. \tag{10.4.1}$$

for:modlaw

This is equivalent to the following inequality:

$$(\alpha \vee (\beta \wedge \gamma)) \wedge \gamma \leq (\alpha \wedge \gamma) \vee (\beta \wedge \gamma). \tag{10.4.2}$$

for:modlaw2

Just as we did for distributivity, we need an approximate form of the modular law in which $\circ$ appears instead of $\vee$. It turns out that the most suitable approximation is

$$(\alpha \circ (\beta \wedge \gamma) \circ \alpha) \wedge \gamma \leq \underbrace{(\alpha \wedge \gamma) \circ (\beta \wedge \gamma) \circ (\alpha \wedge \gamma) \circ \cdots}_{n \text{ factors}} \tag{Σ_n}$$

This approximation of modularity has the same shortcoming that we saw for Δ_n in our discussion of distributivity in the previous section, namely, Σ_n does not imply modularity of **CON A** when it is interpreted in the usual way. For a counterexample, see Exercise 10.20.5. However, we shall see from the proof of the next theorem that if we assume (Σ_n) holds for all β and $\gamma \in \operatorname{Con} A$ and for all symetric, reflexive subuniverses of $\mathbf{A}^2$, then **CON A** is modular.

We can now present the main Maltsev condition for congruence modularity.

Day's Theorem

THEOREM 10.7 (A. Day1969)**.** *For a variety $\mathcal{V}$, the following conditions are equivalent:*

i. *$\mathcal{V}$ is congruence modular;*

ii. $\mathbf{Con}\,\mathbf{F}_{\mathcal{V}}(4)$ *is modular;*

iii. *$\mathcal{V}$ satisfies some Σ_n (as a congruence identity);*

iv. *There exist* 4*-ary terms $m_0, \ldots, m_n$ in the language of $\mathcal{V}$ such that the following identities hold in*

$$\begin{aligned}
&m_0(x,y,z,u) \approx x \\
&m_i(x,y,y,x) \approx x && \textit{for all } i \\
&m_i(x,x,u,u) \approx m_{i+1}(x,x,u,u) && \textit{for } i \textit{ even} \\
&m_i(x,y,y,u) \approx m_{i+1}(x,y,y,u) && \textit{for } i \textit{ odd} \\
&m_n(x,y,z,u) \approx u
\end{aligned} \tag{M_n}$$

Moreover, $\mathcal{V}$ satisfies (iii) for a fixed value of n if and only if $\mathcal{V}$ satisfies (iv) for the same value of n.

Proof. We first prove that for fixed n, (iii) and (iv) are both equivalent to the auxiliary condition:

v. If α, β, and γ are the principal congruences on $\mathbf{F}_{\mathcal{V}}(x,y,z,u)$ generated by $\{\langle x,y\rangle,\langle z,u\rangle\}$, $\{\langle y,z\rangle\}$, and $\{\langle x,u\rangle,\langle y,z\rangle\}$, then

$$\langle x,u\rangle \in \underbrace{(\alpha\wedge\gamma)\circ\beta\circ(\alpha\wedge\gamma)\circ\cdots}_{n \text{ factors}} \qquad (10.4.3)$$ for:mod1

We obviously have (iii) implies (v). To see that (v) implies (iv) note that (10.4.3) means that there are terms $m_0,\ldots,m_n$ such that

$$x = m_0^{\mathbf{F}}(x,y,z,u)\ \alpha\wedge\gamma\ m_1^{\mathbf{F}}(x,y,z,u)\ \beta\ m_2^{\mathbf{F}}(x,y,z,u)$$
$$\alpha\wedge\gamma\ m_3^{\mathbf{F}}(x,y,z,u)\ \beta\ \cdots\ m_n^{\mathbf{F}}(x,y,z,u) = u$$

where $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y,z,u)$. Since $\beta \le \gamma$ we have $x\ \gamma\ m_i^{\mathbf{F}}(x,y,z,u)\ \gamma\ m_i^{\mathbf{F}}(x,y,y,x)$. Since γ is trivial on the subalgebra of $\mathbf{F}$ generated by x and y, $\mathcal{V}$ satisfies $x \approx m_i(x,y,y,x)$. Similar reasoning shows that the other equations of (iv) hold.

Possibly make a general lemma to this effect for §1 or 2.

To see that (iv) implies (iii), let α, β, and $\gamma \in \mathbf{CON\,A}$ for some $\mathbf{A}\in\mathcal{V}$ and suppose that $\langle a,d\rangle \in \gamma\wedge(\alpha\circ(\beta\vee\gamma)\circ\alpha)$. Hence there are b and $c\in A$ so that the relations in Figure 10.3.

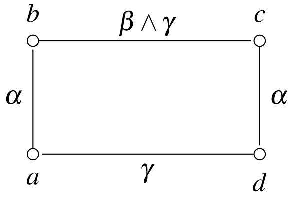

Figure 10.3: fig:modfig1

Let

$$e_i = m_i^{\mathbf{A}}(a,b,c,d) \qquad \text{for } 0\le i\le n$$

Now (iv) yields

$$\begin{aligned}
e_0 &= m_0^{\mathbf{A}}(a,b,c,d) = a \\
e_i &= m_i^{\mathbf{A}}(a,b,c,d)\ \gamma\ m_i^{\mathbf{A}}(a,b,b,a) = a && \text{for all } i \\
e_i &= m_i^{\mathbf{A}}(a,b,c,d)\ \alpha\ m_i^{\mathbf{A}}(a,a,d,d) = m_{i+1}^{\mathbf{A}}(a,a,d,d) \\
&\quad \alpha\ m_{i+1}^{\mathbf{A}}(a,b,c,d) = e_{i+1} && \text{for } i \text{ even} \\
e_i &= m_i^{\mathbf{A}}(a,b,c,d)\ \beta\ m_i^{\mathbf{A}}(a,b,b,d) = m_{i+1}^{\mathbf{A}}(a,b,b,d) \\
&\quad \beta\ m_{i+1}^{\mathbf{A}}(a,b,c,d) = e_{i+1} && \text{for } i \text{ odd} \\
e_n &= m_n^{\mathbf{A}}(a,b,c,d) = d
\end{aligned}$$

Thus we have

$$a = e_0\ (\alpha \wedge \gamma)\ e_1\ (\beta \wedge \gamma)\ e_2\ (\alpha \wedge \gamma)\ e_3\ (\beta \wedge \gamma)\ e_4 \cdots e_n = d,$$

showing that (Σ_n) holds. Thus (iii), (iv), and (v) are equivalent.

Now the fact that (i) implies (ii), and that (ii) implies (v) are both trivial. We will show that (iv) implies (i), which will show that (i)–(v) are equivalent.

Let us first notice that if, in the above proof that (iv) implies (iii), we weaken the assumption that α is transitive and assume only that α is a reflexive symmetric subuniverse of $\mathbf{A}^2$, then we still obtain the slightly weaker conclusion that

$$\langle a,d\rangle \in (\alpha \wedge \gamma) \circ (\alpha \wedge \gamma) \circ (\beta \wedge \gamma) \circ (\alpha \wedge \gamma) \circ \cdots, \qquad (10.4.4)$$ for:mod2

with the pattern of two $(\alpha \wedge \gamma)$–factors followed by one β–factor repeating as long as necessary. (To the above argument, we need only add the observation that $m_i^{\mathbf{A}}(a,b,c,d)\ \gamma\ m_i^{\mathbf{A}}(a,a,d,d)$, which is true since both are γ-related to a.) In proving the modular law (10.4.2) from (iv) we will use (10.4.4) with α replaced by the reflexive symmetric subuniverse

$$\Gamma_k = \alpha \circ (\beta \wedge \gamma) \circ \alpha \circ (\beta \wedge \gamma) \circ \cdots \circ \alpha \qquad \text{with } 2k+1 \text{ factors}$$

Since $\bigcup_{k<\omega}(\Gamma_k \wedge \gamma) = (\alpha \vee (\beta \wedge \gamma)) \wedge \gamma$, modularity will follow from the inclusions

$$\Gamma_k \wedge \gamma \leq (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$$

which we will prove by induction on k. The inclusion is obvious for $k = 0$, and for the inductive step, we first observe that $\Gamma_{k+1} = \Gamma_k \circ (\beta \wedge \gamma) \circ \alpha$, and so if $\langle a,d\rangle \in \Gamma_{k+1} \wedge \gamma$ we have that for some b, $c \in A$ the situation of Figure 10.4 holds.

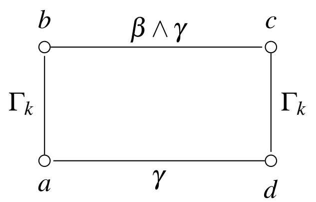

Figure 10.4: fig:modfig2

Thus by (10.4.4) we have $\langle a,d\rangle$ is in a finite relational product of the relations $\Gamma_k \wedge \gamma$ and $\beta \wedge \gamma$. Now by induction, we have $\langle a,d\rangle \in (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$. This completes the proof of the theorem. ■

It is easy to see that the sublattice of $\mathbf{F}_{\mathcal{V}}(x,y,z,u)$ generated by the congruences α, β, and γ defined in condition (v) is a homomorphic image of the lattice diagrammed in Figure 10.5.

Obviously if $\mathcal{V}$ is congruence modular, the pentagon of Figure 10.5 must collapse. One of the surprising consequences of Day's result is that this is sufficient.

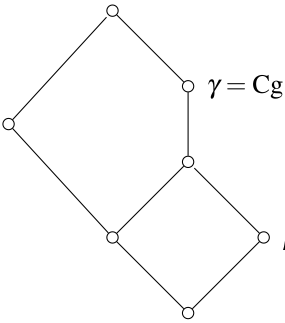

Figure 10.5: fig:modfig3

pentagon

COROLLARY 10.8. *Let α, β, and γ be the congruences on $\mathbf{F}_{\mathcal{V}}(x,y,z,u)$ generated by $\{\langle x,y\rangle,\langle z,u\rangle\}$, $\{\langle y,z\rangle\}$, and $\{\langle x,u\rangle,\langle y,z\rangle\}$ (see Figure 10.5). Then $\mathcal{V}$ is congruence modular if and only if $\beta \vee (\alpha \wedge \gamma) = \gamma$, which is equivalent to*

$$\langle x,u\rangle \in \beta \vee (\alpha \wedge \gamma)$$

■

By Theorem 10.7, $\mathcal{V}$ is congruence modular if and only if **CON** $\mathbf{F}_{\mathcal{V}}(4)$ is modular. This statement is false if 4 is replaced by 3. Indeed, the variety of sets is not congruence modular, but **CON** $\mathbf{F}_{\mathcal{V}}(3)$ is the partition lattice on a three element set, which is $\mathbf{M}_3$.

Notice that the congruence α of the corollary is the kernel of the endomorphism of $\mathbf{F}_{\mathcal{V}}(x,y,z,u)$ which maps $x \mapsto x$, $y \mapsto x$, $z \mapsto u$, and $u \mapsto u$. By the same token, γ is also the kernel of a homomorphism from $\mathbf{F}_{\mathcal{V}}(4)$ onto $\mathbf{F}_{\mathcal{V}}(2)$. Since the copy of $\mathbf{N}_5$ of Figure 10.5 lies above $\alpha \wedge \gamma$, the $\mathbf{N}_5$ lies in the congruence lattice of a subdirect product of two copies of $\mathbf{F}_{\mathcal{V}}(2)$. Thus the next result follows from the last corollary.

mod3

COROLLARY 10.9. *A variety $\mathcal{V}$ is modular if and only if the variety generated by $\mathbf{F}_{\mathcal{V}}(2)$ is.* ■

Mention Walter's problem that shows that distributivity cannot be defined by with 2 variable terms—in the last section. More intro here too.

We will now use Theorem 10.7 to investigate other conditions equivalent to congruence modularity for varieties. We begin with an important lemma.

mod3.5

LEMMA 10.10. *Let $\mathcal{V}$ be a modular variety. Then there is a term $d(x,y,z)$ such that*

i. *$\mathcal{V}$ satisfies $d(x,x,z) \approx z$.*

ii. *If $\mathbf{A} \in \mathcal{V}$ and α, β, and $\gamma \in \mathrm{Con} A$, and if $\langle a,b\rangle \in \alpha \wedge (\beta \vee \gamma)$ then*

$$a \ \ (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \ \ d(a,b,b).$$

In a modular variety it is easy to see that a term satisfying (i) and (ii) will also satisfy: $x\ \theta\ y$ implies $x\,[\theta,\theta]\,d(x,y,y)$. Indeed, in $\mathbf{A}(\theta)$, $\langle x,x\rangle\ \eta_0 \wedge (\Delta_{\theta,\theta} \vee \eta_1)\ \langle x,y\rangle$. Hence by (ii), $\langle x,x\rangle\ \Delta_{\theta,\theta}\ \langle x,d(x,y,y)\rangle$. Now the result follows from 4.9 of Freese and Ralph McKenzie 1987.

Proof. Since $\mathcal{V}$ is modular there are terms such that $\mathcal{V}$ satisfies (M_n). Define terms $q_i(x,y,z)$ inductively by letting $q_0(x,y,z) = z$ and

$$q_{i+1}(x,y,z) = \begin{cases} m_{i+1}(q_i(x,y,z),x,y,q_i(x,y,z)) & i \text{ even} \\ m_{i+1}(q_i(x,y,z),y,x,q_i(x,y,z)) & i \text{ odd} \end{cases}$$

Now define $d(x,y,z) = q_n(x,y,z)$. A straighforward induction shows that $\mathcal{V}$ satisfies $q_i(x,x,z) \approx z$ from which (i) follows.

Let $\theta = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$. Suppose that $\langle x,y\rangle \in \alpha \wedge (\beta \vee \gamma)$. Then $x\ \alpha\ y$ and there are elements $x_j \in A$ such that $x = x_0\ \beta\ x_1\ \gamma\ x_2 \cdots x_k = y$. We will show by induction on i that

$$q_i(x,y,y)\ \theta\ \begin{cases} m_i(y,y,x,x) & i \text{ even} \\ m_i(y,y,y,x) & i \text{ odd} \end{cases}$$

This is trivial for $i = 0$. Assume i is even. Then

$$\begin{aligned} m_{i+1}(q_i(x,y,y),x,x,q_i(x,y,y)) &= q_i(x,y,y) \\ &\ \theta\ m_i(y,y,x,x) \\ &= m_{i+1}(y,y,x,x). \end{aligned}$$

Hence we have the following relations.

$$\begin{array}{ccc} m_{i+1}(q_i(x,y,y),x,x,q_i(x,y,y)) & \theta & m_{i+1}(y,y,x,x) \\ \beta & & \beta \\ m_{i+1}(q_i(x,y,y),x,x_1,q_i(x,y,y)) & \alpha & m_{i+1}(y,y,x_1,x) \\ \gamma & & \gamma \\ m_{i+1}(q_i(x,y,y),x,x_2,q_i(x,y,y)) & \alpha & m_{i+1}(y,y,x_2,x) \\ \beta & & \beta \\ \vdots & & \vdots \\ m_{i+1}(q_i(x,y,y),x,y,q_i(x,y,y)) & \alpha & m_{i+1}(y,y,y,x) \end{array}$$

Since $\theta = \alpha \wedge (\beta \vee \theta) = \alpha \wedge (\gamma \vee \theta)$ by modularity, the first α can be replaced with θ. Then the second α can be replaced with θ. Finally, using the definition of q_i and the last row, we obtain

$$q_{i+1}(x,y,y) = m_{i+1}(q_i(x,y,y),x,y,q_i(x,y,y))\ \theta\ m_{i+1}(y,y,y,x).$$

For i odd the proof is similar (although somewhat more awkward since you use the sequence $x\ \beta\ x_1\ \gamma\ x_2\ \cdots\ y$ backwards).

Thus when $i = n$, we obtain $x\ \theta\ d(x,y,y)$ as desired. ■

The next corollary shows that in an algebra in a modular variety, the congruences $\alpha \wedge (\beta \vee \gamma)$ and $\gamma \vee (\alpha \wedge \beta)$ always permute. This is a surprising result about modular varieties. But notice that in a distributive variety $\alpha \wedge (\beta \vee \gamma) \leq$

$\gamma \vee (\alpha \wedge \beta)$, and so obviously permute. This phenomenom of a result which is nontrivial for modular varieties but trivial for distributive varieties is really part of the commutator theory introduced in Chapter 4 of the first volume. We will take up a thorough study of the commutator in a later volume.

cor1to10.8 **COROLLARY 10.11.** *Let α, β, and $\gamma \in \mathrm{Con}A$, where* **A** *is an algebra in a congruence modular variety $\mathcal{V}$. Then $\alpha \wedge (\beta \vee \gamma)$ and $\gamma \vee (\alpha \wedge \beta)$ permute.*

Proof. As before let $\theta = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and assume that

$$x \;\; \alpha \wedge (\beta \vee \gamma) \;\; y \;\; \gamma \vee (\alpha \wedge \beta) \;\; z$$

Then we calculate

$$x \; \theta \; d(x,y,y) \; \gamma \vee (\alpha \wedge \beta) \; d(x,y,z) \; \alpha \wedge (\beta \vee \gamma) \; d(x,x,z) = z.$$

Since $\theta \leq \gamma \vee (\alpha \wedge \beta)$, this shows that $\gamma \vee (\alpha \wedge \beta)$ and $\alpha \wedge (\beta \vee \gamma)$ permute. ■

COROLLARY 10.12. *Suppose that* **A** *lies in a congruence modular variety and that* **L** *is a sublattice of* **CON A** *isomorphic to* $\mathbf{M}_3$. *Then all the members of L permute with each other. If in addition the least and greatest elements of* **L** *are the least and greatest elements of* **CON A**, *then the term $d(x,y,z)$ of Lemma 10.10 is a Maltsev term for* **A**, *and thus* $\mathbf{V}(\mathbf{A})$ *is permutable.*

Proof. The first part follows directly form the previous corollary. To see the rest, let β, γ, and δ be three congruences pairwise meet to 0_A and pairwise join to 1_A. By the first part, these congruences permute with oneanother and so $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$, where $\mathbf{B} = \mathbf{A}/\beta$ and $\mathbf{C} = \mathbf{A}/\gamma$. If $b \in B$ and c, $c' \in C$, then $\langle b,c\rangle \; \beta \wedge (\gamma \vee \delta) \; \langle b,c'\rangle$ and so Lemma 10.10

$$\langle b,c\rangle \;\; (\beta \wedge \gamma) \vee (\beta \wedge \delta) \;\; d(\langle b,c\rangle, \langle b,c'\rangle, \langle b,c'\rangle).$$

Since this congruence is 0_A, $c = d(c,c',c')$, showing that $d(x,y,z)$ is a Maltsev term for **C**. Similarly it is a Maltsev term for **B**, and thus for **A**. ■

Abelian algebras are defined in section 4.13 and it is shown that an Abelian algebra in a congruence permutable variety is polynomially equivalent to a module over a ring (and conversely). It is pointed out that this is actually true under the weaker assumption of congruece modularity. The next corollary shows that if **A** is an Abelian algebra in a congruence modular variety, then $\mathbf{V}(\mathbf{A})$ is congruence permutable and thus we see that an Abelian algebra in a congruence modular variety is polynomially equivalent to a module over a ring.

COROLLARY 10.13. *Let* **A** *be an Abelian algebra in a congruence modular variety, then* $\mathbf{V}(\mathbf{A})$ *is congruence permutable.*

Proof. Definition 5.151 defines $\Delta(\mathbf{A})$, a congruence on $\mathbf{A}^2$, by

$$\Delta(\mathbf{A}) = \mathrm{Cg}^{\mathbf{A}^2}(\{\langle\langle x,x\rangle, \langle y,y\rangle\rangle : x, y \in A\})$$

Let $\Delta = \Delta(\mathbf{A})$. We claim that the projection kernels, η_0 and η_1, both join with Δ to $\mathbf{1}_{\mathbf{A}^2}$ and meet with it to $\mathbf{0}_{\mathbf{A}^2}$. The calculations for the join are straightforward. For the meet suppose $\langle\langle x,y\rangle,\langle x,z\rangle\rangle \in \eta_0 \wedge \Delta$, then $\langle\langle y,y\rangle,\langle y,z\rangle\rangle \in \eta_1 \circ (\eta_0 \wedge \Delta) \circ \eta_1$. Hence by modularity

$$\langle\langle y,y\rangle,\langle y,z\rangle\rangle \in \eta_0 \wedge (\eta_1 \vee (\eta_0 \wedge \Delta)) = (\eta_0 \wedge \eta_1) \vee (\eta_0 \wedge \Delta) = \eta_0 \wedge \Delta$$

Since $\mathbf{A}$ is Abelian, Theorem 4.152 implies $y = z$ and so $\eta_0 \wedge \Delta = \mathbf{0}_{\mathbf{A}^2}$.

The result now follows from the previous corollary. ■

Rework; it's awkward now. Possible give the ref to gumm: 1978 (Math Z.).

In the introduction we pointed out that congruence permutability could be viewed as completing a parallelagram, see Figure 10.1. In modular varieties such parallelagrams can be completed provided we have certain additional points. We now investigate some of these geometric conditions which are equivalent to congruence modularity. The first condition is due to H.-P. Gumm, who initiated this type of condition.

We call a term $d(x,y,z)$ a **difference term** or a **3-ary difference term** or a **Gumm difference term for** $\mathcal{V}$ if for any $\mathbf{A} \in \mathcal{V}$, α, β, $\gamma \in \operatorname{Con} A$, and x, y, z, u, and $v \in \mathbf{A}$, the implication of Figure 10.6 holds.

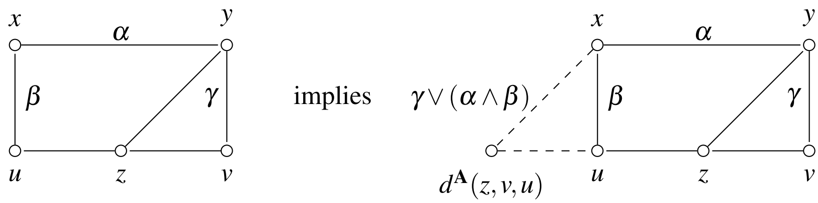

Figure 10.6:

fig:gumm

gummdiff

THEOREM 10.14. *A variety $\mathcal{V}$ is congruence modular if and only if $\mathcal{V}$ has a difference term. Moreover, if there is a term $d(x,y,z)$ such that $\mathcal{V}$ satisfies conditions (i) and (ii) of Lemma 10.10, then $d(x,y,z)$ is a difference term for $\mathcal{V}$.*

Proof. Assume $\mathcal{V}$ is congruence modular. Then it satisfies condition (iv) of Theorem 10.7. Let $d(x,y,z)$ be a term satisfying conditions (i) and (ii) of Lemma 10.10. Assume that $\mathbf{A} \in \mathcal{V}$ and that the relations of the left side of Figure 10.6 hold in $\mathbf{A}$. It follows easily from condition (i) that $d(z,v,u)\ \alpha\ u$. The following relations are easily checked.

$$d(z,v,u)\ \gamma\ d(y,v,u) \tag{10.4.5}$$
for:gumm1

$$d(y,v,u)\ \beta\ d(y,y,x) = x \tag{10.4.6}$$
for:gumm2

$$d(y,v,u)\ \alpha\ d(x,u,u) \tag{10.4.7}$$
for:gumm3

Since $\langle x,u\rangle \in \beta \wedge (\alpha \vee \gamma)$, condition (ii) implies that $d(x,u,u)\ (\alpha \wedge \beta) \vee (\beta \wedge \gamma)\ x$. Hence by (10.4.7), $d(y,v,u)\ \alpha \vee (\beta \wedge \gamma)\ x$. So by (10.4.6), $d(y,v,u)\ \beta \wedge (\alpha \vee$

$(\beta \wedge \gamma))\, x$. But $\beta \wedge (\alpha \vee (\beta \wedge \gamma)) = (\alpha \wedge \beta) \vee (\beta \wedge \gamma)$ and thus by (10.4.5), $d(z,v,u)\ \gamma \vee (\alpha \wedge \beta)\ x$, as desired.

For the converse, see Exercise 10.20.1. ■

The name below may not be the right choice. Mention that the reason for this name will be seen later. I wanted to call the term below a 4-ary difference term. But I can't seem to derive all the needed properties of a 4 ary diff term just from the implication of the figure. Interestingly I can if $\gamma \vee (\alpha \wedge \beta)$ is replaced by $\gamma \vee [\alpha, \beta]$.

A useful variant of Gumm's difference term is a four variable term discovered by E. W. Kiss. This variant only requires 4 points but the term does depend on all 4 of its variables.

THEOREM 10.15. *The following are equivalent for a variety $\mathcal{V}$.*

i. *$\mathcal{V}$ is congruence modular.*

ii. *If α, β, and $\gamma \in$ **CON A**, **A** $\in \mathcal{V}$, then*

$$(\alpha \vee \gamma) \wedge (\beta \vee \gamma) \subseteq (\gamma \vee (\alpha \wedge \beta)) \circ \alpha.$$

iii. *There is a term $q(x,y,z,u)$ such that if α, β, and $\gamma \in$ **CON A**, **A** $\in \mathcal{V}$, then the implication of Figure 10.7 holds.*

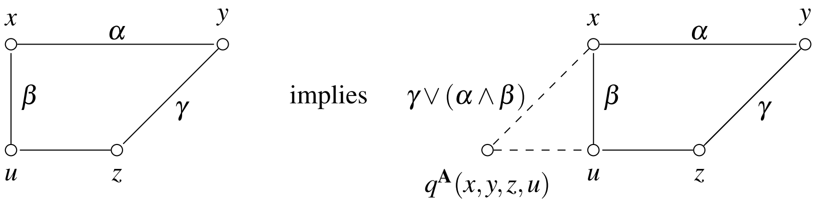

Figure 10.7: fig:kiss

Proof. Let α, β, and $\gamma \in$ **CON A**, **A** $\in \mathcal{V}$. By modularity and Corollary 10.11 we have

$$\begin{aligned}(\alpha \vee \gamma) \wedge (\beta \vee \gamma) &= (\alpha \wedge (\beta \vee \gamma)) \vee \gamma \\ &= (\alpha \wedge (\beta \vee \gamma)) \vee \gamma \vee (\alpha \wedge \beta) \\ &= [\gamma \vee (\alpha \wedge \beta)] \circ [(\alpha \wedge (\beta \vee \gamma))] \\ &\subseteq [\gamma \vee (\alpha \wedge \beta)] \circ \alpha,\end{aligned}$$

proving that (i) implies (ii).

Assume (ii) and let α, β, and γ be the congruences on $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y,z,u)$ generated by $\{\langle x,y\rangle, \langle z,u\rangle\}$, $\{\langle x,u\rangle\}$, and $\{\langle y,z\rangle\}$, respectively. The inclusion of (ii) immediately implies that there is an element $q^{\mathbf{F}}(x,y,z,u) \in \mathbf{F}_{\mathcal{V}}(x,y,z,u)$ satisfying the indicated relations of Figure 10.7. It is easy to see that $q(x,y,z,u)$ satisfies (iii).

To see that (iii) implies (i), let α, β, and γ be the congruences in Figure 10.5. Now the relations of the first part of Figure 10.7 hold with the roles of β and γ reversed. Thus by (iii) we have

$$x\ (\beta \vee (\alpha \wedge \gamma))\ q(x,y,z,u)\ \alpha\ u.$$

Since $\beta \leq \gamma$, we have that $u\,\gamma\,x\,\gamma\,q(x,y,z,u)$ and hence, $u\,\alpha \wedge \gamma\,q(x,y,z,u)$. Thus $x\,\beta \vee (\alpha \wedge \gamma)\,u$ and the modularity of $\mathcal{V}$ now follows from Corollary 10.8. ■

mod6 **LEMMA 10.16.** *Let $q(x,y,z,u)$ be a term which satisfies condition (Iii) of the last theorem. Then*

i. *$q(x,x,u,u) \approx u$ is an identity of $\mathcal{V}$.*

ii. *$q(x,y,y,x) \approx x$ is an identity of $\mathcal{V}$.*

iii. *$r(x,y,z,u) = q(z,y,x,u)$ also satisfies condition (iii) of Theorem 10.15.*

Proof. If we let $\beta = \gamma = 0$ then $\gamma \vee (\alpha \wedge \beta) = 0$, and so the implication of Figure 10.7 with $x = u$ and $y = z$ gives $q(x,y,y,x) = x$. A similar argument proves (ii).

To see (iii), suppose that the relations indicated in the left side of Figure 10.7 hold for some $\mathbf{A} \in \mathcal{V}$. Note

$$r(x,y,y,u)\ \beta\ r(x,y,y,x) = q(y,y,x,x) = x$$
$$q(x,y,y,u)\ \beta\ q(x,y,y,x) = x$$

Moreover, $r(x,y,y,u)\ \alpha\ r(x,x,x,u) = q(x,x,x,u)\ \alpha\ q(x,y,y,u)$. Thus $r(x,y,y,u)\ \alpha \wedge \beta\ q(x,y,y,u)$. Since $r(x,y,z,u)\ \gamma\ r(x,y,y,u)$ and $q(x,y,z,u)\ \gamma\ q(x,y,y,u)$, we have that $r(x,y,z,u)\ \gamma \vee (\alpha \wedge \beta)\ q(x,y,z,u)$. Moreover, $r(x,y,z,u)\ \alpha\ r(x,x,u,u) = q(u,x,x,u) = u$. From this it follows that $q(x,y,z,u)$ satisfies condition (iii) of Theorem 10.15. ■

The next theorem shows that we can form a difference term from a 4-ary difference term.

mod7 **THEOREM 10.17.** *If $q(x,y,z,u)$ is a term satisfing condition (iii) of Theorem 10.15, then $d(x,y,z) = q(x,y,z,z)$ is a 3-ary difference term.*

Proof. Suppose that $\mathbf{A}$ is an algebra in $\mathcal{V}$ which has elements and congruences satisfying the relations of the left side of Figure 10.6. By part (iii) of the last theorem, the relations of the right side of Figure 10.6 hold if we replace $d^{\mathbf{A}}(z,w,u)$ by $q^{\mathbf{A}}(z,y,x,u)$. But $q^{\mathbf{A}}(z,y,x,u)$ and $d^{\mathbf{A}}(z,w,u) = q^{\mathbf{A}}(z,w,u,u)$ are both α-related to u, and clearly they are β-related. This implies that the relations of Figure 10.6 hold. ■

more intro here

Put some of walters remarks here. Also point out that Gumm's contition is the same as thm 4.144 with one equation removed. Possibley put the p at the end so that it is a Gumm difference term.

Now we come to Gumm's Maltsev condition for congruence modularity.

gummterms

THEOREM 10.18. *For a variety $\mathcal{V}$ the following conditions are equivalent.*

i. *$\mathcal{V}$ is congruence modular.*

ii. *For any $\mathbf{A} \in \mathcal{V}$ and any α, β, and $\gamma \in \operatorname{Con} A$,*

$$(\alpha \circ \beta) \wedge \gamma \leq \beta \circ \alpha \circ [(\alpha \wedge \gamma) \vee (\beta \wedge \gamma)].$$

iii. *For some k, there exist terms p, $d_1, \ldots, d_k$ in the language of $\mathcal{V}$ such that the following equations hold identically in $\mathcal{V}$:*

$$\begin{aligned} x &\approx p(x,z,z) \\ p(x,x,z) &\approx d_1(x,x,z) \\ d_i(x,y,x) &\approx x && \textit{for all } i \\ d_i(x,x,z) &\approx d_{i+1}(x,x,z) && \textit{for } i \textit{ even} \\ d_i(x,z,z) &\approx d_{i+1}(x,z,z) && \textit{for } i \textit{ odd} \\ d_k(x,y,z) &\approx z \end{aligned} \tag{G_k}$$

Proof. To see (i) implies (ii), suppose that α, β, and $\gamma \in \operatorname{Con} A$ for some $\mathbf{A} \in \mathcal{V}$ and that $\langle a,c\rangle \in \gamma$, $\langle a,b\rangle \in \alpha$, and $\langle b,c\rangle \in \beta$. Since $\mathcal{V}$ is modular, it has a term $d(x,y,z)$ satisfying conditions (i) and (ii) of Lemma 10.10. Using this we calculate

$$a = d(c,c,a)\ \beta\ d(c,b,a)\ \alpha\ d(c,a,a)\ (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)\ c.$$

Now assume (ii) holds. In $\mathbf{F}_{\mathcal{V}}(x,y,z)$ let $\alpha = \operatorname{Cg}(x,y)$, $\beta = \operatorname{Cg}(y,z)$, and $\gamma = \operatorname{Cg}(x,z)$. By (ii) there are three variable terms p and d_i, for $i = 1, \ldots, k$, such that $x\ \beta\ p(x,y,z)$, $p(x,y,z)\ \alpha\ d_1(x,y,z)$, $d_i(x,y,z)\ \alpha \wedge \gamma\ d_{i+1}(x,y,z)$, for i even, and $d_i(x,y,z)\ \beta \wedge \gamma\ d_{i+1}(x,y,z)$, for i odd. These fact imply that $\mathcal{V}$ satisfies the equations (G_k) by the usual arguments.

Given Gumm terms satisfying (iii), we can define Day tems by:

$$\begin{aligned} m_0(x,y,z,u) &= m_1(x,y,z,u) = x \\ m_2(x,y,z,u) &= p(x,y,z) \\ m_3(x,y,z,u) &= d_1(x,y,u) \\ m_4(x,y,z,u) &= d_1(x,z,u) \end{aligned}$$

and for $i > 0$:

$$
\begin{aligned}
m_{4i+1}(x,y,z,u) &= d_{2i}(x,z,u)\\
m_{4i+2}(x,y,z,u) &= d_{2i}(x,y,u)\\
m_{4i+3}(x,y,z,u) &= d_{2i+1}(x,y,u)\\
m_{4i+4}(x,y,z,u) &= d_{2i+1}(x,z,u)
\end{aligned}
$$

It is straightforward to verify that these terms satisfy (M_{2k+1}). ■

We close this section with Gumm's Shifting Lemma Gumm 1978a, which plays an important role in the commutator theory.

Walter included some more exact relations between day and gumm which I think I'll put in the exercises. The stuff on Wille Pixley should go in the congruence identities chapter. Also check where gumm first did the shifting lemma.

shiftinglemma

THEOREM 10.19 (THE SHIFTING LEMMA)**.** *If* **CON A** *is modular and* α, β, $\gamma \in \operatorname{Con} A$, *then the implication of Figure 10.8 holds for* x, y, z, *and* $u \in A$. *Moreover, if the implication of Figure 10.8 holds for all algebras in a variety* $\mathcal{V}$, *then* $\mathcal{V}$ *is congruence modular.*

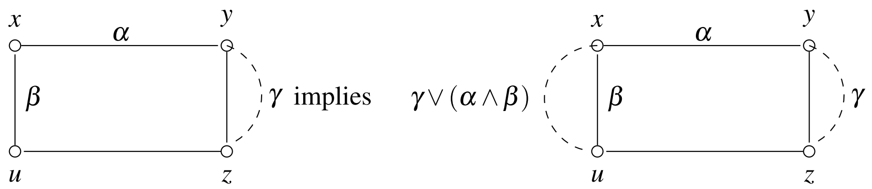

Figure 10.8:

fig:shifting

Proof. To see the first statement, note that $\langle x,u\rangle \in \beta \wedge (\alpha \vee (\beta \wedge \gamma))$ and so by modularity $\langle x,u\rangle \in (\alpha\wedge\beta)\vee(\beta\wedge\gamma)$. For the second statement, see Exercise 10.20.6. ■

ex10.4

Exercises 10.20

exer:gummmod

1. Use Corollary 10.8 to show that if a variety $\mathcal{V}$ has a difference term, then $\mathcal{V}$ is modular.

2. (Alan Day 1969) Show how to derive terms satisfying (M_2) from a Maltsev term $p(x,y,z)$. Conversely, given terms satisfying (M_2) derive a Maltsev term.

3. Show that if $\mathcal{V}$ is 3-permutable with terms p_1 and p_2 which satisfy condition (iii) of Theorem 10.1 (with $k=3$), then $\mathcal{V}$ satisfies (M_3).

4. Show that the variety of lattices satisfies (M_3). Thus the converse of the previous exercise is not true.

exer:notmod

5. Show that Polin's algebra, given in Example 10.2, satisfies (Σ_3), but is not modular.

hiftinglemma

6. Prove that if the implication of Figure 10.8 holds for all algebras in a variety $\mathcal{V}$, then $\mathcal{V}$ is congruence modular.

10.5 Congruence Regularity and Uniformity

A family F of equivalence relations on a set A is called **regular** if each $\theta \in F$ is determined (within F) by each of its blocks, i.e., if θ, $\phi \in F$ and $a/\theta = a/\phi$ for some $a \in A$, then $\theta = \phi$. This property is evident for the family of partitions of an affine space into parallel flats, and thus also for the congruence relations on a vector space. Thus, in general, we call an algebra **A** **congruence regular** if and only if Con$A$ is a regular family of equivalence relations, and as usual, we will call a variety congruence regular if and only if each of its algebras is congruence regular. As we will see in §10.7, congruence regularity of a variety $\mathcal{V}$ is stronger than modularity, but independent of distributivity; it also implies that $\mathcal{V}$ is congruence k-permutable for some k. Later in this section we will consider the even stronger properties of congruence uniformity and coherence.

Our first lemma shows that regularity can be established for an entire variety by checking the above condition only for $\theta = 0$.

reg1 **LEMMA 10.21.** *$\mathcal{V}$ is congruence regular if and only if the following condition holds for all* $\mathbf{A} \in \mathcal{V}$*,* $\phi \in$ Con$A$*, and all* $a \in A$*:*

$$a/\phi = \{a\} \Rightarrow \phi = 0.$$

Proof. The condition is a special case of the definition (namely $\theta = 0$), and hence follows from regularity. Conversely, let us suppose that the condition holds for $\mathcal{V}$. To prove regularity, let $\mathbf{A} \in \mathcal{V}$, and θ, $\phi \in$ Con$A$ and suppose that $a/\theta = a/\phi$ for some $a \in A$. It easily follows that $a/(\theta \wedge \phi) = a/(\theta \vee \phi)$. Thus $\theta \vee \phi$ considered as a congruence on $\mathbf{A}/\theta \wedge \phi$ has a singleton block. The condition tells us that $\theta \vee \phi$ is the zero congruence on $\mathbf{A}/\theta \wedge \phi$. Thus $\theta \vee \phi = \theta \wedge \phi$, so $\theta = \phi$, establishing regularity. ■

Do we need parentheses around these joins?

Before stating our Maltsev condition for regularity, we review congruence generation in a form suitable for the application here and later in this chapter. This is an easy reformulation of the Congruence Generation Theorem (4.13).

reg2 **LEMMA 10.22.** *For any algebra* **A** *and any* $Z \subseteq A^2$*, the congruence* $\mathrm{Cg}^{\mathbf{A}}(Z)$ *generated by* Z *consists of all pairs* $\langle c,d\rangle$ *such that for some* $\bar{e} \in A^k$*, some* $(k{+}2)$*-ary terms* $t_1,\ldots,t_m$*, and some pairs* $\langle a_i,b_i\rangle \in Z$*, we have*

$$\begin{aligned} c &= t_1^{\mathbf{A}}(a_1,b_1,\bar{e}) \\ t_1^{\mathbf{A}}(b_1,a_1,\bar{e}) &= t_2^{\mathbf{A}}(a_2,b_2,\bar{e}) \\ &\vdots \\ t_m^{\mathbf{A}}(b_m,a_m,\bar{e}) &= d. \end{aligned}$$

■

Before presenting a Maltsev condition for regularity, let us remark that well

This needs to be made clear. Also the reference to Vaught1961 before the theory of Maltsev conditions, R. L. Vaught 1961 made some general model-theoretic observations on regularity. His remarks, which were based on Beth's Theorem, held that, under regularity, there must exist some formulas of first order logic yielding one congruence block in terms of another one. Of course condition (iii) of the following Theorem yields such formulas very explicitly (interpreting $=$ as an arbitrary congruence).

Give credits here to Csakany and year Wille. Should Gratzer or Hashimoto get credit also?

THEOREM 10.23. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

i. *$\mathcal{V}$ is congruence regular;*
ii. *$\mathbf{F}_{\mathcal{V}}(3)$ is congruence regular;*
iii. *there are ternary terms $g_1,\ldots,g_n$ in the language of $\mathcal{V}$ such that $\mathcal{V}$ satisfies the implication*
$$\left[\bigwedge_{i=1}^{n} g_i(x,y,z) \approx z\right] \longleftrightarrow x \approx y.$$
iv. *$\mathcal{V}$ has ternary terms $g_1,\ldots,g_n$ and 5-ary terms $f_1,\ldots,f_n$ such that the following are identities of $\mathcal{V}$:*
$$\begin{aligned} g_i(x,x,z) &\approx z \qquad 1 \le i \le n \\ x &\approx f_1(x,y,z,z,g_1(x,y,z)) \\ f_1(x,y,z,g_1(x,y,z),z) &\approx f_2(x,y,z,z,g_2(x,y,z)) \\ f_2(x,y,z,g_2(x,y,z),z) &\approx f_3(x,y,z,z,g_3(x,y,z)) \\ &\vdots \\ f_n(x,y,z,g_n(x,y,z),z) &\approx y. \end{aligned}$$

REMARK 10.24. In §10.7 we will be able to derive a slightly simpler version of this Maltsev condition, one in which the f_i's do not depend on their fifth variables.

Proof. The implications (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (ii) are immediate. For (iii) $\Rightarrow$ (i), we will assume (iii) and derive the condition of Lemma 10.21. Given a singleton congruence block a/ϕ, we will use the terms g_i supplied by (iii) to prove that ϕ is $0_{\mathbf{A}}$. For any $\langle b,c\rangle \in \phi$ we have, for each i,
$$g_i(b,c,a) \;\phi\; g_i(b,b,a) = a$$
and so, by assumption
$$g_i(b,c,a) = a$$
for each i. Now, by (iii), $b = c$, and thus $\phi = 0_{\mathbf{A}}$

To complete the proof we will show that (ii) implies (iv). We will apply regularity to two congruences θ and ϕ on the free algebra $\mathbf{F}_{\mathcal{V}}(x,y,z)$, which we now define. First, θ is the kernel of the endomorphism extending the map
$$x,y \mapsto x \qquad z \mapsto z.$$

And then ϕ is the congruence generated by

$$Z = \{\langle z, u\rangle : u \in z/\theta\}.$$

It is obvious that $\phi \subseteq \theta$ and $z/\theta \subseteq z/\phi$; hence $z/\theta = z/\phi$. By regularity $\theta = \phi$ and hence in particular, $\langle x, y\rangle \in \phi$. By Lemma 10.22, we have

$$\begin{aligned} x &= t_1(z, g_1, e_1, \ldots, e_m) \\ t_1(g_1, z, e_1, \ldots, e_m) &= t_2(z, g_2, e_1, \ldots, e_m) \\ &\vdots \\ t_n(g_n, z, e_1, \ldots, e_m) &= y \end{aligned}$$

for some $e_1, \ldots, e_m, g_1, \ldots, g_n \in \mathbf{F}_{\mathcal{V}}(x, y, z)$ such that $\langle z, g_i\rangle \in \theta$ for each i. From the definition of θ it is clear that for each i, $\mathcal{V}$ identically satisfies $g_i(x, x, z) \approx z$. Now if we define

$$f_i(x, y, z, u, v) = t_i(u, v, e_1(x, y, z), \ldots, e_m(x, y, z)),$$

then we obtain the equations (iv) by the usual reasoning of this chapter. ■ more details?

Groups of course are congruence regular; in fact the equations (iv) obtain with $n = 1$, by defining

$$g_1(x, y, z) = x^{-1}yz \qquad f_1(x, y, z, u, v) = xuz^{-1}.$$

For some subtler examples of regularity, the reader is referred to Exercises 3, 4, 18, 21, 22, 22, 24, 26, and Example 10.28 below. check these numbers

A Maltsev condition is **linear** if it is defined by equations of the simple form $t(\text{variables}) \approx x$ or $t(\text{variables}) \approx s(\text{variales})$, where t and s are function symbols.. The Maltsev conditions for k-permutability, for congruence distributivity and modularity are all linear whereas the one for congruence regularity give in condition (iv) of Theorem 10.23 is not. The next theorem shows that congruence regularity cannot be defined by a linear Maltsev condition. This theorem uses a clever idea of Libor Barto. cite Hobby McKenzie

is there a coauthor; also maybe say why this is important

regnotlinear **THEOREM 10.25.** *Congruence regularity cannot be defined by a linear Maltsev condition.*

Proof. We begin with Barto's construction. Let $\mathbf{B}$ be an algebra and let A be a set. Let $f : B \twoheadrightarrow A$ and $g : A \rightarrowtail B$ be maps with $f(g(x)) = x$. So A is a set retraction of B. For each term $t(x_1, \ldots, x_n)$ of $\mathbf{B}$ we define an operation $t^{\mathbf{A}}$ by

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = f(t^{\mathbf{B}}(g(a_1), \ldots, g(a_n))).$$

$\mathbf{A}$ is the algebra with these operations. It is easy to check that if $\mathbf{B}$ satisfies a linear Maltsev condition then $\mathbf{A}$ also satisfies it. The reader can check, for example, that if $p(x, y, z)$ is a Maltsev term for $\mathbf{B}$ then $p^{\mathbf{A}}(x, y, z)$ is a Maltsev operation for $\mathbf{A}$.

One subtle point of this construction is that if, for example, $q(x,y) = t(s(x,y), r(x,y))$ it is not necessarily the case that $q^{\mathbf{A}}(x,y) = t^{\mathbf{A}}(s^{\mathbf{A}}(x,y), r^{\mathbf{A}}(x,y))$ which is why nonlinear Maltsev condtions of **B** may fail in **A**.

Now let $B = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$ with the 3-place operation $x+y+z$ modulo 2. As above the variety generated by **B** is congruence regular. Let $A = \{0,1,2\}$ and define maps $f(\langle x,y\rangle) = x+y$ and $g(0) = \langle 0,0\rangle$, $g(1) = \langle 1,0\rangle$ and $g(2) = \langle 1,1\rangle$.

The terms for **B** have the form $t(x_1,\dots,x_n) = y_1 + \cdots + y_m$ modulo 2, where the y_j's are a subset of the x_i's and m is odd (so that t is idempotent).

We claim that the partition θ with blocks $[0,2]$ and $[1]$ is a congruence of **A**. By symmetry it is enough to show $t^{\mathbf{A}}(0,a_2,\dots,a_n)\ \theta\ t^{\mathbf{A}}(2,a_2,\dots,a_n)$. We may assume $t^{\mathbf{A}}$ depends on its first variable. If $t^{\mathbf{B}}(\langle 0,0\rangle, b_2,\dots,b_n) = \langle u,v\rangle$ then $t^{\mathbf{B}}(\langle 1,1\rangle, b_2,\dots,b_n) = \langle u+1, v+1\rangle$ modulo 2. Since $f(\langle 0,1\rangle) = f(\langle 1,0\rangle)$ it follows that θ is a congruence on **A**. We leave the details for the reader.

Thus **A** does not lie in a congruence regular variety which shows that congruence regularity cannot be defined by a linear Maltsev condition. ■

Notice that although condition (iii) of the last theorem is simpler than (iv), condition (iii) involves implications while (iv) is expressed in terms of equations (and thus (iv) is a Maltsev condition). The next theorem shows that it is always possible to convert a varietal condition expressed with implications into a Maltsev condition. It is an immediate consequence of Lemma 10.22.

Horn2eq

THEOREM 10.26. *Let $\mathcal{V}$ be a variety and let s_i and t_i, $i = 0,\dots,n$, be k-ary terms. The the following are equivalent:*

i. *$\mathcal{V}$ satisfies the implication*

$$\left[\bigwedge_{i=1}^{n} s_i(\bar{x}) \approx t_i(\bar{x})\right] \to s_0(\bar{x}) \approx t_0(\bar{x})$$

ii. *For some m, $\mathcal{V}$ has $(k+2)$-ary terms $f_1,\dots,f_m$ such that*

$$\begin{aligned} s_0(\bar{x}) &\approx f_1(\bar{x}, s_{i_1}(\bar{x}), t_{i_1}(\bar{x})) \\ f_1(\bar{x}, t_{i_1}(\bar{x}), s_{i_1}(\bar{x})) &\approx f_2(\bar{x}, s_{i_2}(\bar{x}), t_{i_2}(\bar{x})) \\ &\vdots \\ f_m(\bar{x}, t_{i_m}(\bar{x}), s_{i_m}(\bar{x})) &\approx t_0(\bar{x}) \end{aligned}$$

■

We now turn our attention to the closely related topic of congruence uniformity. As we mentioned in the introduction to this chapter, a variety $\mathcal{V}$ is **congruence uniform** if and only if each congruence θ on every algebra $\mathbf{A} \in \mathcal{V}$ is a **uniform equivalence relation** in the sense that all blocks of θ have the same cardinality. Now it is readily apparent from Lemma 10.21 that every congruence uniform variety is congruence regular. As we will see in the exercises, uniformity itself is not Maltsev definable, and in fact remains rather mysterious (even though it figured importantly in Chapter 9, where we saw it deduced from narrowness). Thus it turns out to be more fruitful, in the present context, to study a slightly weaker, but Maltsev definable, property which is nevertheless still strong enough to imply regularity. check this reference

We define a variety $\mathcal{V}$ to be **weakly congruence uniform** if and only if there exists a cardinal function f, mapping the class of cardinals into itself, such that $|b/\theta| \leq f(|a/\theta|)$ whenever $\mathbf{A} \in \mathcal{V}$, a, $b \in A$, and $\theta \in \operatorname{Con} A$. Uniformity, of course, corresponds to the special case $f(\kappa) = \kappa$. Let us define the **iterated exponential** $\beth_n(\kappa)$, $n \in \omega$, by

$$\beth_0(\kappa) = \kappa \qquad \beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}.$$

We call a variety **congruence n-uniform** if it satisfies the definition of congruence weakly congruence uniform with $f(\kappa) = \beth_n(\kappa + \aleph_0)$. We shall show that a variety is weakly congruence uniform if and only if it is congruence n-uniform for some n. Exercise 10.34.20 shows that if one block of a conguence of an algebra in a weakly congruence uniform variety is finite, then all blocks are finite. Hence the strongest of these conditions, congruence 0-uniformity, guarantees that all the blocks of a congruence with an infinite block will have the same size. Notice that even this is weaker than conguence uniformity. The next theorem will show that these concepts are stronger than congruence regularity.

reg4 **THEOREM 10.27.** *Every weakly congruence uniform variety is congruence regular.*

Proof. Suppose that $\mathcal{V}$ is weakly congruence uniform with f as in the definition. By way of contradiction, we will assume that $\mathcal{V}$ is not congruence regular. Thus by Lemma 10.21 above, $\mathcal{V}$ has an algebra $\mathbf{A}$ which has a congruence ϕ with both a one-element block and a block with more than one element: $|a/\phi| = 1$ and $|b/\phi| = \mu \geq 2$. Let ν be a cardinal satisfying $2^\nu > f(|A|)$. Let $\mathbf{C} = \mathbf{A}^\nu$ and let $\overline{\phi}$ be the congruence on $\mathbf{C}$ that identifies $\overline{x}$ and $\overline{y}$ if and only if $x_i \ \phi \ y_i$ for $1 \leq i < \nu$ (no restriction on the 0 coordinate). If we let $\overline{a}$ and $\overline{b}$ be the ν-tuples all of whose coordinates are a and b, respectively, then, since $\overline{a}$ is congruent modulo ϕ only to ν-tuples which differ in at most the zeroth coordinate, $|\overline{a}/\overline{\phi}| = |A|$ and

$$|\overline{b}/\overline{\phi}| \geq \mu^\nu \geq 2^\nu > f(|A|) = f(|\overline{a}/\overline{\phi}|),$$

in contradiction to the definition of weak uniformity. ■

reg5 **EXAMPLE 10.28.** This example shows that there is a congruence regular variety which is not weakly congruence uniform.

Let $\mathcal{V}$ be the variety with operation symbols h, g_1, and g_2 defined by the equations

$$g_1(x,x,z) \approx g_2(x,x,z) \approx z$$
$$h(g_1(x,y,z),x,y,z) \approx x$$
$$h(g_2(x,y,z),x,y,z) \approx y.$$

It is obvious that $\mathcal{V}$ satisfies (iii) of Theorem 10.23, and hence is congruence regular. For the failure of weak uniformity, we let μ be any ordinal ≥ 2, and define ternary operations g_1 and g_2 on $\mu \times \{0,1\}$ as follows:

$$g_1(\langle a,i\rangle,\langle b,j\rangle,\langle c,k\rangle) = \begin{cases} \langle c,k\rangle & \text{if } \langle a,i\rangle = \langle b,j\rangle \\ \langle 0, i+j+k\rangle & \text{otherwise} \end{cases}$$

$$g_2(\langle a,i\rangle,\langle b,j\rangle,\langle c,k\rangle) = \begin{cases} \langle c,k\rangle & \text{if } \langle a,i\rangle = \langle b,j\rangle \\ \langle 1,k\rangle & \text{otherwise.} \end{cases}$$

Add a phantom to the above display

The addition here is modulo 2, of course. Notice that every set B with $2 \times 2 \subseteq B \subseteq \mu \times 2$ is closed under g_1 and g_2. Given such a B, we consider the following four conditions on a quaternary operation h,

$$h(\langle 0,i+j+k\rangle,\langle a,i\rangle,\langle b,j\rangle,\langle c,k\rangle) = \langle a,i\rangle$$
$$h(\langle 1,k\rangle,\langle a,i\rangle,\langle b,j\rangle,\langle c,k\rangle) = \langle b,j\rangle$$
$$h(\langle c,k\rangle,\langle a,i\rangle,\langle a,i\rangle,\langle c,k\rangle) = \langle a,i\rangle$$
$$h(\langle *,s\rangle,\langle *,i\rangle,\langle *,j\rangle,\langle *,k\rangle) = \langle *,s+j+k\rangle$$

where the $*$'s represent arbitrary elements.

The first three conditions specify h on some special elements of B^4. It is easy to check that where these specifications overlap with one another they agree, and so we easily obtain a partial operation obeying the first three conditions. But the fourth condition holds for this partial operation (as one easily checks), and so it is evidently possible to find a total operation h satisfying all four conditions.

It is now trivial to check that the algebra $\mathbf{B} = \langle B,h,g_1,g_2\rangle$ satisfies the equations defining $\mathcal{V}$. Moreover it is apparent from the fourth condition that second coordinate projection defines a congruence θ on $\mathbf{B}$. Since the image of the second projection has only two elements, this congruence has exactly two blocks,

$$B_i = B \cap (\mu \times \{i\}) \qquad \text{for } i = 0,1.$$

By construction, the cardinalities of these blocks are completely arbitrary except for the obvious constraint that $2 \leq |B_i| \leq \mu$. Since μ itself is arbitrary, we have an obvious failure of weak uniformity.

We remark that any congruence regular variety which is not weakly congruence uniform must be residually large see Exercise 10.34.17.

The few examples of uniformity which we know (see, for example, Exercises 10.34.21, 24 and 26) seem to have a general flavor reminiscent of group theory, especially the cancellation available in group theory. But weak uniformity, in its most general form, has a surprisingly different flavor, namely that of infinitary combinatorics. In order to develop a Maltsev condition for weak uniformity, we need a couple of lemmas from Ramsey theory. This theory began with the famous theorem of F. P. Ramsey 1930, and was later developed by P. Erdős, A. Hajnal and R. Rado.

Let $X^{(n)}$ denote the set of all n-element subsets of X. For finite n and cardinals κ, λ, μ, the symbolic expression

$$\kappa \to (\lambda)^n_\mu$$

abbreviates the following assertion: if $\kappa^{(n)} = \bigcup \mathcal{M}$ with $|\mathcal{M}| \leq \mu$, then there exists $D \in \mathcal{M}$ and $Y \subseteq \kappa$ with $|Y| = \lambda$, such that $Y^{(n)} \subseteq D$. The famous theorem of F. P. Ramsey (which appeared as a lemma to a theorem much less remembered) was that for all finite l, m, n, there exits a finite k such that

$$k \to (l)^n_m.$$

In fact, $m^{ml^{\cdot^{\cdot^{\cdot^{ml}}}}}$ (ml appearing n times) is large enough (see, e.g., pages 7–9 of Graham, Rothschild, and Spencer 1980). is this right

reg6 **THEOREM 10.29.** *(P. Erdős and R. Rado 1956)*

$$(\beth_n(\mu))^+ \to (\mu^+)^{n+1}_\mu$$

for every $n \in \omega$ and every infinite μ. ■

reg7 **THEOREM 10.30.** *(P. Erdős, A. Hajnal and R. Rado 1965) The assertion*

$$\beth_n(\mu) \to (n+2)^{n+1}_\mu$$

fails for all $n \in \omega$ and all μ. ■

The next theorem will use Theorem 10.29 to show that the class of weakly congruence uniform varieties is defined by a Maltsev condition. Rather than explicitly presenting this Maltsev condition, which would be rather messy, we will show that a variety $\mathcal{V}$ is weakly congruence uniform if and only if it satisfies certain implications. Since, by Theorem 10.26 these implications are equivalent to a Maltsev condition, we will be content to present the more perspicuous implications.

For $m \geq n \geq 0$ and $0 \leq i < j \leq m$, terms $F_1, \ldots, F_p$ of rank $n+2$ will be called (a set of weak) **uniformity terms for $\mathcal{V}$ of type $\langle n,m,p,i,j \rangle$** if $\mathcal{V}$ obeys the equations

$$F_t(z,x,\ldots,x) \approx z$$

and the (universal closure of the) implication

$$\left[\bigwedge_{t=1}^{p} \bigwedge F_t(z,\bar{x}) \approx F_t(z,\bar{y})\right] \rightarrow x_i \approx x_j$$

where the second conjunction is over all of the (n+1)-sequences $\bar{x}$ and $\bar{y}$ from $\{x_0,\ldots,x_m\}$ with strictly increasing subscripts. For instance, if $n = 0$ or $n = m$, then the only varieties having uniformity terms of type $\langle n,m,\ldots\rangle$ are the trivial ones (those satisfying $x_0 \approx x_1$). For $n = 1$, $m = 2$, $p = 1$, $i = 0$ and $j = 2$, our implication is simply

$$[F_1(z,x_0,x_1) \approx F_1(z,x_0,x_2) \approx F_1(z,x_1,x_2)] \rightarrow x_0 \approx x_2.$$

reg8 **THEOREM 10.31.** *(W. Taylor 1974) The following statements are equivalent for a variety* $\mathcal{V}$.

i. $\mathcal{V}$ *is weakly congruence uniform.*

ii. *For some* $0 \le n \le m$, $0 \le i < j \le m$, *and* p, $\mathcal{V}$ *has uniformity terms of type* $\langle n,m,p,i,j\rangle$.

iii. *For some* n, $\mathcal{V}$ *has* n*-uniform congruences.*

Moreover, if (ii) holds for some n, *then (iii) holds with that same* n.

Proof. We prove the last assertion first. Thus we are given uniformity terms $F_1,\ldots,F_p$. Let $\mathbf{A} \in \mathcal{V}$ and let $\theta \in \mathrm{Con}A$. By way of contradiction, let us suppose that $|a/\theta| \le \mu$, where $\mu \ge \aleph_0$, and that $|b/\theta| > \beth_n(\mu)$. Let $<$ be a total order on b/θ. We divide $(b/\theta)^{(n+1)}$ into $\mu = \mu^p$ classes

$$(b/\theta)^{(n+1)} = \bigcup\{D_\alpha : \alpha \in (a/\theta)^p\} \qquad (10.5.1)$$ for:regEq1

where

$$D_\alpha = \big\{\{b_0,\ldots,b_n\} : b_0 < \cdots < b_n,\ F_t(a,b_0,\ldots,b_n) = \alpha_t,\ \text{for } 1 \le t \le p\big\}.$$

(And thus (10.5.1) follows easily from the fact that $F_t(z,x,\ldots,x) = z$ for each t.) Now Lemma 10.29 yields the existence of an infinite set, and hence a set Y with $|Y| = m+1$, such that $Y^{(n+1)} \subseteq D_\alpha$ for a fixed α. Taking $Y = \{b_0,\ldots,b_m\}$ with $b_0 < \cdots < b_m$, we see by definition of D_α that, for each t and for each $\{b_{i_0},\ldots,b_{i_n}\} \subseteq \{b_0,\ldots,b_m\}$, with $b_{i_0} < \cdots < b_{i_n}$, all values $F_t(a,b_{i_0},\ldots,b_{i_n})$ are equal. Therefore by the defining implications for uniformity terms, $b_i = b_j$, in contradiction to the previously established relation that $b_i < b_j$. This contradiction establishes the desired conclusion that $|b/\theta| \le \beth_n(\mu)$.

Clearly (iii) implies (i). To complete the proof of the theorem, we assume that $\mathcal{V}$ is weakly congruence uniform and show that $\mathcal{V}$ has uniformity terms. Let f be the function defined on the cardinals which witnesses the weak uniformity of $\mathcal{V}$. Clearly we may assume that f is monotone. Let μ be an infinte cardinal at least as large as the cardinality of the set of operation symbols for $\mathcal{V}$ and let $\nu = f(\mu)$.

Let $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x_i : i < \nu^+)$ be the free algebra on ν^+ generators. We then define θ to be the kernel of the endomorphism of $\mathbf{F}$ which is specified as follows:

$$x_0 \mapsto x_0 \quad \text{and} \quad x_i \mapsto x_1 \quad \text{if } i > 0.$$

We further define $Z \subseteq F^2$ to be the set of all pairs

$$\langle g^{\mathbf{F}}(x_0, x_{i_1}, \ldots, x_{i_s}), g^{\mathbf{F}}(x_0, x_{j_1}, \ldots, x_{j_s})\rangle$$

where g is any $\mathcal{V}$-term such that $\mathcal{V}$ identically satisfies $g(x_0, x_1, \ldots, x_1) \approx x_0$, and where the subscripts are as follows: $0 < i_1 < \cdots < i_s$ and $0 < j_1 < \cdots < j_s$. Defining ψ to be the congruence on $\mathbf{F}$ generated by Z, it is obvious that $Z \subseteq \theta$ and hence that $\psi \subseteq \theta$. Therefore θ may be regarded as a congruence on $\mathbf{F}/\psi$.

Let a and b denote the images of x_0 and x_1 in $\mathbf{F}/\psi$. Clearly $|a/\theta|$ is the number of ψ-blocks contained in the θ-block x_0/θ. Every member of x_0/θ is clearly given by a term $g(x_0, x_{i_1}, \cdots)$ with $0 < i_1 < \cdots$ and with $g(x_0, x_1, \ldots, x_1) \approx x_0$ holding in $\mathcal{V}$. Regardless of the sequence $i_1, i_2, \ldots$, formally identical g's yield the same ψ-class; therefore the number of ψ-blocks in x_0/θ is limited by the cardinality of the language of $\mathcal{V}$. Hence $|a/\theta| \leq \mu$, and thus $|b/\theta| \leq \nu$.

Now the θ-class of x_1 in $\mathbf{F}$ obviously contains the ν^+ distinct elements x_i, for $0 < i < \nu^+$ (since we may assume $\mathcal{V}$ nontrivial). As we just saw, the image of this class in $\mathbf{F}/\psi$ has at most ν elements, and so we obviously have $x_i \;\psi\; x_j$ for some i, j with $0 < i < j < \nu^+$. Thus $\langle x_i, x_j\rangle$ is in the congruence generated by a finite subset of pairs from Z. One may add fictitious variables so that the finitely many g appearing in these pairs all have the same arity, which we may take to be $n+1$ (thus defining n). Taking $F_t(z, x_1, \ldots, x_n)$ to be $g(z, x_1, \ldots, x_n)$ (with one F_t for each of the formally distinct g's appearing), and considering, without loss of generality, each of the finitely many subscripts appearing to be a *finite* ordinal, it is routine to verify that the desired implications hold in $\mathcal{V}$. ■

In the case of a countable set of operations, the implication (i) implies (iii) follows from a general model-theoretic result of R. L. Vaught 1965. (See also Theorem 7.2.6 of Chang and Keisler 1990.) Exercises 22 and 22 will show that (ii) of Theorem 10.31 does not imply (iii) for a smaller n.

I don't like the change to z here. Walter is this correct?, I changed to the 3rd edition so the page number was wrong (so I eliminated it) but the number 7.2.6 seems right. I couldn't find this Geiger paper.

We round out this section with a look at D. Geiger's notion of coherence. It is a Maltsev definable property which implies 0-uniformity, the strongest of our weak uniformity conditions. An algebra $\mathbf{A}$ will be called **coherent** if the following condition holds for all $\phi \in \mathrm{Con} A$ and all subalgebras $\mathbf{B}$ of $\mathbf{A}$: if B contains a ϕ-block, then B is a union of ϕ-blocks. Naturally, a variety $\mathcal{V}$ is called coherent if each $\mathbf{A} \in \mathcal{V}$ is coherent.

reg9 **THEOREM 10.32.** *(D. Geiger* **Geiger1974***) A variety $\mathcal{V}$ is coherent if and only if there exist terms H, $F_1, \ldots, F_n$ such that the following are identities of $\mathcal{V}$:*

$$F_i(x, x, z) \approx z \quad \text{for } 1 \leq i \leq n$$
$$H(y, F_1(x, y, z), \ldots, F_n(x, y, z)) \approx x.$$

Proof. See Exercise 27. ■

reg10

THEOREM 10.33. *Every coherent variety is weakly congruence uniform and permutable; in fact, it is congruence* 0*-uniform.*

Proof. Let $\mathcal{V}$ be a coherent variety. To prove that it is congruence 0-uniform, we need to show that if a, $b \in \mathbf{A} \in \mathcal{V}$ and $\theta \in \operatorname{Con} A$ with a/θ infinite, then $|b/\theta| \leq |a/\theta|$. This follows from the fact that if $F_1, \ldots, F_n$ are the terms of Theorem 10.32 then

$$x \mapsto \langle F_1(x,b,a), \ldots, F_n(x,b,a) \rangle$$

is a bijection from b/θ into $(a/\theta)^n$, which is easily verified.

To see that $\mathcal{V}$ is permutable, let

$$p(x,y,z) = H(z, F_1(x,y,z), \ldots, F_n(x,y,z)).$$

It is easy to verify that $p(x,y,z)$ is a Maltsev term. ■

Exercise 28 gives a version of the last theorem for individual algebras.

ex10.5

Exercises 10.34

1. The variety of distributive lattices is not congruence regular.

remember that this may end up in ch 8

2. The variety of implication algebras (see Exercise 10.4.5) is not congruence regular. (It helps to remember that every family of sets closed under set difference forms an implication algebra by defining $x \to y$ to mean $y - x$.)

exer:regEx3

3. If $\mathcal{V}$ has terms F and H obeying the equations

$$F(x,x,z) \approx z$$
$$H(F(x,y,z),y,z) \approx x,$$

then $\mathcal{V}$ is both coherent and congruence uniform. Thus, for example, the variety of quasigroups (defined in §3.4) is both coherent and congruence uniform. Hint: Look at the proof of Theorem 10.33.

exer:regEx4

4. (Varlet 1972). The variety of three-valued Lukasiewicz algebras is congruence regular. This is the variety generated by $\langle 3, \wedge, \vee, ^+, ^*, 0 \rangle$ where $\wedge$ and $\vee$ are the are ordinary infimum and supremum on $3 = \{0,1,2\}$ ordered as

usual, and $^+$ and * are defined by the following table

	$*$	$+$
0	2	2
1	0	2
2	0	0

5. (Grätzer 1970). **A** is congruence regular if and only if for all a, b, $c \in A$ there exists a finite sequence $\langle d_0, \ldots, d_m \rangle$ of elements of A such that

$$\operatorname{Cg}(a,b) = \bigvee_{i=0}^{m} \operatorname{Cg}(c,d_i).$$

this is the one Paolo says was proved by Hashimoto in 1962

6. Give a direct proof that (iii) implies (iv) in Theorem 10.23.

7. For the $\mathcal{V}$ given in Example 10.28 find 5-ary terms f_1, f_2, ... satisfying condition (iv) of Theorem 10.23.

8. (Grätzer 1970 [Hagemann, unpublished]; see also Fichtner 1968.) A variety $\mathcal{V}$ with constant terms $c_1, \ldots, c_n$ is said to be **weakly congruence regular** with respect to $c_1, \ldots, c_n$ if and only if each congruence θ on arbitrary $\mathbf{A} \in \mathcal{V}$ is determined by its n blocks c_1/θ, ..., c_n/θ. That is, if θ, $\psi \in \operatorname{Con} A$ and $c_i/\theta = c_i/\psi$, for $i = 1, \ldots, n$, then $\theta = \psi$. Prove that $\mathcal{V}$ is weakly congruence regular with respect to $c_1, \ldots, c_n$ if and only if there exist binary terms $s_1, t_1, s_2, t_2, \ldots, s_m, t_m$ and ternary terms $r_1, \ldots, r_m$ such that the following equations hold identically in $\mathcal{V}$:

Fichtner has 2 1968 articles both with almost the same name. We need to check if we have the right one although it looks like either will do.

$$\begin{aligned} s_i(x,x) \approx t_i(x,x) &\approx c_j \qquad \text{for } i \equiv j \pmod{m} \\ r_1(s_1(x,y),x,y) &\approx x \\ r_i(t_i(x,y),x,y) &\approx r_{i+1}(s_{i+1}(x,y),x,y) \\ r_m(t_m(x,y),x,y) &\approx y. \end{aligned}$$

9. ([Hagemann, unpublished]). Define a variety $\mathcal{V}$ to be **weakly congruence regular with respect to unary operations** g_1, ..., g_n if and only if for each $\mathbf{A} \in \mathcal{V}$ and each $\theta \in \operatorname{Con} A$, the block x/θ is determined by the n blocks $g_1(x)/\theta$, ..., $g_n(x)/\theta$. Give a necessary and sufficient condition for this form of regularity, in the style of the condition of the previous exercise.

10. ([Hagemann, unpublished]). The variety of k-Boolean algebras (defined in Exercise 10.4.8) is weakly congruence regular with respect to the constant operations c_1, ..., c_{k-1}.

11. ([Hagemann, unpublished]). The variety of k-Boolean algebras is not weakly congruence regular with respect to fewer than $k-1$ constants.

In the Exercises 10.4.17–22, we presented some exercises indicating how some features of the theory of topological groups could be replicated for topological algebras in permutable varieties. Here we continue this development for topological algebras in congruence regular varieties. These results are due to W. Taylor 1974 and 1977. Our assumptions on the algebra **A** and the topology are exactly as stated before Exercise 10.4.17, except that we are not assuming permutability.

12. If **A** lies in a congruence regular variety and $\langle A, \mathcal{T} \rangle$ has an isolated point, then $\langle A, \mathcal{T} \rangle$ is discrete.

13. If **A** lies in a congruence regular variety and $\langle A, \mathcal{T} \rangle$ is an infinite compact metric space, then $|A| = 2^{\aleph_0}$. Hint: Use the Baire category theorem. It is known that every uncountable compact metric space has power at least $2^{\aleph_0}$.

14. If **B** is a topological algebra in a congruence regular variety and **A** is a topological subalgebra of $\mathbf{B}^I$ such that **A** has an isolated point, then there exists a finite set $J \subseteq I$ such that the projection of $\mathbf{B}^I$ onto $\mathbf{B}^J$ is one-to-one on A.

In the next volume, we will see that the following exercise on congruence regular topological algebras will extend to equationally compact algebras.

***15.** If **A** lies in a congruence regular variety and **A** is embeddable in some algebra which carries a compact Hausdorff topology, then $|B_2| \leq 2^{B_1}$ for B_1 and B_2 any two infinite blocks of any congruence $\theta \in \mathrm{Con} A$. (See p. 351 of Taylor 1974 for a solution.)

16. If **A** is a congruence regular algebra, $\theta \in \mathrm{Con} A$ and a, $b \in A$, then any set of congruences on **A** which separates points of a/θ will also separate points of b/θ.

er:residsmall **17.** Let $\mathcal{V}$ be a residually small, congruence regular variety in which each subdirectly irreducible algebra has power $\leq \mu$. If $\mathbf{A} \in \mathcal{V}$ and a, $b \in A$, then $|b/\theta| \leq \mu^{|a/\theta|}$ for a/θ infinte. Hint: use the previous exercise together with the method of proof for the Subdirect Representation Theorem (4.44).

exer:regEx18 **18.** Prove that the variety $\mathcal{V}$ presented in Example 10.28 has the following property which is stronger than what we required there. For any family of cardinals $\mu_i \geq 2$, for $i \in I$, there exists an algebra $\mathbf{B} \in \mathcal{V}$ and a congruence θ on $\mathbf{B}$ which partitions B into blocks B_i with $|B_i| = \mu_i$ for all i. Prove moreover that the above property still holds if we enrich $\mathcal{V}$ by adding binary operations $\vee$ and $\wedge$ and add the axioms of lattice theory. Thus regularity and distributivity together do not imply weak uniformity.

19. Give a syntactic proof of Theorem 10.27 That is, deduce (iii) of Theorem 10.23 from the existence of uniformity terms.

exer:wufinite

20. If $\mathcal{V}$ is weakly congruence uniform, then there exists a function $f : \omega \to \omega$ such that $|b/\theta| \leq f(|a/\theta|)$ whenever $\mathbf{A} \in \mathcal{V}$, a, $b \in A$, $\theta \in \mathrm{Con} A$, and a/θ is finite. (But this condition does not guarantee uniformity.) (A first proof arises more or less directly from the definition, by using the compactness theorem. A more sophisticated argument will yield an explicit formula for f (depending on the n, m and p appearing in uniformity terms for $\mathcal{V}$). Here one will need the explicit form of Ramsey's Theorem mentioned prior to Theorem 10.29.)

weak univormity?

exer:uniform1

21. Prove that every discriminator variety is coherent. (Hint: there exist very simple Geiger terms in the ternary discriminator, with $n = 2$.) Then prove that every discriminator variety is congruence uniform. (In the case of a congruence having a finite block, a special argument will be required, utilizing our knowledge of the structure of congruences in a discriminator variety—see ?? In the case of a finitely generated discriminator variety, uniformity follows from narrowness of the spectrum, according to a result in Chapter ??–see Theorem ??.) Also use the ternary discriminator to obtain regularity terms for such a variety.

put some references here. fill this in

exer:regEx22

22. Let $\mathcal{V}$ be defined by the equations:

$$F(x,x,z) \approx z$$
$$G(w,w,w,x_0,x_1) \approx x_0$$
$$G(F(x_0,x_1,z),F(x_0,x_2,z),F(x_1,x_2,z),x_0,x_1) \approx x_1$$

Clearly $\mathcal{V}$ has a uniformity term of type $\langle 2,3,1,0,1\rangle$, and hence $|b/\theta| \leq 2^{|a/\theta|}$ whenever $\mathbf{A} \in \mathcal{V}$, $\theta \in \mathrm{Con} A$, and a,$b \in A$. The task here is to show that the above estimate is best possible for this variety, i.e., for all infinite μ, there exist $\mathbf{A}$, θ, a,and b as above such that $|a/\theta| = \mu$ and $|b/\theta| = 2^{\mu}$. Hint: Let A be the subset of $\{0,1\}^{\mu}$ consisting of sequences α such that either α has finite support or $\alpha_0 = 0$. Now define

$$F(\alpha,\beta,\gamma) = \begin{cases} \text{the majority value} & \text{if } |\{\alpha,\beta,\gamma\}| < 3\text{; otherwise:} \\ \alpha+\beta+\gamma \pmod 2 & \text{unless exactly two of } \alpha_0,\ \beta_0, \text{ and } \gamma_0 \\ & \text{are 0, in which case:} \\ \{0,m\}+\gamma & \text{where } m \text{ is the first place the two that} \\ & \text{agree at 0 differ.} \end{cases}$$

(Here, by $\{0,m\}$ we mean the characteristic function, i.e. δ such that $\delta(0) = \delta(m) = 1$ and $\delta(x) = 0$ otherwise.) One may check that the kernel of first coordinate projection is a congruence θ for this operation, and that a corresponding G may be found so that θ remains a congruence.

exer:regEx23 ***23.** For $2 \leq n < \omega$, there exists a variety $\mathcal{V}_n$ which has a uniformity term of type $\langle n, n+1, 1, 0, 1\rangle$, and such that for each infinite μ, $\mathcal{V}_n$ has an algebra **A** with a congruence θ which has block of power μ and $\beth_n(\mu)$. Hint: The equations involved form an obvious generalization of those of the previous exercise. The construction of **A** uses the Erdős-Hajnal-Rado result (10.30); it seems to be necessarily more complicated than it was for $n = 2$ in the previous exercise. For a detailed solution, see pages 346–348 of Taylor 1974.

24. Weak uniformity is not definable by a strong Maltsev condition. (Use the previous exercise.)

exer:uniform2 ***25.** Let $\mathcal{V}$ be defined by the following infinite collection of equations:

$$\begin{aligned} G_1(x,x,z) \approx G_2(x,x,z) &\approx z & & \\ K(G_1(x,y,z), G_2(x,y,z), y, z) &\approx x & & \\ F_i(x,x,z) &\approx z & & \text{for } i \in \omega \\ H_i(F_i(x,y,z), x, y, z) &\approx x & & \text{for } i \in \omega \\ H_i(F_j(x,y,z), x, y, z) &\approx y & & \text{for } i \neq j \end{aligned}$$

Prove that $\mathcal{V}$ is coherent and congruence uniform, but that every finite subset of these equations defines a variety which is not congruence uniform. (For a proof of the final assertion, see pages 353–355 of Taylor 1974.) Thus, although coherence implies 0-uniformity (any congruence with an infinite block must be uniform), it does not imply uniformity. Moreover, uniformity is not equivalent to a Maltsev condition. Hint: we readily satisfy the Maltsev condition for coherence by taking F_1 (of the coherence condition) to be G_1, F_2 to be G_2 and F_3 to be z.

exer:uniform3 Should this hint be given?

26. The variety axiomatized by the following equations is congruence uniform and congruence permutable but not coherent:

$$F(x,x,z) \approx z$$
$$H(u,u,x,y,w,z) \approx x$$
$$H(F(x,w,z),F(y,w,z),x,y,w,z) \approx y.$$

Hint: show that $H(F(z,z,y),F(x,y,y),z,x,z,y)$ is a Maltsev term. (See page 356 of Taylor 1974 for the proof that this variety is uniform and not coherent.)

er:proofofThm

27. Prove Theorem 10.32.

exer:coherent

28. Prove that if $\mathbf{A}$ has a congruence with blocks of power μ and ν, where μ is an infinite cardinal at least as large as the cardinality of the basic operations of $\mathbf{A}$ and $\nu > \mu$, then $\mathbf{A}$ is not coherent.

29. Prove that the variety of Heyting algebras (define in Exercise 18 of §4.5) is not congruence regular. Hint: Show that there is a three element Heyting algebra which has a two-element homomorphic image.

Possible add some exercises from Davey's paper in the Huhn volume.

10.6 Congruence Identities

In this section we give a brief overview of the subject of lattice equations satisfied by the congruence lattices of all the algebras in a variety. A more thorough check this treatment will be contained in a later volume. As mentioned in §10.4, a lattice equation valid in all the congruence lattices of every member of a variety $\mathcal{V}$ is called a **congruence identity for** $\mathcal{V}$. Modularity and distributivity are the two most important examples. We have seen, by the results of B. Jónsson and A. Day, that both of these equations have a Maltsev condition associated with them. After these results were proved, there was hope that, for any lattice equation ε, the class of varieties whose congruence lattices satisfied ε would be Maltsev definable. This question is still open. The only general result result we have is the result of Pixley and Wille that such a class is weakly Maltsev definable. However, it is still possible that every lattice equation has a Maltsev condition associated with it; that is, we know of no lattice equation which is only weakly Maltsev definable.

In his thesis, J. B. Nation (see Nation 1974) showed that there are lattice equations ε, weaker than modularity, i.e., satisfied by some nonmodular lattice, which nevertheless have the property that any variety for which ε is a congruence identity is congruence modular. Let $s \approx t$ be a lattice equation and let Σ be a set of lattice equations. Then the notation

$$\Sigma \models_{\text{con}} s \approx t$$

indicates that if $\mathcal{V}$ is any variety of algebras, such that all of the congruence lattices of all the members of $\mathcal{V}$ satisfy the equations in Σ, then these congruence lattices will also satisfy $s \approx t$. In this case we say that $s \approx t$ is a **congruence consequence** of Σ. Thus, in this terminology, Nation's result was that there exists a lattice equation ε such that modularity is a congruence consequence of ε, but not a consequence of ε. Of course the statement ε_2 is a congruence consequence of ε_1 will mean $\{\varepsilon_1\} \models_{\text{con}} \varepsilon_2$. We call two sets Σ_0 and Σ_1 of lattice equations **congruence equivalent** if each element of Σ_0 is a congruence consequence of Σ_1 and vice versa. At the time of Nation's thesis only four congruence inequivalent lattice equations were known: $x \approx x$, $x \approx y$, the distributive equation, and the modular equation. However, using modular lattice theory, it is possible to construct a lattice equation for each prime number p which holds in the congruence lattice of a vector space over a field $\mathbf{F}$ if and only if the characteristic of $\mathbf{F}$ is p; see Exercise 10.45.1. Thus all of these equations are congruence inequivalent.

E. Gedeonová 1972 and P. Mederly 1975 were able to show that for certain lattice equations ε the class of varieties having ε as a congruence identity is Maltsev definable. Moreover, there are nonmodular lattices which satisfy ε, but not every lattice satisfies ε. However Day, using the techniques of Nation, was able to show that each such ε is congruence equivalent to modularity, and thus his Maltsev condition for modularity would also define the class of varieties having ε as a congruence identity. The question of the existence of a lattice equation ε, not congruence equivalent to distributivity or modularity, such that the class of varieties whose congruence lattice satisfy ε is Maltsev definable was settled by

Freese in Freese and McKenzie 1987. Using the commutator theory it was shown that there are infinitely many congruence inequivalent lattice equations ε such that the class of varieties satisfying ε is Maltsev definable.

Another surprising result in this area of congruence identities is the fact that if a variety is congruence modular it's congruence lattices must also satisfy certain stronger equations as well. In this section we will introduce the Arguesian equation and prove the result of Freese and Jónsson that if a variety is congruence modular then it satisfies the Arguesian equation, see Theorem 10.43 below. maybe leave out the ref to freese and jonsson

We begin with a proof of a version of Nation's result. His original proof took the weak Maltsev condition associated with a certain lattice equation and derived the Maltsev condition for modularity, much in the same spirit as several of the proofs of this chapter. The proof we give is local (as described in §10.1). Starting with an algebra $\mathbf{A}$ with a nonmodular congruence lattice we use certain algebraic constructions to build an algebra $\mathbf{B} \in \boldsymbol{V}(\mathbf{A})$ from $\mathbf{A}$. Now **CON A** contains $\mathbf{N}_5$ and we will show that **CON B** contains a larger subdirectly irreducible, nonmodular lattice. Nation's result can easily be derived from this.

10.1 **THEOREM 10.35.** *Let* $\mathbf{A}$ *be an algebra with a nonmodular congruence lattice. Then there is an algebra* $\mathbf{B} \in \boldsymbol{S}(\mathbf{A} \times \mathbf{A})$ *such that* **CON B** *has a sublattice isomorphic to the lattice diagramed in Figure 10.9.*

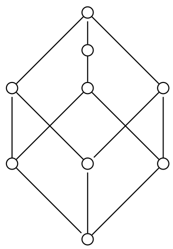

Figure 10.9: fig:conidFig

Proof. Since **CON A** is nonmodular, it has a sublattice isomorphic to $\mathbf{N}_5$, which we label as indicated in Figure 10.10.

Of course γ is a subuniverse of $\mathbf{A} \times \mathbf{A}$. We let $\mathbf{B}$ denote the corresponding subalgebra. Since each of the projection homomorphisms is onto $\mathbf{A}$, $\mathbf{B}$ is a subdirect product of two copies of $\mathbf{A}$. We let η_0 and $\eta_1 \in \operatorname{Con} B$ denote the kernels of the restrictions of the projection homomorphisms. If θ is any congruence on $\mathbf{A}$, we let

$$\theta_i = \{\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle \in B \times B : \langle a_i, b_i\rangle \in \theta\} \qquad \text{for } i = 0, 1. \qquad (10.6.1)$$ for:conidEq1

First note that if $\theta \geq \gamma$ then $\theta_0 = \theta_1$. For if $\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle \in \theta_0$ then $\langle a_0, b_0\rangle \in \theta$. But $\langle a_0, a_1\rangle$ and $\langle a_0, a_1\rangle$ are in γ. Since $\gamma \leq \theta$, a_0, a_1, b_0, b_1 are all in the same θ class; thus $\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle \in \theta_1$.

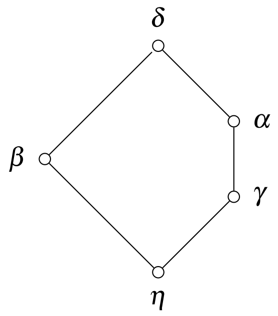

Figure 10.10:

By the Correspondence Theorem (Theorem 4.12) the map $\theta \mapsto \theta_i$ is a lattice isomorphism of **CON A** onto the interval $\mathbf{I}[\eta_i, 1_{\mathbf{B}}]$ of **CON B**, $i = 0, 1$. In particular, $\theta_i \vee \psi_i = (\theta \vee \psi)_i$, for $i = 0, 1$. Next observe that $\eta_0 \vee \eta_1 = \gamma_0\ (= \gamma_1)$. Indeed if $\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle \in \gamma_0$ then all four elements are in the same γ-class. Thus $\langle a_0, b_1\rangle \in B$. Hence

$$\langle a_0, a_1\rangle\ \eta_0\ \langle a_0, b_1\rangle\ \eta_1\ \langle b_0, b_1\rangle.$$

This proves that $\gamma_0 \le \eta_0 \vee \eta_1$. The other inequality is obvious, since $\gamma_0 = \gamma_1$.

To see that $\beta_0 = (\beta_0 \wedge \beta_1) \vee \eta_0$, let $\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle \in \beta_0$. Then

$$\langle a_0, a_1\rangle\ \eta_0\ \langle a_0, a_0\rangle\ \beta_0 \wedge \beta_1\ \langle b_0, b_0\rangle\ \eta_0\ \langle b_0, b_1\rangle.$$

Using these facts it is easy to verify that **CON B** has the sublattice of Figure 10.11.

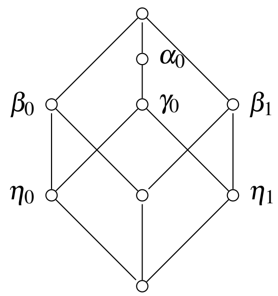

Figure 10.11:

This completes the proof. ■

It is easy to check that $\mathbf{N}_5$ satisfies the following equation, see Exercise 10.45.3.

$$(x \wedge y) \vee (x \wedge z) \approx x \wedge [(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)] \tag{10.6.2}$$

COROLLARY. *If $\mathcal{V}$ is a variety all of whose congruence lattices satisfy* (10.6.2)*, then $\mathcal{V}$ is congruence modular.*

Proof. If $\mathcal{V}$ is not congruence modular, then by the last theorem there is an algebra $\mathbf{B} \in \mathcal{V}$ whose congruence lattice has a sublattice isomorphic to the lattice pictured in Figure 10.9. It is easy to check that this lattice fails (10.6.2). ■

After Nations's original result several authors were able to show that there is a wide class of lattice equations each of which has modularity as a congruence consequence. It turned out to be difficult to construct a variety which was not congruence modular, but for which some nontrivial lattice equation is a congruence identity. The first such example was produced by S. V. Polin 1977. Polin's variety $\mathcal{P}$ is described in Example 10.2. That example showed that $\mathcal{P}$ had 4-permutable congruences and that it is not congruence modular. Moreover $\mathcal{P}$ had a binary operation symbol $\wedge$ which is a semilattice operation on each algebra in $\mathcal{P}$.

10.2 **THEOREM 10.36.** *The following equation is a congruence identity for Polin's variety* $\mathcal{P}$.

$$x\wedge(y\vee z) \leq [x\wedge(y\vee(x\wedge(z\vee(x\wedge(y\vee(x\wedge z))))))] \vee [x\wedge(z\vee(x\wedge(y\vee(x\wedge(z\vee(x\wedge y))))))]. \tag{10.6.3}$$

for:conidEq3

$\mathbf{M}_3$ *fails this equation.*

What's interesting is that it hints at another example of failure of join-primeness in the interpretability lattice, much like the one you told me about years ago. For example, suppose we knew that not every 4-permutable variety congruence-satisfies (63), and that the variety of semilattices doesn't

Proof. It is easy to see that this equation fails in $\mathbf{M}_3$. Let α, β, and γ be congruences of an algebra $\mathbf{B} \in \mathcal{P}$, and let $\langle a,b\rangle \in \alpha\wedge(\beta\vee\gamma)$. Since $\mathcal{P}$ is 4-permutable, there are elements c, d, and $e \in B$ such that the congruences indicated in Figure 10.12 hold.

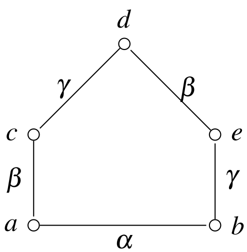

Figure 10.12: fig:conidFig

The following relations hold.

$$
\begin{array}{c}
a \wedge b \quad \alpha \quad b \\
a \wedge b \wedge c \quad (\alpha \wedge \beta) \quad b \wedge c \\
a \wedge b \wedge d \quad (\alpha \wedge (\gamma \vee (\alpha \wedge \beta))) \quad b \wedge d \\
a \wedge b \wedge e \quad (\alpha \wedge (\beta \vee (\alpha \wedge (\gamma \vee (\alpha \wedge \beta))))) \quad b \wedge e \\
a \wedge b = a \wedge b \wedge b \quad (\alpha \wedge (\gamma \vee (\alpha \wedge (\beta \vee (\alpha \wedge (\gamma \vee (\alpha \wedge \beta))))))) \quad b \wedge b = b.
\end{array}
$$

The first relation is obvious since $a \wedge b \; \alpha \; b \wedge b = b$. Each of the others follows easily from the previous ones.

By symmetry we have

$$a \quad (\alpha \wedge (\beta \vee (\alpha \wedge (\gamma \vee (\alpha \wedge (\beta \vee (\alpha \wedge \gamma))))))) \quad a \wedge b,$$

Thus $\langle a, b \rangle$ is in the right side of the equation, proving the theorem. ■

Actually $\mathcal{P}$ satisfies a simpler congruence identity, namely:

$$x \wedge (y \vee z) \leq y \vee (x \wedge (z \vee (x \wedge y))), \tag{10.6.4}$$

for:conidEq4

possibly make this into an exercise. Also that semilattices have no congr. identities might make an exercise. In fact this might make a good subsection.

see Theorem 7.1 in Alan Day and Freese 1980. The dual of this equation is also a congruence identity of $\mathcal{P}$. Moreover, they show that

$$
\begin{aligned}
y \wedge ([x \wedge (z \vee (x \wedge y))] \vee (y \wedge z)) \\
\leq [x \wedge (y \vee (z \wedge (x \vee y)))] \vee (z \wedge (x \vee y))
\end{aligned} \tag{10.6.5}
$$

for:conidEq5

is a congruence identity of $\mathcal{P}$ but that the dual of (10.6.5) is not a congruence identity of $\mathcal{P}$. Thus the set of congruence identities of a variety need not be self dual.

The subspaces of a vector space form a complemented, algebraic, atomic modular lattice, i.e., a projective geometry. These lattices actually satisfy an equation stronger than modularity known as the Arguesian equation. As we shall see, this equation is closely related to Desargues' Law of projective geometry, and thus the name. Let $\mathbf{L}$ be a lattice. A **triangle** in $\mathbf{L}$ is an element of L^3. Associated with any two triangles $\overline{a} = \langle a_0, a_1, a_2 \rangle$ and $\overline{b} = \langle b_0, b_1, b_2 \rangle$ are the elements

$$
\begin{aligned}
p &= (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \\
c_i &= (a_j \vee a_k) \wedge (b_j \vee b_k) \quad \text{for} \quad \{i, j, k\} = \{1, 2, 3\}.
\end{aligned} \tag{10.6.6}
$$

for:argEq0

Give defs of c_0, c_1, and c_2 separately?

Triangles $\overline{a}$ and $\overline{b}$ are said to be **centrally perspective** if

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \leq a_2 \vee b_2 \tag{10.6.7}$$

for:argEq0.5

and are said to be **axially perspective** if $c_2 \leq c_0 \vee c_1$. **Desargues' implication** is the statement that any two centrally perspective triangles are axially perspective.

When the triangles $\overline{a}$ and $\overline{b}$ are centrally perspective, we call the element p the **center of perspectivity**. A lattice **L** is said to be **Arguesian** if for all a, b, and $c \in L$ we have

Perhaps define Desargues "Theorem."

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \leq a_0 \vee (b_0 \wedge (b_1 \vee (c_2 \wedge (c_0 \vee c_1)))). \quad (10.6.8)$$

for:argEq1

Of course this is equivalent to a lattice equation which is known as the **Arguesian equation**.

argmod

LEMMA 10.37. *Every Arguesian lattice is modular but not conversely.*

Proof. See Exercises 10.45.4 and 10.45.5. ■

If we add more exercises constructing nonarg lattices, mention them here. Put in general principles of modular law calculations here?

In Theorem 4.67 we saw Jónsson's result that a lattice of 3-permuting equivalence relations is modular. The next result, also due to Jónsson, shows that a lattice of permuting equivalence relations is Arguesian. Thus a permutable variety is congruence Arguesian. In Theorem 10.43 we will prove the much stronger result that if a variety is congruence modular then it is congruence Arguesian. Along the way we will develop some of the theory of Arguesian lattices.

argaaa.0

THEOREM 10.38. *Every lattice of permuting equivalence relations is Arguesian. If an algebra* **A** *has permuting congruences, then* **CON A** *is Arguesian.*

Proof. Suppose that **L** is a lattice of permuting equivalence relations on a set S, and that a_i and b_i, $i = 0,1,2$, are elements of L. Let $\langle u,v \rangle$ be in the left side of (10.6.8). Then $\langle u,v \rangle \in a_i \vee b_i = a_i \circ b_i$ for $i = 0,1,2$. Hence there are elements $z_i \in S$ such that the relations of Figure 10.13 hold. Note that $\langle z_0, z_1 \rangle \in c_2 \wedge (c_0 \vee c_1)$. Using this and Figure 10.13 it is easy to verify that $\langle u,v \rangle$ is in the right side of (10.6.8). ■

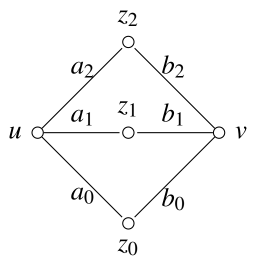

Figure 10.13:

fig:argFig1

argaaa.1

THEOREM 10.39. *Let* **L** *be a modular lattice. Then the following are equivalent.*

more credits?

i. **L** *is Arguesian,*

ii. *any two centrally perspective triangles of* $\mathbf{L}$ *are axially perspective,*

iii. *any two centrally perspective triangles* $\overline{a}$*,* $\overline{b}$ *of* $\mathbf{L}$ *which satisfy*

$$p \vee a_i = p \vee b_i = a_i \vee b_i \quad i = 0,1,2 \tag{10.6.9}$$ for:argEq2

are axially perspective.

Proof. Clearly (ii) implies (iii). We will first show that (iii) implies (i). Assume (iii). We need to show that (10.6.8) holds for arbitrary triangles $\overline{a}$ and $\overline{b}$ in $\mathbf{L}$. It suffices to show that (10.6.8) holds for triangles that satisfy (10.6.9). To see this, suppose that (10.6.8) holds whenever $\overline{a}$ and $\overline{b}$ satisfy (10.6.9), and let $\overline{a}$ and $\overline{b}$ be arbitrary triangles of $\mathbf{L}$ and let

$$a_i' = a_i \wedge (b_i \vee p), \quad b_i' = b_i \wedge (a_i \vee p), \quad i = 0,1,2.$$

Successive applications of modularity will establish that $a_i' \vee b_i' \geq p$ for each i. From this it readily follows that the new p', defined like p, but using the primed triangles, satisfies $p' \geq p$. On the other hand, the primed quantities are evidently all below their unprimed counterparts, and so in fact $p' = p$. Thus the primed triangles satisfy (10.6.9) and so, by our assumption, they satisfy (10.6.8). Hence

$$p \leq a_0' \vee (b_0' \wedge (b_1' \vee m'))$$

where $m' = c_2' \wedge (c_0' \vee c_1')$. Since the primed elements are smaller than their unprimed counterparts, it follows that (10.6.8) holds for the original (unprimed) triangles.

Now assume that our centrally perspective triangles $\overline{a}$ and $\overline{b}$ satisfy (10.6.9). Let $\overline{b'} = \overline{b}$ and $\overline{a'} = \langle a_0, a_1, a_2 \vee (a_0 \wedge (a_1 \vee b_1))\rangle$. Using (10.6.9) (for the unprimed triangles) it is easy to show that the primed triangles are centrally perspective. Now notice that by modularity

$$\begin{aligned} p' &= (a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2 \vee (a_0 \wedge (a_1 \vee b_1))) \\ &= p \vee (a_0 \wedge (a_1 \vee b_1)). \end{aligned}$$

From this it is straightforward to verify that the primed triangles satisfy (10.6.9). Thus we can apply (iii). Since $c_2 = c_2'$ and $c_1 = c_1'$, we have

$$\begin{aligned} c_2 &\leq c_1 \vee ((b_1 \vee b_2) \wedge (a_1 \vee a_2 \vee (a_0 \wedge (a_1 \vee b_1)))) \\ &= c_1 \vee ((b_1 \vee b_2) \wedge (a_2 \vee [(a_0 \vee a_1) \wedge (a_1 \vee b_1)])) \\ &= c_1 \vee ((b_1 \vee b_2) \wedge (a_1 \vee a_2 \vee (b_1 \wedge (a_0 \vee a_1)))) \\ &= c_1 \vee ((b_1 \vee b_2) \wedge (a_1 \vee a_2)) \vee (b_1 \wedge (a_0 \vee a_1)) \\ &= c_0 \vee c_1 \vee (b_1 \wedge (a_0 \vee a_1)). \end{aligned}$$

Now if we meet both sides with $a_0 \vee a_1$ and then join both sides with b_1 we obtain $b_1 \vee c_2 = b_1 \vee (c_2 \wedge (a_0 \vee a_1)) \leq b_1 \vee ((a_0 \vee a_1) \wedge (c_0 \vee c_1))$. But by modularity $b_1 \vee c_2 = (a_0 \vee a_1 \vee b_1) \wedge (b_0 \vee b_1)$. By two applications of (10.6.9) we have $a_0 \vee a_1 \vee b_1 = a_0 \vee p \vee b_1 = a_0 \vee b_0 \vee b_1 \geq b_0$. Thus $b_1 \vee c_2 = b_0 \vee b_1$ and hence $b_0 \vee b_1 \leq b_1 \vee ((a_0 \vee a_1) \wedge (c_0 \vee c_1))$. Meeting with b_0 we get

another extra step below?

$$
\begin{aligned}
b_0 &\leq b_0 \wedge [b_1 \vee ((a_0 \vee a_1) \wedge (c_0 \vee c_1))] \\
&\leq b_0 \wedge (b_0 \vee b_1) \wedge [b_1 \vee ((a_0 \vee a_1) \wedge (c_0 \vee c_1))] \\
&= b_0 \wedge [b_1 \vee ((b_0 \vee b_1) \wedge (a_0 \vee a_1) \wedge (c_0 \vee c_1))] \\
&= b_0 \wedge [b_1 \vee (c_2 \wedge (c_0 \vee c_1))] \\
&= b_0 \wedge [(a_0 \wedge b_0) \vee b_1 \vee (c_2 \wedge (c_0 \vee c_1))] \\
&= (a_0 \wedge b_0) \vee (b_0 \wedge [b_1 \vee (c_2 \wedge (c_0 \vee c_1))])
\end{aligned}
$$

Thus

$$p \leq a_0 \vee b_0 \leq a_0 \vee (b_0 \wedge [b_1 \vee (c_2 \wedge (c_0 \vee c_1))])$$

as desired.

To see that (i) implies (ii), let $\overline{a}$ and $\overline{b}$ be centrally perspective triangles of L. Then it follows from (10.6.8) and the central perspectivity that

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \leq a_0 \vee b_1 \vee (c_2 \wedge (c_0 \vee c_1)).$$

Now we join $a_0 \vee b_1$ to both sides and then we meet both sides with c_2. The left side becomes

$$c_2 \wedge (a_0 \vee b_0 \vee b_1) \wedge (a_0 \vee a_1 \vee b_1) = c_2$$

Thus we have

$$
\begin{aligned}
c_2 &\leq c_2 \wedge (a_0 \vee b_1 \vee (c_2 \wedge (c_0 \vee c_1))) \\
&= [c_2 \wedge (a_0 \vee b_1)] \vee [c_2 \wedge (c_0 \vee c_1)] \\
&= c_2 \wedge (c_0 \vee c_1 \vee (c_2 \wedge (a_0 \vee b_1))) \\
&\leq c_0 \vee c_1 \vee [a_0 \wedge (b_0 \vee b_1)] \vee [b_1 \wedge (a_0 \vee a_1)] \\
&= [(a_1 \vee a_2) \wedge (b_1 \vee b_2)] \vee [(a_0 \vee a_2) \wedge (b_0 \vee b_2)] \\
&\qquad \vee [a_0 \wedge (b_0 \vee b_1)] \vee [b_1 \wedge (a_0 \vee a_1)] \\
&= [(b_1 \vee b_2) \wedge (a_1 \vee a_2 \vee (b_1 \wedge (a_0 \vee a_1)))] \\
&\qquad \vee [(a_0 \vee a_2) \wedge (b_0 \vee b_2 \vee (a_0 \wedge (b_0 \vee b_1)))] \\
&= [(b_1 \vee b_2) \wedge (a_2 \vee ((a_1 \vee b_1) \wedge (a_0 \vee a_1)))] \\
&\qquad \vee [(a_0 \vee a_2) \wedge (b_2 \vee ((a_0 \vee b_0) \wedge (b_0 \vee b_1)))] \\
&= [(b_1 \vee b_2) \wedge (a_1 \vee a_2 \vee (a_0 \wedge (a_1 \vee b_1)))] \\
&\qquad \vee [(a_0 \vee a_2) \wedge (b_0 \vee b_2 \vee (b_1 \wedge (a_0 \vee b_0)))]
\end{aligned}
$$

Since $\overline{a}$ and $\overline{b}$ are centrally perspective, $a_0 \wedge (a_1 \vee b_1) \leq a_0 \wedge (a_2 \vee b_2)$ and $b_1 \wedge (a_0 \vee b_0) \leq b_1 \wedge (a_2 \vee b_2)$. Thus we have

$$
\begin{aligned}
c_2 \le{} & [(b_1 \vee b_2) \wedge (a_1 \vee a_2 \vee (a_0 \wedge (a_2 \vee b_2)))] \\
& \qquad \vee [(a_0 \vee a_2) \wedge (b_0 \vee b_2 \vee (b_1 \wedge (a_2 \vee b_2)))] \\
={} & [(b_1 \vee b_2) \wedge (a_1 \vee a_2 \vee (b_2 \wedge (a_0 \vee a_2)))] \\
& \qquad \vee [(a_0 \vee a_2) \wedge (b_0 \vee b_2 \vee (a_2 \wedge (b_1 \vee b_2)))] \\
={} & c_0 \vee (b_2 \wedge (a_0 \vee a_2)) \vee c_1 \vee (a_2 \wedge (b_1 \vee b_2)) \\
={} & c_0 \vee c_1,
\end{aligned}
$$

completing the proof. ■

The last theorem has two interesting corollaries. The first was discovered by A. Day and D. Pickering 1984, and the second by B. Jónsson 1972.

COROLLARY 1. *A lattice is Arguesian if and only if it satisfies*

$$(a_0 \vee b_0) \wedge (a_1 \vee b_1) \wedge (a_2 \vee b_2) \le a_0 \vee b_1 \vee (c_2 \wedge (c_0 \vee c_1)). \qquad (10.6.10)$$ for:argEq3

Proof. By Exercise 10.45.4 each of (10.6.8) and (10.6.10) imply modularity. Clearly (10.6.8) implies (10.6.10). Now our proof in the last theorem that (i) implies (ii) only used (10.6.10), not (10.6.8). Thus (10.6.10) implies (ii) which is equivalent to (i), the Arguesian equation. ■

COROLLARY 2. *The class of Arguesian lattices is self-dual.*

Proof. Suppose that **L** is an Arguesian lattice, and let $\overline{a}$, and $\overline{b}$ be centrally perspective triangles in the dual of **L**. Then, $(a_0 \wedge b_0) \vee (a_1 \wedge b_1) \ge a_2 \wedge b_2$. Let

$$\overline{a'} = \langle a_0 \wedge a_2, b_0 \wedge b_2, a_0 \wedge b_0 \rangle \qquad \overline{b'} = \langle a_1 \wedge a_2, b_1 \wedge b_2, a_1 \wedge b_1 \rangle$$

Then $(a'_0 \vee b'_0) \wedge (a'_1 \vee b'_1) \le a_2 \wedge b_2$, and by the dual central perspectivity, this is less than or equal to $a'_2 \vee b'_2$, showing that $\overline{a'}$ and $\overline{b'}$ are centrally perspective in **L**. Thus, by hypothesis and Theorem 10.39, they are axially perspective. Hence

$$
\begin{aligned}
[(a_0 \wedge a_2) & \vee (b_0 \wedge b_2)] \wedge [(a_1 \wedge a_2) \vee (b_1 \wedge b_2)] \\
&= (a'_0 \vee a'_1) \wedge (b'_0 \vee b'_1) \\
&\le [(a'_0 \vee a'_2) \wedge (b'_0 \vee b'_2)] \vee [(a'_1 \vee a'_2) \wedge (b'_1 \vee b'_2)] \\
&\le (a_0 \wedge a_1) \vee (b_0 \wedge b_1).
\end{aligned}
$$

Thus $\overline{a}$ and $\overline{b}$ are axially perspective in the dual of **L**. Hence, by the previous theorem, the dual of **L** is Arguesian, as was to be shown. ■

Projective geometries and the lattices associated with them are defined in §4.8 of the first volume. Exercise 10.45.5 outlines the construction of a projective plane whose lattice is not Arguesian. On the other hand the lattice associ-

If we add more examples, reference them here.

ated with any projective geometry, which is not a plane, is Arguesian, see Exercise 10.45.6. Roughly the argument runs as follows. It is easy to see that if $\overline{a}$ and $\overline{b}$ are centrally perspective triangles which do not lie in the same plane, then they are axially perspective. In fact c_0, c_1 and c_2 lie in the intersection of the planes determined by $\overline{a}$ and $\overline{b}$. If $\overline{a}$ and $\overline{b}$ do lie in the same plane, but there is a point of the geometry not in that plane, then it is possible to construct a third triangle $\overline{d}$ not in the plane but centrally perspective to both $\overline{a}$ and $\overline{b}$. Thus $\overline{d}$ is axially perspective to both $\overline{a}$ and $\overline{b}$ and from this one can show that $\overline{a}$ and $\overline{b}$ are axially perspective.

The next lemma, which is due to Jónsson, uses some of these ideas. In fact the triangle $\overline{d}$, constructed in the proof, can be shown to be centrally perspective to both $\overline{a}$ and $\overline{b}$, see Exercise 10.45.7.

argaaa.2 **LEMMA 10.40.** *Let* **L** *be a modular lattice and let $\overline{a}$ and $\overline{b}$ be centrally perspective triangles in* **L** *which satisfy* (10.6.9). *Let $u = a_0 \vee a_1 \vee a_2 \vee b_0 \vee b_1 \vee b_2$, and suppose that there exist q, $r \in L$ such that*

$$p \vee q = p \vee r = q \vee r, \qquad u \wedge q = p \wedge a_2, \qquad u \wedge r = p \wedge b_2.$$

Then $\overline{a}$ and $\overline{b}$ are axially perspective.

Proof. Let $\overline{d}$ be the triangle with $d_i = (a_i \vee q) \wedge (b_i \vee r)$. Figure 10.14 gives a geometric representation of this situation. We begin by showing that

$$a_0 \wedge (b_0 \vee b_1) \leq d_0 \vee d_1. \qquad (10.6.11)$$ for:argEq4

This inclusion is proved with the following calculations.

$$\begin{aligned}
d_0 \vee d_1 &\geq [(a_0 \vee q) \wedge (b_0 \vee r)] \vee (q \wedge (b_1 \vee r)) \\
&= (a_0 \vee q) \wedge [b_0 \vee r \vee (q \wedge (b_1 \vee r))] \\
&= (a_0 \vee q) \wedge [b_0 \vee r \vee (b_1 \wedge (q \vee r))] \\
&\geq (a_0 \vee q) \wedge (b_0 \vee r \vee (b_1 \wedge p)) \\
&= (a_0 \vee q) \wedge (b_0 \vee r \vee (b_1 \wedge (a_0 \vee b_0))) \\
&\geq (a_0 \vee q) \wedge (b_0 \vee b_1) \wedge (a_0 \vee a_1) \\
&\geq a_0 \wedge (b_0 \vee b_1).
\end{aligned}$$

Now we show that

$$(a_0 \vee q) \wedge (b_0 \vee b_1 \vee r) \leq d_0 \vee d_1. \qquad (10.6.12)$$ for:argEq5

To see this note

$$\begin{aligned}
(a_0 \vee q) \wedge (b_0 \vee b_1 \vee r) &\leq (a_0 \vee p \vee q) \wedge (b_0 \vee b_1 \vee r) \\
&= [(a_0 \vee p \vee q) \wedge (b_0 \vee b_1)] \vee r
\end{aligned}$$

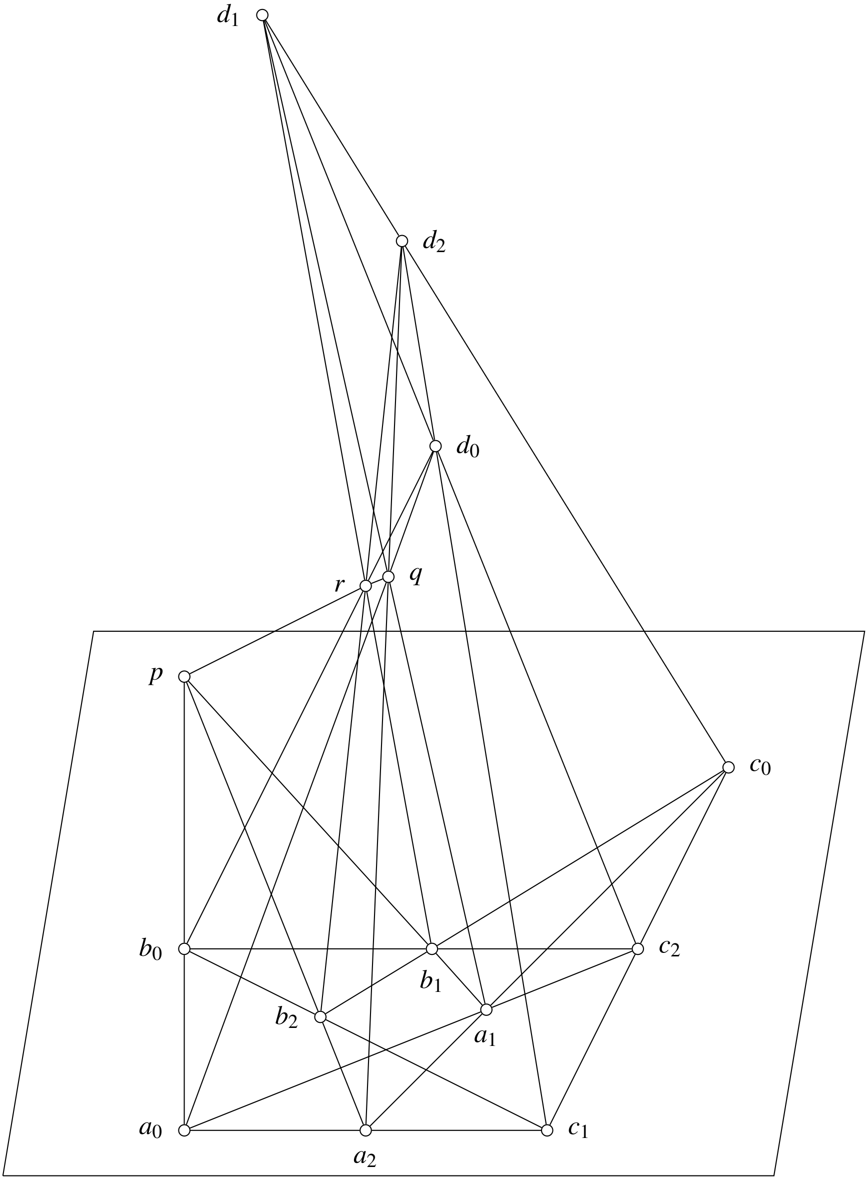

Figure 10.14:

fig:bigargfi

Using the fact that $b_0 \vee b_1 \leq u$ and $u \wedge q \leq p \leq a_0 \vee b_0$, we see that the right side is equal to $[(a_0 \vee b_0) \wedge (b_0 \vee b_1)] \vee r = (a_0 \wedge (b_0 \vee b_1)) \vee b_0 \vee r$. Thus

$$\begin{aligned}(a_0 \vee q) \wedge (b_0 \vee b_1 \vee r) &\leq (a_0 \vee q) \wedge [(a_0 \wedge (b_0 \vee b_1)) \vee b_0 \vee r] \\ &= [a_0 \wedge (b_0 \vee b_1)] \vee [(a_0 \vee q) \wedge (b_0 \vee r)] \\ &= [a_0 \wedge (b_0 \vee b_1)] \vee d_0.\end{aligned}$$

Now (10.6.12) follows from (10.6.11).

Using (10.6.12), we derive

$$\begin{aligned}d_0 \vee d_1 &\geq [(a_0 \vee q) \wedge (b_0 \vee b_1 \vee r)] \vee d_1 \\ &= (a_0 \vee q \vee d_1) \wedge (b_0 \vee b_1 \vee r) \\ &= (a_0 \vee q \vee a_1) \wedge (b_0 \vee b_1 \vee r) \\ &\geq c_2.\end{aligned} \tag{10.6.13}$$

for:argEq6

By an argument similar to the proof of (10.6.12) we obtain:

$$(a_1 \vee a_2 \vee q) \wedge (b_2 \vee r) \leq c_0 \vee d_2.$$

Thus

$$\begin{aligned}c_0 \vee d_2 &\geq [(a_1 \vee a_2 \vee q) \wedge (b_2 \vee r)] \vee c_0 \\ &= (a_1 \vee a_2 \vee q) \wedge (b_2 \vee c_0 \vee r) \\ &= (a_1 \vee a_2 \vee q) \wedge (b_1 \vee b_2 \vee r) \\ &\geq d_1.\end{aligned} \tag{10.6.14}$$

for:argEq7

Similarly we have

$$d_0 \leq c_1 \vee d_2. \tag{10.6.15}$$

for:argEq8

Now by (10.6.13), (10.6.14), and (10.6.15) we have

$$\begin{aligned}c_2 &\leq u \wedge (d_0 \vee d_1) \\ &\leq u \wedge (c_0 \vee c_1 \vee d_2) \\ &= c_0 \vee c_1 \vee (d_2 \wedge u).\end{aligned}$$

Since we have $q \wedge u \leq a_2$ and $r \wedge u \leq b_2$, $d_2 \wedge u = (a_2 \vee q) \wedge u \wedge (b_2 \vee r) \wedge u = a_2 \wedge b_2$. Thus $c_2 \leq c_0 \vee c_1$, completing the proof. ■

Let **A** be an algebra and let α, β, γ, and $\delta \in \operatorname{Con} A$. Of course α is a subuniverse of $\mathbf{A} \times \mathbf{A}$. Let **B** denote the corresponding subalgebra. Let θ be any congruence of **A**. As in the proof of Theorem 10.35, we let $\theta_i \in \operatorname{Con} B$, $i = 0$, 1, be the congruences $\{\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle : \langle a_i, b_i\rangle \in \theta\}$. With this notation we have the following lemma.

argaaa.2.5 **LEMMA 10.41.** *Let* **A** *and* **B** *be as described above. If* α, β, γ, *and* δ *satisfy*

$$\delta \le \alpha \wedge \beta = \alpha \wedge \gamma = \beta \wedge \gamma \tag{10.6.16}$$

for:argEq10.5

then we have

$$\alpha_0 \wedge \beta_1 = \alpha_0 \wedge \gamma_1 = \beta_1 \wedge \gamma_1 \tag{10.6.17}$$
for:argEq9
$$\beta_1 \vee \delta_0 = \alpha_0 \vee \beta_0 \tag{10.6.18}$$
for:argEq10
$$\gamma_1 \vee \delta_0 = \alpha_0 \vee \gamma_0. \tag{10.6.19}$$
for:argEq11

Proof. The proof of Theorem 10.35 showed that $\eta_0 \vee \eta_1 = \alpha_0 = \alpha_1$, where η_0 and η_1 are the kernels of the projection homomorphisms. By assumption (10.6.17) is true if we change all the subscripts to 0. Now the map $\theta_0 \mapsto \theta_1$ is a lattice isomorphism of $\mathbf{I}[\eta_0, 1_{\mathbf{B}}]$ onto $\mathbf{I}[\eta_1, 1_{\mathbf{B}}]$. Thus (10.6.17) holds if all the subscripts are changed to 1. Since $\alpha_0 = \alpha_1$ it follows that (10.6.17) holds.

Walter suggests a picture here, but I'm not sure I could make a decent one.

To see (10.6.18) first note $\theta_0 \le \eta_0 \vee \theta_1$ for any congruence θ on **A**. For suppose that $\langle a_0, a_1\rangle\ \theta_0\ \langle b_0, b_1\rangle$. Then $a_0\ \theta\ b_0$ and thus

$$\langle a_0, a_1\rangle\ \eta_0\ \langle a_0, a_0\rangle\ \theta_1\ \langle b_0, b_0\rangle\ \eta_0\ \langle b_0, b_1\rangle.$$

Now (10.6.18) and (10.6.19) can be derived from this, symmetry, and the fact that $\eta_0 \vee \eta_1 = \alpha_0 = \alpha_1$. ■

argaaa.3 **LEMMA 10.42.** *Let* $\mathcal{V}$ *be a variety and let* $\mathcal{K}$ *be the class of all lattices which can be embedded into the dual of* **CON A** *for some* $\mathbf{A} \in \mathcal{V}$*. Then, for any* $\mathbf{L} \in \mathcal{K}$*, and any* $p, s, t, u \in L$ *which satisfy* $p \vee s = p \vee t = s \vee t \le u$*,* **L** *has an extension* $\mathbf{L}' \in \mathcal{K}$ *such that, for some* q*,* $r \in L'$*,*

$$p \vee q = p \vee r = q \vee r, \qquad q \wedge u = p \wedge s, \qquad r \wedge u = p \wedge t.$$

Proof. This is immediate from the last lemma. ■

Put some introductory remarks here.

With these lemmas we can prove the following theorem of Freese and Jónsson 1976.

argaaa.4 **THEOREM 10.43.** *Let* $\mathcal{V}$ *be a congruence modular variety. Then* $\mathcal{V}$ *is congruence Arguesian.*

Proof. Let $\mathbf{A} \in \mathcal{V}$ and let **L** be the dual of **CON A**. Let $\overline{a}$ and $\overline{b}$ be centrally perspective triangles in **L** which satisfy (10.6.9). Applying Lemma 10.42, with $s = a_2$ and $t = b_2$, we embed **L** into a modular lattice $\mathbf{L}'$ possessing elements q and r satisfying the condition stated in Lemma 10.42. Now by Lemma 10.40 $\overline{a}$ and $\overline{b}$ are axially perspective. By Theorem 10.39, **L** is Arguesian, and so, by the second corollary to that theorem, **CON A** is Arguesian. ■

The proof of the last theorem shows that it is really a local theorem:

COROLLARY 10.44. *If all the subalgebras of* $\mathbf{A} \times \mathbf{A}$ *have modular congruence lattices, then* **CON A** *is Arguesian.* ■

Ex10.6

Exercises 10.45

er:inftymany

1. This exercise shows that there are infinitely many congruence inequivalent lattice equations.

credit herrmann and Huhn?

a. The concept of a spanning n-frame in a lattice is defined in Exercise 4.89.15 in the first volume. Let $x_0, \ldots, x_n$ be elements of a modular lattice and define

$$u = \bigwedge_{i=0}^{n} \bigvee_{j \neq i} x_j$$

$$v = \bigvee_{i=0}^{n} \bigwedge_{j \neq i} \bigvee_{k \neq i,j} x_k$$

$$a_i = (x_i \vee v) \wedge u = (x_i \wedge u) \vee v. \tag{10.6.20}$$

for:ai

Show that $\langle a_0, \ldots, a_n \rangle$ is an n-frame spanning the interval sublattice $\mathbf{I}[v,u]$.

b. Let $\langle A_0, \ldots, A_n \rangle$ be a spanning n-frame in the lattice of subspaces of a vector space $\mathbf{V}$. Show that

$$\mathbf{V} \cong \mathbf{A}_0 \times \cdots \times \mathbf{A}_{n-1}.$$

Let $\{f_\lambda : \lambda \in \Lambda\}$ be a basis for $\mathbf{A}_n$. By the above there are elements $e_{i,\lambda} \in A_i$ such that $f_\lambda = \sum_{i=0}^{n-1} e_{i,\lambda}$. Show that $\{e_{i,\lambda} : \lambda \in \Lambda\}$ is a basis for $\mathbf{A}_i$.

Put a hint here? That $\mathbf{A}_n$ and $\sum \mathbf{A}_i$ are complements.

c. Let $\langle a_0, a_1, a_2, a_3 \rangle$ be a 3-frame in a modular lattice. Let

$$c = [(a_1 \vee a_2) \wedge (a_0 \vee a_3)] \vee [(a_0 \vee a_1) \wedge (a_2 \vee a_3)]$$

and define $r_0 = a_2$ and inductively

$$r_{k+1} = (a_2 \vee a_3) \wedge (a_0 \vee (c \wedge (a_1 \vee r_k))).$$

Now let $\langle \mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 \rangle$ be a spanning 3-frame in the lattice of subspaces of a vector space $\mathbf{V}$ over a field $\mathbf{F}$. Using the previous part, show that the subspace corresponding to r_k as defined above is spanned by

$$\{ke_{0,\lambda} + ke_{1,\lambda} + e_{2,\lambda} : \lambda \in \Lambda\}.$$

Hint: By the last part we can find a basis, $\{e_{i,\lambda} : \lambda \in \Lambda\}$ of $\mathbf{A}_i$, for $i = 0, 1, 2$, so that $\{e_{0,\lambda} + e_{1,\lambda} + e_{2,\lambda} : \lambda \in \Lambda\}$ is a basis of $\mathbf{A}_3$. Using this show that the subspace corresponding to c defined above is

$$\left\{ \sum_{\lambda \in \Lambda} c_\lambda e_{0,\lambda} + (c_\lambda + d_\lambda) e_{1,\lambda} + d_\lambda e_{2,\lambda} : c_\lambda, d_\lambda \in \mathbf{F}, \lambda \in \Lambda \right\}.$$

Is this hint necessary?

Now use induction to prove the result.

d. Let $a_0, \ldots, a_n$ be the lattice terms in the variables $x_0, \ldots, x_n$ defined by (10.6.20). Then r_k can be viewed as a term in these variables. If $\mathbf{F}$ is a field and $\mathcal{V}_{\mathbf{F}}$ is the variety of all vector spaces over $\mathbf{F}$, then $\mathcal{V}_{\mathbf{F}}$ satisfies the congruence equation

$$r_k \approx a_2$$

if and only if the characteristic of $\mathbf{F}$ divides k. Thus there are infinitely many congruence inequivalent congruence equations.

2. (A. Day and J. B. Nation, see B. Jónsson 1980) Let x, y, and z be variables and let $y_1 = y$, $z_1 = z$, $y_{n+1} = y \vee (x \wedge z_n)$, and $z_{n+1} = z \vee (x \wedge y_n)$. Let δ_n be the inclusion

$$x \wedge (y \vee z) \leq (x \wedge y_n) \vee (x \wedge z_n).$$

Show that if $\mathbf{L}$ is a $2n$-permutable sublattice of the congruence lattice of a semilattice then $\mathbf{L}$ satisfies δ_{2n}. Notice that Theorem 10.36 follows from this exercise.

exer:verify

3. Verify that (10.6.2) holds in $\mathbf{N}_5$. Hint: First show that it holds in any distributive lattice and thus one can assume that in any failure x, y, and z are assigned to a generating set of $\mathbf{N}_5$.

exer:argmod

4. Show that every Arguesian lattice is modular. Also show that any lattice satisfying (10.6.10) is modular. Hint: Derive the modular law by applying either (10.6.8) or (10.6.10) to the triangles $\overline{a} = \langle x \wedge y \wedge z, x, x \rangle$ and $\overline{b} = \langle y \vee z, y \wedge z, y \rangle$.

exer:nonarg

5. Recall from Definition 4.78 and Exercise 4.89.6 that a projective plane is a pair $\boldsymbol{\pi} = \langle \mathrm{P}, \Lambda \rangle$ such that P is a set and Λ is a set of subsets of P and the following axioms are satisfied.

i. Any two distinct points belong to one and only one line.
ii. Any two distinct lines contain one and only one point.
iii. There exist four points, no three of which are collinear.

A **partial projective plane** is a pair $\boldsymbol{\pi} = \langle \mathrm{P}, \Lambda \rangle$ satisfying

i′. Any two distinct points belong to at most one line.

ii′. Any two distinct lines contain at most one point.

(Notice that **ii**.′ follows from **i**.′) Let $\boldsymbol{\pi}_0 = \langle \mathrm{P}_0, \Lambda_0 \rangle$ be a partial projective plane satisfying (iii). This partial plane can be completed to a projective plane using the free plane construction of Marshall Hall described below. Inductively define $\boldsymbol{\pi}_{n+1} = \langle \mathrm{P}_{n+1}, \Lambda_{n+1} \rangle$ from $\boldsymbol{\pi}_n = \langle \mathrm{P}_n, \Lambda_n \rangle$ as follows. If n is even, enlarge Λ_n by adding a new line for each pair of points of P_n not contained on a line. At this stage, each of these new lines contains only two points. If n is odd, the dual construction is used: a new point is added for each pair of nonintersecting lines. Let $\mathrm{P} = \bigcup \mathrm{P}_n$ and $\Lambda = \bigcup \Lambda_n$ and $\boldsymbol{\pi} = \langle \mathrm{P}, \Lambda \rangle$.

a. Show that $\boldsymbol{\pi}$ is a projective plane.

b. Let $\boldsymbol{\pi}$ be the plane obtained from the above construction starting with P_0 a four element set and $\Lambda_0 = \varnothing$. Show that the lattice $\mathbf{L}^{\pi}$ associated with this plane (defined in §4.8) is modular but not Arguesian. Hint: A *confined configuration* in a (partial) projective plane is a pair $\langle \mathrm{Q}, \Gamma \rangle$ where Q is a finite, nonempty subset of the set of points and Γ is a finite, nonempty subset of the set of lines such that each element of Q lies on at least three lines in Γ and each line in Γ contains at least three points of Q. Show that $\boldsymbol{\pi}$ contains no confined configuration. Then show that if $\langle a_0, a_1, a_2 \rangle$ and $\langle b_0, b_1, b_2 \rangle$ are triangles in a projective plane which are centrally perspective from p, and which are also axially perspective, then the ten points p, a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, and c_2 (the c_i are defined by (10.6.6)) are contained in a confined configuration which also has ten lines.

A generalization of 10.62

exer:3arg

6. The lattice associated with a projective geometry is defined in §4.8. Show that the lattice associated with a projective geometry, which is not a projective plane, is Arguesian. Hint: See the remarks before Lemma 10.40.

I have not done this or the next yet.

exer:3arg2

7. Show that the triangle $\overline{d}$ defined in the proof of Lemma 10.40 is centrally perspective with both $\overline{a}$ and $\overline{b}$ of that lemma.

10.7 Relationships

In this section we examine briefly the relative strengths of some of the more important classes which comprise our classification scheme for varieties. The varietal properties under consideration will include the following ones from earlier work in this chapter:

$$
\begin{aligned}
\mathrm{U} &: \text{ congruence uniformity}\\
\mathrm{C} &: \text{ coherence}\\
\mathrm{WU} &: \text{ weak uniformity}\\
\mathrm{R} &: \text{ regularity}\\
\mathrm{P} &: \text{ permutibility of congruences}\\
\mathrm{P}_3 &: \text{ 3-permutability}\\
\mathrm{P}_* &: \text{ } k\text{-permutibility for some } k\\
\mathrm{D} &: \text{ distributivity of congruences}\\
\mathrm{M} &: \text{ modularity of congruences}
\end{aligned}
$$

We will use these same letters to denote the class of all varieties satisfying the corresponding property. Thus R is also used to denote the class of all regular varieties. We will also introduce some new properties, E, NL and K_n, $n > 1$. E is the class of all varieties which have (10.6.3) as a conguence identity. The definitions of the other properties will be given later. We propose to establish all the implications indicated in Figure 10.15.

fix the downright arrow

$$
\begin{array}{ccccccccc}
 & & & & \mathrm{D} & \longrightarrow & \mathrm{E} & & \\
 & & & & & \searrow & & & \\
\mathrm{U} & \longrightarrow & \mathrm{WU} & \longrightarrow & \mathrm{R} & \longrightarrow & \mathrm{M} & & \\
 & \nearrow & & & & \times & & & \\
\mathrm{C} & \longrightarrow & \mathrm{P} & \longrightarrow & \mathrm{P}_3 & \longrightarrow & \mathrm{P}_* & \longrightarrow & \mathrm{NL} \\
 & & & & & & & \nearrow & \\
 & & \cdots & \longrightarrow & \mathrm{K}_3 & \longrightarrow & \mathrm{K}_2 & &
\end{array}
$$

Figure 10.15: fig:relsFig1

Moreover, we will see that none of the arrows can be reversed. It may be possible that there are some true relationships not implied by this diagram; for example, it is possible that $\mathrm{K}_2 \to \mathrm{P}_*$. However, from our examples it will be clear that very few further implications could hold.

How about $\mathrm{U} \to \mathrm{P}$.

We first concentrate on the ten implications which do not involve the new properties. Of these, five are trivial:

$$\mathrm{U} \to \mathrm{WU} \qquad \mathrm{D} \to \mathrm{M} \qquad \mathrm{D} \to \mathrm{E} \qquad \mathrm{P} \to \mathrm{P}_3 \to \mathrm{P}_*$$

Jónsson's implication $\mathrm{P}_3 \to \mathrm{M}$ was established in Theorem 4.67. By Theorem 10.27, $\mathrm{WU} \to \mathrm{R}$. Theorem 10.33 gives $\mathrm{C} \to \mathrm{WU}$ and $\mathrm{C} \to \mathrm{P}$. It remains for us to show that $\mathrm{R} \to \mathrm{M}$ and $\mathrm{R} \to \mathrm{P}_*$. Both of these are unpublished results of J. Hagemann dating from about 1973. The first result has both a local proof and a global proof. We present the global one first.

rels1 **THEOREM 10.46.** *Every congruence regular variety is congruence modular.*

Proof. If $\mathcal{V}$ is congruence regular, then $\mathcal{V}$ has terms f_i, g_i, for $1 \le i \le n$, satisfying condition (iv) of Theorem 10.23. Now we define new $\mathcal{V}$-terms $m_0, \ldots, m_{2n+1}$ as follows:

$$\begin{aligned}
m_0(x,y,z,w) &= x \\
m_{2i-1}(x,y,z,w) &= f_i(x,w,w,w,g_i(y,z,w)) \quad \text{for } 1 \le i \le n \\
m_{2i}(x,y,z,w) &= f_i(x,w,w,g_i(y,z,w),w) \quad \text{for } 1 \le i \le n \\
m_{2n+1}(x,y,z,w) &= w.
\end{aligned}$$

We claim that these terms satisfy Day's equations for congruence modularity given in Theorem 10.7. To verify that these equations hold, we make the following deductions from the regularity equations of Theorem 10.23.

$$\begin{aligned}
m_{2i-1}(x,y,y,x) &\approx f_i(x,x,x,x,g_i(y,y,x)) \\
&\approx f_i(x,x,x,x,x) \\
&\approx x.
\end{aligned}$$

A similar calculation yields $m_{2i}(x,y,y,x) \approx x$. Now,

$$\begin{aligned}
m_0(x,x,w,w) &\approx x \\
&\approx f_1(x,w,w,w,g_1(x,w,w)) \\
&\approx m_1(x,x,w,w) \\
m_{2n}(x,x,w,w) &\approx f_n(x,w,w,g_n(x,w,w),s) \\
&\approx w \\
&\approx m_{2n+1}(x,y,z,w).
\end{aligned}$$

Finally

$$\begin{aligned} m_{2i-1}(x,y,y,w) &\approx f_i(x,w,w,w,g_i(y,y,w)) \\ &\approx f_i(x,w,w,g_i(y.y.w),w) \\ &\approx m_{2i}(x,y,y,w) \\ m_{2i}(x,x,w,w) &\approx f_i(x,w,w,g_i(x,w,w),w) \\ &\approx f_{i+1}(x,w,w,w,g_{i+1}(x,w,w)) \\ &\approx m_{2i+1}(x,x,w,w). \end{aligned}$$

■

The above theorem tells us that if **CON A** is nonmodular, then some $\mathbf{B} \in \boldsymbol{HSP}(\mathbf{A})$ has nonregular congruences. The following local result tells us that such a **B** may be found in $\boldsymbol{S}(\mathbf{A}^2)$. This result is due to S. Bulman-Fleming, A. Day and W. Taylor 1974.

rels2 **THEOREM 10.47.** *If every subalgebra of* $\mathbf{A}^2$ *is congruence regular, then* **CON A** *is modular.*

Proof. Let α, β, and γ be elements $\mathrm{Con}\,A$ with $\beta \le \gamma$. We will show by induction that

$$(\alpha \circ^n \beta) \wedge \gamma \le (\alpha \wedge \gamma) \vee \beta. \tag{10.7.1}$$ for:relsEq1

We extend the definition of $\alpha \circ^n \beta$ given in §10.2 to include the case $n = 0$ by letting $\alpha \circ^0 \beta = 0_{\mathbf{A}}$. Since $\alpha \circ^1 \beta = \alpha$, the cases $n = 0$ and 1 are trivial. Now assume that (10.7.1) holds for a particular n, where $n > 1$. Of course $\alpha \circ^n \beta$ is a subuniverse of $\mathbf{A}^2$. We let **B** be the corresponding subalgebra. As in §10.6, for any $\psi \in \mathrm{Con}\,A$, and for $i = 0$, 1, we let $\psi_i \in \mathrm{Con}\,B$ be defined by $\langle a_0, a_1\rangle\ \psi_i\ \langle b_0, b_1\rangle$ if $a_i\ \psi\ b_i$. Let $\theta = (\alpha \wedge \gamma) \vee \beta$, and note that $\theta \le \gamma$. Thus $\theta_0 \wedge \theta_1 \le \gamma_0 \wedge \theta_1$. Now if

$$\langle a,a\rangle\ \gamma_0 \wedge \theta_1\ \langle b_0, b_1\rangle$$

then $a\ \theta\ b_1$. Since $\langle b_0, b_1\rangle \in B$, (10.7.1) implies that $\langle b_0, b_1\rangle \in \theta$. Thus $a\ \theta\ b_0$ and hence

$$\langle a,a\rangle\ \theta_0 \wedge \theta_1\ \langle b_0, b_1\rangle.$$

This shows that the $\langle a,a\rangle$-block of $\theta_0 \wedge \theta_1$ and $\gamma_0 \wedge \theta_1$ are the same. Thus $\theta_0 \wedge \theta_1 = \gamma_0 \wedge \theta_1$ by regularity.

Now suppose that $\langle a,c\rangle \in (\alpha \circ^{n+1} \beta) \wedge \gamma$. Then there are elements b and d such that the relations of Figure 10.16 hold.

Clearly $\langle a,b\rangle$ and $\langle c,d\rangle$ are in B and $\langle a,b\rangle\ \gamma_0 \wedge \theta_1\ \langle c,d\rangle$. Hence $\langle a,b\rangle\ \theta_0 \wedge \theta_1\ \langle c,d\rangle$. Thus $a\ \theta\ c$, completing our inductive argument. Of course modularity follows from (10.7.1). ■

COROLLARY. *If every finite algebra in a variety* $\mathcal{V}$ *is congruence regular, then every finite algebra in* $\mathcal{V}$ *is congruence modular.* ■

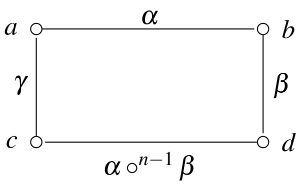

Figure 10.16: fig:relsFig2

Of course, Hagemann's Theorem 10.46 is also a corollary of the last theorem; in fact the proof of the theorem, when applied to the appropriate free algebra, yields Hagemann's terms seen in the proof of Theorem 10.46. (See Exercise 10.51.1.) The exponent 2 appearing in the theorem is best possible in the sense that there exists an eight element algebra **A** with **CON A** nonmodular, but such that every subalgebra of **A** is congruence regular (see Exercise 10.51.2). Be careful these are absolute references

rels3 **THEOREM 10.48.** *(J. Hagemann) Every congruence regular variety is congruence n-permutable for some n.*

Proof. If $\mathcal{V}$ is congruence regular, then $\mathcal{V}$ has terms f_i, g_i, for $i = 1, \ldots, n$, satisfying the equations in condition (iv) of Theorem 10.23. Now we define new $\mathcal{V}$-terms $p_1, \ldots, p_n$ as follows:

$$p_i(x,y,z) = f_i(x,z,z,g_i(y,z,z),g_i(x,y,z))$$

It is easy to verify that the Hagemann-Mitschke equations of Theorem 10.1 hold. ■

We still do not know any local proof for this last result. Nevertheless, it has an interesting corollary which simplifies the equations for regularity given in Theorem 10.23.

COROLLARY. *A variety $\mathcal{V}$ is congruence regular if and only if there exist ternary terms $g_1, \ldots, g_n$ and 4-ary terms $f_1, \ldots, f_n$ such that the following equations hold identically in $\mathcal{V}$.*

$$\begin{aligned} g_i(x,x,z) &\approx z \qquad \textit{for } 1 \le i \le n \\ x &\approx f_1(x,y,z,z) \\ f_1(x,y,z,g_1(x,y,z)) &\approx f_2(x,y,z,z) \\ &\vdots \\ f_n(x,y,z,g_n(x,y,z)) &\approx y \end{aligned}$$

Proof. We return to the proof of Theorem 10.23, especially the part (ii) implies (iv). As before, we take ϕ to be the congruence generated by Z. But instead of applying Lemma 10.22, we apply H. Lakser's special description of congruence generation for congruence n-permutable varieties given in Exercise 10.4.14 of §10.2. The details are left to the reader. ■

Check this reference.

Now we turn to the new concepts introduced in Figure 10.15. NL is simply the class of varieties in which no algebra of more than one element admits a linear ordering. (We say an algebra **A** admits an order $\leq$ if $\langle A, \leq \rangle$ is a partially ordered set such that each basic operation of **A** is monotone. That is, if $x_i \leq y_i$ for $1 \leq i \leq n$, then, for each basic operation f, we have

$$f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n).$$

We let K_n be the class of varieties in which no algebra has a universe which can be written as the union of n proper subuniverses. Thus K_{n+1} implies K_n *a fortiori*. The remaining implications of Figure 10.15, $P_* \to$ NL and $K_2 \to$ NL, follows from the next result. This result, which is local, is due to S. Bulman-Fleming and W. Taylor 1976.

rels4 **THEOREM 10.49.** *If* **A** *admits a linear order and* $|A| \geq 2$*, then* $\mathbf{A}^2$ *is the union of two proper subuniverses, and for each* $n \geq 2$*, there exists a subalgebra* **B** *of* $\mathbf{A}^n$ *with* **CON B** *not n-permutable.*

Proof. If **A** admits the linear order $\leq$, it is obvious that

$$A^2 = \{\langle x, y \rangle : x \leq y\} \cup \{\langle x, y \rangle : y \leq x\}.$$

Moreover, each of these two sets is a subuniverse of $\mathbf{A}^2$. If $|A| \geq 2$, each is a proper subuniverse.

For a failure of n-permutability, we define **B** to be the subalgebra of $\mathbf{A}^n$ with universe

$$B = \{\langle x_1, \ldots, x_n \rangle : x_1 \leq x_2 \leq \cdots \leq x_n\}.$$

Define congruences θ and ϕ on **B** as follows

$$\langle x_1, \ldots, x_n \rangle \ \theta \ \langle y_1, \ldots, y_n \rangle \quad \text{if } x_1 = y_1, x_3 = y_3 \cdots$$

and

$$\langle x_1, \ldots, x_n \rangle \ \phi \ \langle y_1, \ldots, y_n \rangle \quad \text{if } x_2 = y_2, x_4 = y_4 \ \cdots$$

Since $|A| \geq 2$, there exist a, $b \in A$ with $a < b$. If we let $\overline{a} = \langle a, \ldots, a \rangle$ and $\overline{b} = \langle b, \ldots, b \rangle$, then it is easy to check that $\overline{a} \ \theta \vee \phi \ \overline{b}$. Suppose that

$$\overline{b} = \overline{c}_0 \ \theta \ \overline{c}_1 \ \phi \ \overline{c}_2 \ \cdots \ \overline{c}_n = \overline{a}$$

for some $\overline{c}_0, \ldots, \overline{c}_n \in B$. Then, if we let π_j be the j^{th} projection,

$$b = \pi_1(\overline{c}_1) \leq \pi_2(\overline{c}_1) = \pi_2(\overline{c}_2) \leq \pi_3(\overline{c}_2) = \pi_3(\overline{c}_3) \leq \cdots \leq \pi_n(\overline{c}_n) = a$$

This implies that $b \leq a$, a contradiction. Thus θ and ϕ do not n-permute. ■

Now we will give examples showing that none of the implications of Figure 10.15 can be reversed. In most cases either well known or previous examples show that the reverse implication fails. For example the variety of groups shows that K_2 does not imply K_3. The idempotent reduct of groups shows that NL does not imply K_2. The variety of lattices shows that M does not imply NL and hence M implies neither R nor P_3.

E. T. Schmidt's variety of k-Boolean algebras was presented in Exercise 10.4.8. The variety of 3-Boolean algebras and the variety of implication algebras (see Exercise 10.4.5) show that P_3 implies neither R nor P. It follows that neither M nor P_* imply R. Heyting algebras show that P does not imply R, and hence P does not imply C. Polin's variety (Example 10.2) and 4-Boolean algebras show that P_* does not imply P_3. Example 10.28 shows that R does not imply WU; Exercise 10.34.22 shows that WU does not imply U. Finally, Exercise 10.34.26 shows that WU does not imply C. make sure this is correct.

The only possible reverse implication remaining for consideration is NL implies P_*. For an example blocking this implication, take $\mathcal{V}$ to be the variety of algebras $\langle A, \vee, \wedge, 0, 1, f \rangle$, where $\langle A, \vee, \wedge, 0, 1 \rangle$ is a lattice with least element 0, and greatest element 1, and f is a unary operation obeying $f(0) \approx 1$ and $f(1) \approx 0$. Using finite chains, it is easy to see that $\mathcal{V}$ does not have n-permutable congruences for any n, that is $\mathcal{V} \notin P_*$. Now suppose $\mathbf{A} \in \mathcal{V}$, with $|A| \geq 2$, admits a linear order $\leq$ (not to be confused with the lattice order). Thus $1 \neq 0$, and so, without loss of generality, $0 < 1$ in the linear order on $\mathbf{A}$. Hence, $1 = f(0) < f(1) = 0$. This contradiction establishes that such an $\mathbf{A}$ cannot admit a linear order, and hence that $\mathcal{V} \in$ NL.

Although we do not propose to consider every possibility, we will conclude this section with examples blocking two possible further implications in Figure 10.15. The first example is Polin's variety which we saw in Example 10.2 is 4-permutable but not modular. Thus we conclude that P_* does not imply M.

Finally, we will close the section with the result of E. T. Schmidt that regularity does not imply permutability. We extend this result by showing that weak uniformity of congruences does not imply permutability. Perhaps the most important remaining problem in this area is whether uniformity implies permutability. We do know that this holds for locally finite varieties, see ?? where is this?

Schmidt wrote 2 papers in 1970; check that this is the right one.

rels5 **THEOREM 10.50** (Schmidt 1970)**.** *There is a variety which is weakly congruence uniform (and hence regular) but not permutable.*

Proof. We take $\mathcal{V}$ to be the variety generated by two four-element algebras $\mathbf{A}_0$ and $\mathbf{A}_1$. Each $\mathbf{A}_i$ is a lattice (as depicted below) with six unary operations, μ_0, μ_1, λ_0, λ_1, c_0, and c_1 and four constants 0, 1, a_0, and a_1. We define the lattice structure of $\mathbf{A}_0$ by Figure 10.17.

The unary operations are defined by

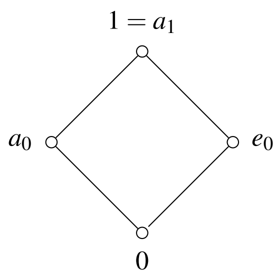

Figure 10.17: fig:relsFig3

$$
\begin{aligned}
\mu_0(x) &= x \\
\mu_1(x) &= \begin{cases} 1 & \text{if } x = 1 \\ a_0 & \text{otherwise} \end{cases} \\
\lambda_0(x) &= x \vee a_0 \\
\lambda_1(x) &= \begin{cases} 0 & \text{if } x = a_0 \\ x & \text{otherwise} \end{cases} \\
c_1(x) &= \neg x \\
c_0(x) &= (\neg x) \vee a_0.
\end{aligned}
$$

Here $\neg x$ denote the complement of x. For the purposes of this proof only, we define the *dual* of a term (or formula) to be the usual lattice theoretic dual term except that we also interchange the subscripts 0 and 1 of a_0 and a_1 and the six unary operations. Now $\mathbf{A}_1$ is the lattice diagrammed in Figure 10.18.

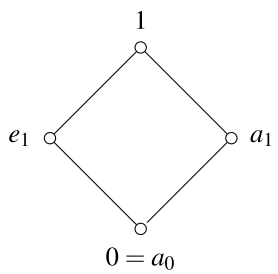

Figure 10.18: fig:relsFig4

The unary operations are defined by duality as described above. Thus, for example, the formula in $\mathbf{A}_1$ we have

$$
\lambda_0(x) = \begin{cases} 1 & \text{if } x = a_1 \\ x & \text{otherwise} \end{cases}.
$$

Obviously, the variety $\mathcal{V}$ defined in this way is self-dual: its set of identities is closed under the formation of dual equations.

In order to see that $\mathcal{V}$ is not congruence permutable, we leave it to the reader to check that the sublattice of $\mathbf{A}_0 \times \mathbf{A}_1$ given in Figure 10.19 is in fact a subalgebra. Now the failure of permutability is evident from the fact that this subuniverse contains $\langle 0, a_0\rangle$, $\langle a_0, a_0\rangle$, and $\langle a_0, 1\rangle$ but not $\langle 0, 1\rangle$, and thus the two coordinate projections do not permute.

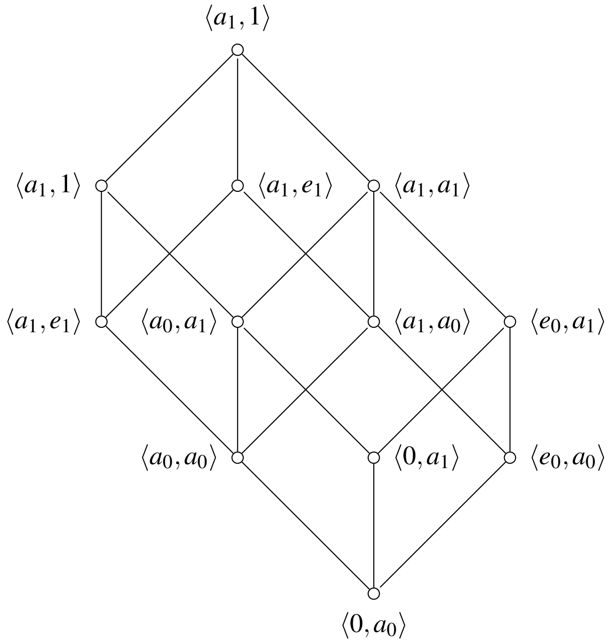

Figure 10.19: fig:relsFig5

By Jónsson's Theorem 7.??, $\mathcal{V}$ is residually finite. Therefore, by Exercise 10.34.17 weak uniformity of $\mathcal{V}$ will follow from the regularity of $\mathcal{V}$. Now to prove regularity, we will establish condition (iii) of Theorem 10.23. That is, we will describe ternary $\mathcal{V}$-terms g_i, for $1 \leq i \leq 4$, such that, for any x, $y \in \mathbf{A} \in \mathcal{V}$, $x = y$ if and only if $g_i(x,y,z) = z$ for all $z \in A$. We will in fact continue to use our notion of duality; for this reason we will prefer to denote our four $\mathcal{V}$-terms by g_i and g_i' for $i = 0, 1$, giving explicit definitions only for $i = 0$, and allowing duality to take care of the case $i = 1$. We proceed by first defining some auxiliary terms.

get this reference check this

$$
\begin{aligned}
t_0(x,y) &= [c_0(x \vee a_0) \wedge (y \vee a_0)] \vee [(x \vee a_0) \wedge c_0(y \vee a_0)] \\
s_0(x,y) &= \mu_0(t_0(x,y)) \\
\hat{z} &= a_1 \wedge (z \vee a_0) \\
q_0(x,y,z) &= s_0(x,y) \vee \hat{z} \\
r_0(x,y,z) &= s_0(x,y) \wedge \hat{z} \\
g_0(x,y,z) &= z \vee (q_0(x,y,z) \wedge c_1(\hat{z})) \\
g_0'(x,y,z) &= z \wedge c_0(r_0(x,y,z))
\end{aligned}
$$

Before proving that the terms g_0, g_0', g_1, and g_1' are as required, we first leave it to the reader to verify that $\mathbf{A}_0$ satisfies the following equations and their duals.

$$
\begin{array}{rr}
c_1(x) \leq a_1 & x \vee c_1(x) \approx x \vee a_1 \\
c_1(a_1) \approx 0 & x \wedge c_1(x) \approx 0 \\
\lambda_0(\mu_0(x)) \geq x & \mu_0(x) \leq a_1 \\
a_0 \approx \mu_0(a_0) \approx \lambda_0(a_0) & \mu_0(x \vee a_0) \geq a_0
\end{array}
$$

and thus these equations and their duals hold on $\mathcal{V}$. It is now easy to verify the following are identities of $\mathcal{V}$.

$$
\begin{aligned}
t_0(x,y) &\geq a_0 \\
s_0(x,y) &\geq a_0 \\
r_0(x,y,z) &\geq a_0 \\
t_0(x,x) &\approx a_0 \\
s_0(x,x) &\approx a_0 \\
q_0(x,x,z) &\approx a_0 \vee \hat{z} \approx \hat{z} \\
r_0(x,x,z) &\approx a_0 \wedge \hat{z} \approx a_0 \\
g_0(x,x,z) &\approx z \vee (\hat{z} \wedge c_1(\hat{z})) \approx z \\
g_0'(x,x,z) &\approx z \wedge c_0(a_0) \approx z
\end{aligned}
$$

Thus, if $x = y$ we have (by duality)

$$
g_0 = g_0' = g_1 = g_1' = z. \tag{10.7.2}
$$

for:relsEq2

To complete the proof we will assume (10.7.2) and use the identities of $\mathcal{V}$ to prove that $x = y$. We first deduce that $q_0(x,y,z) = \hat{z}$ and dually, as follows.

$$
\begin{aligned}
z \wedge a_1 &= g_0(x,y,z) \wedge a_1 \\
&= [(q_0(x,y,z) \wedge c_1(\hat{z})) \vee z] \wedge a_1 \\
&= (q_0(x,y,z) \wedge c_1(\hat{z})) \vee (z \wedge a_1)
\end{aligned}
$$

In the last step we have used the distributivity of $\mathcal{V}$ and the fact that

$$q_0(x,y,z) \wedge c_1(\hat{z}) \leq a_1.$$

And now, continuing,

$$\begin{aligned}\hat{z} &= (z \wedge a_1) \vee a_0 \\ &= (q_0(x,y,z) \wedge c_1(\hat{z})) \vee (z \wedge a_1) \vee a_0 \\ &= (q_0(x,y,z) \wedge c_1(\hat{z})) \vee \hat{z} \\ &= q_0(x,y,z) \wedge (c_1(\hat{z}) \vee \hat{z}) \\ &= q_0(x,y,z) \wedge a_1 = q_0(x,y,z)\end{aligned}$$

In the last equation we have used the identity $\mu_0(x) \leq a_1$ and the definition of q_0. We next deduce that $r_0(x,y,z) = a_0$, and dually, as follows.

$$\begin{aligned}a_0 \vee z = a_0 \vee g_0'(x,y,z) &= a_0 \vee (z \wedge c_0(r_0(x,y,z))) \\ &= (a_0 \vee z) \wedge c_0(r_0(x,y,z)),\end{aligned}$$

since $a_0 \leq c_0(u)$ for all u. Now, since $a_0 \leq r_0(x,y,z)$, $a_0 = (z \vee a_0) \wedge a_0 \wedge r_0(x,y,z)$. Using one of our first identities $x \wedge c_0(x) \approx x \wedge a_0$, we have

$$\begin{aligned}a_0 &= (z \vee a_0) \wedge c_0(r_0(x,y,z)) \wedge r_0(x,y,z) \\ &= (z \vee a_0) \wedge r_0(x,y,z) = r_0(x,y,z).\end{aligned}$$

Now, recalling that $q_0(x,y,z) = \hat{z}$ from above, we have

$$\begin{aligned}s_0(x,y) = s_0(x,y) \wedge (s_0(x,y) \vee \hat{z}) &= s_0(x,y) \wedge q_0(x,y,z) \\ &= s_0(x,y) \wedge \hat{z} = r_0(x,y,z) = a_0\end{aligned}$$

By duality, we also have $s_1(x,y) = a_1$. Now observe that

$$\begin{aligned}a_0 &= \lambda_0(a_0) = \lambda_0(a_0) \wedge t_0(x,y) \\ &= \lambda_0(s_0(x,y)) \wedge t_0(x,y) = \lambda_0(\mu_0(t_0(x,y))) \wedge t_0(x,y) = t_0(x,y).\end{aligned}$$

And so, by duality, we have $a_0 = t_0(x,y)$ and $a_1 = t_1(x,y)$. Since t_0 is merely the symmetric difference for the (Boolean) interval $\mathbf{I}[a_0, 1]$, we deduce that $x \vee a_0 = y \vee a_0$. Likewise, by using t_1 on $\mathbf{I}[0, a_1]$, we deduce that $x \wedge a_1 = y \wedge a_1$, and hence also that $x \wedge a_0 = y \wedge a_0$. Now by distributivity it follows that $x = y$, see Theoren 2.51. This completes the proof. ■

relsEx **Exercises 10.51**

1. Show that the method of proof of Theorem 10.47, if applied to appropriate congruences α, β, and γ on $\mathbf{F}_{\mathcal{V}}(x,y,z,w)$, yields precisely the Hagemann formulas occurring in the proof of Theorem 10.46.

2. (A. F. Bravtsev, B. Csákány) Let A be an eight element cube with one edge direction stipulated as "vertical," and let **G** be the eight element group consisting of all rigid motions of A which map each vertical face into a vertical face. (**G** is obviously isomorphic to the dihedral group $\mathbf{D}_4$.) Define **A** to be the algebra with universe A and with each $g \in G$ acting as a unary operation, i.e., a **G**-set, as defined in Chapter 3. Prove that **CON A** is not modular, but that every subalgebra of A has regular congruences.

3. (B. Csákány) If **A** is an algebra with regular but not modular congruences, then **A** has at least eight elements.

4. Show that $\mathrm{P}_* \to \mathrm{NL}$ does not hold for individual algebras.

5. Strengthen the varietal implication $\mathrm{P}_* \to \mathrm{NL}$ by showing that if $\mathcal{V}$ is k-permutable for some k, then no algebra in $\mathcal{V}$ admits any nontrivial ordering, i.e., any ordering with $a < b$ for some a and b with $a \neq b$.

6. Give a *local* proof of the varietal implication $\mathrm{R} \to \mathrm{NL}$. This is of interest, since we still do not know whether there exists a local proof of $\mathrm{R} \to \mathrm{P}_*$.

7. The variety of groups is in K_2 but not in K_3.

Is this defined?

8. The variety of 3-groups is in NL but not in K_2.

9. The variety of lattices is in D but not in K_2. Thus, in fact, we see from the previous two exercises that none of our varietal properties implies K_2, except of course K_n for $n \geq 2$.

10. (Bulman-Fleming and Taylor 1976) $\mathcal{V} \in \mathrm{K}_n$ if and only if the $\mathcal{V}$-free algebra on n generators is not the union of n proper subuniverses.

11. (Bulman-Fleming and Taylor 1976) If $\mathcal{V} = \boldsymbol{HSP}\mathbf{A}$ for **A** a finite quasiprimal algebra, then the relation $\mathcal{V} \in \mathrm{K}_n$ depends on both n and on the subalgebra structure of **A**. For instance, if $|A| = n+1$ and the proper subuniverses of **A** are precisely the singletons, then $\mathcal{V} \in \mathrm{K}_n - \mathrm{K}_{n+1}$.

12. If **A** is primal (quasi-primal with all elements as constants), then $\mathcal{V} = \boldsymbol{HSP}\mathbf{A} \in \mathrm{K}_{m-1} - \mathrm{K}_m$ for $m = \binom{k}{2}$, where $k = |A|$. Consequently, if **A** is *any* finite algebra with $|A| = k$ then $\mathcal{V} \notin \mathrm{K}_m$.

Check this.

13. If $\mathbf{F}_{\mathcal{V}}(2)$ is finite and $\mathcal{V}$ is nontrivial, then $\mathcal{V} \notin \mathrm{K}_n$ for $n = |\mathbf{F}_{\mathcal{V}}(2)|$.

14. Prove that the 12 element subset of $\mathbf{A}_0 \times \mathbf{A}_1$ given in Figure 10.19 is in fact a subuniverse.

15. For the Schmidt variety defined in the proof of Theorem 10.50, exhibit terms f_i of the sort required for condition (iv) of Theorem 10.23.

10.8 Abelian Varieties

In this section we will continue our study of Abelian varieties, which began in §4.13. We will also study certain concepts which are closely related. The properties considered in this section have a somewhat different flavor from those that have come before. They are not defined by Maltsev conditions, and as we shall see, in conjunction with modularity, they define a very special class of varieties, namely the class of affine varieties. A algebra **A** is **affine** if, for some ring **R**, it is polynomially equivalent to an **R**-module with universe A. A variety is affine if all its members are.

Recall that an algebra **A** is **Abelian** if $Z(\mathbf{A}) = 1_{\mathbf{A}}$, where $Z(\mathbf{A})$ is the center of **A** as defined in §4.13. That definition says that **A** is Abelian if and only if it satisfies the implication

$$t(x,\overline{u}) \approx t(x,\overline{v}) \to t(y,\overline{u}) \approx t(y,\overline{v}) \tag{10.8.1}$$ for:abelEq1

for each term t appropriate for **A**. A variety $\mathcal{V}$ is Abelian if all of its algebras are Abelian, i.e., if each implication (10.8.1) follows from the identities of $\mathcal{V}$. The prototypical examples of Abelian varieties are module varieties and unary varieties. The reader can easily check (10.8.1) in these cases.

Such a collection of implications is very different from a Maltsev condition. Somewhat closer in spirit to the rest of this chapter is L. Klukovits's closely related notion of a Hamiltonian variety. An algebra **A** is called **Hamiltonian** if every subuniverse of **A** is a block of some congruence on **A**, and a variety $\mathcal{V}$ is called Hamiltonian if each algebra in **A** is Hamiltonian. This nomenclature honors W. R. Hamilton (1805-1865), who invented quaternions. The closely allied eight element quaternion group is Hamiltonian, i.e., each of its subgroups is normal. The following characterization of Hamiltonian varieties, which is due to L. Klukovits 1975, is not generally regarded as a true Maltsev condition, since it involves the universally quantified function variable f. For a discussion of these more general sorts of conditions, see Chapter 11.

Clone Theory

abel1 **THEOREM 10.52.** *A variety $\mathcal{V}$ is Hamiltonian if and only if for every $\mathcal{V}$-term f there exists a $\mathcal{V}$-term h, such that the following equation is an identity of $\mathcal{V}$.*

$$f(x_0,x_2,\ldots,x_n) \approx h(x_0,x_1,f(x_1,\ldots,x_n))$$

Proof. First, suppose that $\mathcal{V}$ is Hamiltonian, and consider the free algebra $\mathbf{F}_{\mathcal{V}}(x_0,x_1,\ldots,x_n)$. The subuniverse U generated by x_0, x_1, and $f(x_1,\ldots,x_n)$ must be a θ-block for some congruence θ. Since $x_0 \mathrel{\theta} x_1$, we have

$$f(x_0,x_2,\ldots,x_n) \mathrel{\theta} f(x_1,x_2,\ldots,x_n),$$

i.e., $f(x_0,x_2,\ldots,x_n) \in U$. This means that

$$f(x_0,x_2,\ldots,x_n) = h(x_0,x_1,f(x_1,\ldots,x_n))$$

for some ternary term h. The desired equation now follows.

Conversely, let us suppose that the appropriate terms h exist for every f, and then prove that $\mathcal{V}$ is Hamiltonian. Let $\mathbf{B}$ be a subalgebra of an algebra $\mathbf{A} \in \mathcal{V}$, and consider the congruence θ generated on $\mathbf{A}$ by $Z = B^2$. Obviously B is contained in a single θ-block; to complete the proof, it will suffice to show that this θ-block contains nothing outside B. In other words, given $a\ \theta\ b$ with $b \in B$, we must prove that $a \in B$. Now according to Lemma 10.22, we have

$$\begin{aligned} b &= f_1(b_1, b_1', \bar{e}) \\ f_1(b_1', b_1, \bar{e}) &= f_2(b_2, b_2', \bar{e}) \\ &\vdots \\ f_m(b_m', b_m, \bar{e}) &= a \end{aligned}$$

for some $\mathcal{V}$-terms f_i, and some b_i, $b_i' \in B$.

Now our hypothesis (on the existence of h) can be applied both to the term $f_1(x_1, \ldots, x_n)$ and to the term $f_1(x_2, x_1, x_3, \ldots, x_n)$ yielding the identities

$$\begin{aligned} f_1(u, x_2, \ldots, x_n) &\approx h_1(u, x_1, f_1(x_1, \ldots, x_n)) \\ f_1(x_1, w, x_3, \ldots, x_n) &\approx k_1(w, x_2, f_1(x_1, \ldots, x_n)) \end{aligned}$$

for appropriate $\mathcal{V}$-terms h_1 and k_1. Thus we have

$$\begin{aligned} f_1(b_1', b_1, \bar{e}) &= k_1(b_1, b_1', h_1(b_1', b_1, f_1(b_1, b_1', \bar{e}))) \\ &= k_1(b_1, b_1', h_1(b_1', b_1, b)), \end{aligned}$$

and thus $f_1(b_1', b_1, \bar{e}) \in B$. Continuing in like manner, we inductively prove that $f_i(b_i', b_i, \bar{e}) \in B$ and thus $a \in B$. ■

abel2 **LEMMA 10.53.** *Every Hamiltonian variety is Abelian.*

Proof. In order to establish the implication (10.8.1) for a Hamiltonian variety, we take for each term t, another term h such that

$$t(y, \bar{u}) \approx h(y, x, t(x, \bar{u})).$$

The implication (10.8.1) is now evident. ■

We do not know any example of an Abelian variety which fails to be Hamiltonian. We will see that every Abelian algebra in a modular variety is Hamiltonian, see Theorem 10.57. What did Matt prove?

abel3 **LEMMA 10.54.** *If a variety is congruence modular and is Abelian, then it has permutable congruences.*

Proof. By Theorem 10.18, $\mathcal{V}$ has some Gumm terms, $p, d_1, \ldots, d_m$ satisfying the equations

$$\begin{aligned}
x &\approx p(x,z,z) \\
p(x,x,z) &\approx d_1(x,x,z) \\
d_i(x,y,x) &\approx x && \text{for all } i \\
d_i(x,z,z) &\approx d_{i+1}(x,z,z) && \text{for } i \text{ odd} \\
d_i(x,x,z) &\approx d_{i+1}(x,x,z) && \text{for } i \text{ even} \\
d_m(x,y,z) &\approx z.
\end{aligned}$$

Thus $\mathcal{V}$ satisfies

$$d_i(z,x,z) \approx z \approx d_i(z,z,z).$$

Now from (10.8.1) we deduce $d_i(x,x,z) \approx d_i(x,z,z)$, for all i. Using Gumm's equations we obtain $d_i(x,x,z) \approx d_{i+1}(x,x,z)$. Hence

$$p(x,x,z) \approx d_1(x,x,z) \approx \cdots \approx d_m(x,x,z) \approx z,$$

from which it follows that p is a Maltsev term for $\mathcal{V}$. ■

LEMMA 10.55. *An algebra* **A** *is affine if and only if, for some ring* **R***, an* **R***-module* $\mathbf{M} = \langle A, +, -, 0, \ldots \rangle$ *can be defined on* **A** *in such a way that*

$$p \in \operatorname{Clo}\mathbf{A} \subseteq \operatorname{Pol} M$$

where p is the ternary term $p(x,y,z) = x - y + z$.

Proof. Suppose first that **A** is affine, i.e., that $\operatorname{Pol} A = \operatorname{Pol} M$ for some **M** with universe A. Obviously $\operatorname{Clo} A \subseteq \operatorname{Pol} M$, and so it remains to be proved that $\operatorname{Clo} A$ contains p. Since $x - y$ is a polynomial of **M** it is a polynomial of **A**. Thus

$$x - y = F(x, y, a_1, \ldots, a_n)$$

for some **A**-term F and some $a_1, \ldots, a_n \in A$. But F itself must be an **M**-polynomial, and so

$$F(x, y, z_1, \ldots, z_n) = mx + ny + \sum p_i z_i + c$$

for some $c \in A$ and m, n, $p_1, \ldots, p_n \in \mathbf{R}$. By substituting 0 for x and y, we see
more details? that $mx = x$ and $nx = -x$, for all $x \in M$. Thus an easy calculation yields

$$F(x, F(y, z, x, \ldots, x), x, \ldots, x) = x - y + z$$

and so $p \in \operatorname{Clo} A$.

Conversely, suppose that we have an **R**-module **M** defined on A satisfying the conditions of the Lemma. From this it is not hard to see that every unary operation in $\operatorname{Pol} A$ has the form

$$f(x) = mx + a$$

for some $m \in R$ and $a \in A$. One may easily check that the set of all such m forms a (unital) subring $\mathbf{R}_0$ of $\mathbf{R}$, and that $\operatorname{Pol} A = \operatorname{Pol} M_0$, where $\mathbf{M}_0$ is the $\mathbf{R}_0$-reduct of **M**. ■

Now we are ready to present Herrmann's beautiful theorem connecting affine and Abelian algebras. McKenzie (unpublished; see R. McKenzie 1978) and Gumm 1978 had earlier established this result for permutable varieties. The proof in that case was presented in Theorem 4.155.

abel4 **THEOREM 10.56** (Herrmann 1979)**.** *An algebra* **A** *is affine if and only if* **A** *is Abelian and* **HSPA** *is congruence modular.*

Proof. If **A** is affine, then it generates a permutable, hence modular, variety by the previous lemma. Elementary linear algebra establishes the implication (10.8.1) for all operations in $\operatorname{Pol} A = \operatorname{Pol} M$; hence every affine **A** is Abelian.

Conversely, let us suppose that **A** is Abelian and lies in a congruence modular variety $\mathcal{V}$. By Theorem 10.54, $\mathcal{V}$ is congruence permutable. Now the result follows from Theorem 4.155. ■

abel5 **COROLLARY 10.57.** *Let $\mathcal{V}$ be congruence modular. Then $\mathcal{V}$ is Abelian if and only if it is Hamiltonian.*

Proof. If $\mathcal{V}$ is Hamiltonian then it is Abelian by Lemma 10.53. On the other had if $\mathcal{V}$ is modular and Abelian then it is affine, and from this it easily follows that $\mathcal{V}$ is Hamiltonian. ■

In our next corollary, we look at the special form taken by idempotent affine varieties. (Recall that a variety $\mathcal{V}$ is **idempotent** if $\mathcal{V}$ satisfies $F(x,\ldots,x) \approx x$ for each operation F of $\mathcal{V}$.)

At this point Walter presented some results about the clone associated with an abelian variety. I am omitting them for now; but after I see the clone chapters, I may reinstate them.

abel6 **COROLLARY 10.58.** *$\mathcal{V}$ is an idempotent affine variety if and only if $\mathcal{V}$ is equivalent to reduct of a module variety to some operations $\sum r_i x_i$ with $\sum r_i = 1$ and this reduct contains the operation $x - y + z$.*

Proof. See Exercise 14. ■

We now conclude our discussion of Abelian varieties, and our discussion of Maltsev conditions, with a sort of "anti-Abelian" property of varieties. A variety $\mathcal{V}$ is called **semidegenerate** if no algebra in $\mathcal{V}$ has a one-element subuniverse, other than, of course, the one-element algebra itself.

abel7 **THEOREM 10.59** (Csákány 1976; Kollár 1979)**.** *For a variety $\mathcal{V}$, the following conditions are equivalent:*

i. *$\mathcal{V}$ is semidegenerate;*

ii. *if* $\mathbf{A} \in \mathcal{V}$ *and B is a subuniverse of* **A** *with $B \neq A$, then B is not a block of any congruence on* **A***;*

iii. *for each* $\mathbf{A} \in \mathcal{V}$*,* $1_{\mathbf{A}}$ *is a compact element of* **CON A***;*

iv. *for each* $\mathbf{A} \in \mathcal{V}$*,* $1_{\mathbf{A}}$ *is a countably compact in* **CON A***;*

v. *for some n, there exist unary $\mathcal{V}$-terms $f_1,\ldots,f_n$ and $g_1,\ldots,g_n$ and ternary $\mathcal{V}$-terms $p_1,\ldots,p_n$ such that the following are identities of $\mathcal{V}$.*

$$\begin{aligned} x &\approx p_1(x,y,f_1(x)) \\ p_1(x,y,g_1(x)) &\approx p_2(x,y,f_2(x)) \\ &\vdots \\ p_n(x,y,g_n(x)) &\approx y \end{aligned}$$

Proof. We leave the easy equivalence (i) and (ii) to the reader. The rest of the proof will proceed (iii) $\to$ (iv) $\to$ (i) $\to$ (v) $\to$ (iii). The first of these implications is obvious. For (iv) $\to$ (i) let us assume that $\mathcal{V}$ is not semidegenerate, i.e., that some nontrivial $\mathbf{A} \in \mathcal{V}$ has a one-element subuniverse $\{e\}$. To show that (iv) fails for $\mathcal{V}$, we will show that $1_{\mathbf{B}}$ is not countably compact in **CON B**, where **B** is the "weak direct power" consisting of all sequences in A^{ω} which are equal to e in all but finitely many places. For each $i \in \omega$, we define $a\ \theta_i\ b$ if $a_j = b_j$ for all $j \geq i$. It is easy to see that each $\theta_i < 1$ but $\bigvee \theta_i = 1$, contradicting countable compactness.

For (i) $\to$ (v), we assume that $\mathcal{V}$ is semidegenerate, and consider the free algebra $\mathbf{F}_{\mathcal{V}}(x,y)$. Let **B** be the subalgebra generated by x, and θ the congruence generated by $Z = B^2$. Since $\mathbf{B}/\theta$ is a one-element subalgebra of $\mathbf{F}_{\mathcal{V}}(x,y)/\theta$, this latter algebra must itself have only one element, and thus $\langle x,y\rangle \in \theta$. According to Lemma 10.22, we have the following equations holding in $\mathbf{F}_{\mathcal{V}}(x,y)$, for some terms q_i, h_i, and k_i:

$$\begin{aligned} x &= q_1(x,y,h_1(x),k_1(x)) \\ q_1(x,y,k_1(x),h_1(x)) &= q_2(x,y,h_2(x),k_2(x)) \\ q_2(x,y,k_2(x),h_2(x)) &= q_3(x,y,h_3(x),k_3(x)) \\ &\vdots \\ q_m(x,y,k_m(x),h_m(x)) &= y. \end{aligned}$$

We now obtain the Maltsev condition of (v) if we define $n = 2m$, and let

$$\begin{aligned} p_{2i-1}(x,y,u) &= q_i(x,y,u,k_i(x)) \\ p_{2i}(x,y,w) &= q_i(x,y,k_i(x),w) \\ f_{2i-1} &= g_{2i} = h_i \\ g_{2i-1} &= f_{2i} = k_i \end{aligned}$$

for $i = 1,\ldots,m$. We leave the details to the reader.

Finally, for (v) $\to$ (iii), we will use the equations (v) to prove that, on any algebra $\mathbf{A} \in \mathcal{V}$, $1_{\mathbf{A}}$ is compact, i.e., finitely generated. We will in fact show that, for any $a \in A$, the congruence θ generated by the n pairs $\langle f_i(a), g_i(a)\rangle$, with f_i and g_i as in (v), is $1_{\mathbf{A}}$. Thus we have, for arbitrary b, $c \in A$:

$$p_1(a,b,f_1(a)) = a = p_1(a,c,f_1(a))$$
$$p_1(a,b,g_1(a)) \;\theta\; p_1(a,c,g_1(a))$$

By the equations (v), we have

$$p_2(a,b,f_2(a)) \;\theta\; p_2(a,c,f_2(a)),$$

and hence,

$$p_2(a,b,g_2(a)) \;\theta\; p_2(a,c,g_2(a)).$$

Continuing in a like manner, we finally arrive at

$$p_n(a,b,g_n(a)) \;\theta\; p_n(a,c,g_n(a));$$

in other words, $\langle b,c\rangle \in \theta$. Thus $\theta = 1_{\mathbf{A}}$ and the proof is complete. ■

From condition (ii) of this theorem it is evident that the Hamiltonian property is completely incompatible with semidegeneracy, in that the only variety having both properties is the trivial variety defined by $x \approx y$. In the next theorem we prove something a bit stronger. For a related fact concerning congruence distributivity, see Exercise 21.

abel8 **THEOREM 10.60.** *No nontrivial Abelian variety is semidegenerate.*

Proof. Supposing $\mathcal{V}$ to be Abelian and semidegenerate, we will prove $\mathcal{V}$ trivial by an argument almost exactly like the one which ended our last proof. For b, c any elements of any algebra $\mathbf{A} \in \mathcal{V}$, we need to prove $b = c$. Taking any $a \in A$, and taking the terms p_i, f_i, and g_i as supplied by the previous theorem, we have

$$p_1(a,b,f_1(a)) = a = p_1(a,c,f_1(a)).$$

Since $\mathcal{V}$ is Abelian, we have

$$p_1(a,b,g_1(a)) = p_1(a,c,g_1(a)),$$

and hence, by the equations for p_i, f_i, and g_i, we have

$$p_2(a,b,f_2(a)) = p_2(a,c,f_2(a)).$$

Continuing in like manner, we finally obtain $b = c$. ■

One may easily observe that the variety of bounded lattices (i.e., lattices with 0 and 1 explicitly in the similarity type) is semidegenerate since in such a lattice with more than one element, $0 \neq 1$, and hence all subalgebras have at least two elements. A similar argument applies to rings with unit. A few more examples appear in the exercises. We conclude the main body of this chapter with another look at Magari's Theorem. Our argument for (iv) $\to$ (i) in Theorem 10.59 comes from Kollár 1979; his argument turned out to repeat an important step from Magari 1969; thus we now present Magari's Theorem as a corollary to Theorem 10.59.

COROLLARY 10.61 (Magari 1969)**.** *Every nontrivial variety contains a simple algebra.*

Proof. Let $\mathcal{V}$ be a nontrivial variety. If $\mathcal{V}$ is semidegenerate, then $\mathcal{V}$ contains a nontrivial algebra **A** such that $1_{\mathbf{A}}$ is compact. Zorn's Lemma yields a maximal proper congruence θ. Clearly $\mathbf{A}/\theta$ is simple.

Otherwise $\mathcal{V}$ is not semidegenerate, and so has an algebra **A** with both a one-element subuniverse $\{e\}$ and a further element $a \neq e$. Let **Q** be the subalgebra of $\mathbf{A}^2$ generated by $\{\langle e,e\rangle, \langle a,e\rangle, \langle e,a\rangle, \langle a,a\rangle\}$, and take R to be the smallest equivalence relation on A containing Q. (R is not necessarily a congruence.) One easily checks that $B = e/R$ is a subuniverse of **A**, and that in the corresponding subalgebra **B**, the congruence $\operatorname{Cg}(a,e) = 1_{\mathbf{B}}$. Thus $1_{\mathbf{B}}$ is compact, and so **B** has a simple homomorphic image as before. ■

abelEx

Exercises 10.62

1. The Hamiltonian property of varieties is not definable by a Maltsev condition. Likewise the Abelian property.

2. (R. McKenzie) A variety $\mathcal{V}$ is called **strongly Abelian** if and only if $\mathcal{V}$ satisfies the following implication for every $\mathcal{V}$-term t:

$$t(x,\overline{u}) \approx t(y,\overline{v}) \to t(x,\overline{w}) \approx t(y,\overline{w}).$$

Prove that if $\mathcal{V}$ is strongly Abelian, then $\mathcal{V}$ is Abelian, and provide an example showing that the converse is false. Also, give an example of a nontrivial $\mathcal{V}$ which is strongly Abelian.

3. Give an example of a Hamiltonian variety which is not congruence permutable.

possibly promote this to a theorem.

4. (W. Taylor) Every k-permutable Abelian variety is affine. (Hint: Use the Maltsev condition for k-permutability given in Theorem 10.1, while trying to imitate the first part of the proof of Lemma 10.54; for the details, see page 16 of Taylor 1982.)

5. An algebra **A** is affine if and only if $\boldsymbol{HSP}\mathbf{A}$ is congruence permutable and $\{\langle a,a\rangle : a \in A\}$ is a block of a congruence on $\mathbf{A} \times \mathbf{A}$.

Call a class of algebras **κ-categorical**, or **categorical in power** κ, if it contains, up to isomorphism, only one algebra of power κ. In a later volume we will prove that if a variety $\mathcal{V}$ is κ-categorical for sufficiently large κ, then either $\mathcal{V}$ is affine or $\mathcal{V}$ is strongly Abelian (as defined in Exercise 2 above). The following seven exercises provide a description of some κ-categorical affine varieties, and along the way show some interesting examples of reducts of module varieties.

6. Show that if $\mathcal{V}$ is κ-categorical and affine then it is a reduct of a module variety.

exer:defMFG

7. Let **L** be a left ideal of a ring **R** with unit. The variety $\mathcal{V}_{\mathbf{R},\mathbf{L}}$ has ternary operations F_r for each $r \in R$, and unary operations G_α for each $\alpha \in L$. It is defined by the laws

$$\begin{aligned}
F_r(x,x,z) &\approx z \\
F_0(x,y,z) &\approx z \\
F_1(x,z,z) &\approx x \\
F_1(x,z,y) &\approx F_1(y,z,x) \\
F_{rs}(x,z,z) &\approx F_r(F_s(x,z,z),z,z) \\
F_{r+s}(x,y,z) &\approx F_1(F_r(x,y,z),z,F_s(x,y,z)) \\
F_1(x,y,F_1(z,v,w)) &\approx F_1(F_1(x,y,z),v,w) \\
F_r(x,y,z) &\approx F_1(F_1(F_r(x,u,u),u,F_{-r}(y,u,u)),u,z) \\
F_r(F_1(x,u,y),u,u) &\approx F_1(F_r(x,u,u),u,F_r(y,u,u)) \\
F_{1+\alpha}(x,y,z) &\approx F_1(G_\alpha(x),G_\alpha(y),z) \\
G_\alpha(G_\beta(x)) &\approx G_{\alpha+\beta+\alpha\beta}(x)
\end{aligned}$$

for all r, **R** and all $\alpha \in L$. Prove that every algebra $\mathbf{A} \in \mathcal{V}_{\mathbf{R},\mathbf{L}}$ can be embedded in an algebra $\langle M, F_r, G_\alpha \rangle$ where **M** is an **R**-module and, on **M**, we define

$$\begin{aligned}
F_r(x,y,z) &= rx - ry + z \\
G_\alpha(x) &= (1+\alpha)x.
\end{aligned}$$

Conversely each such $\langle M, F_r, G_\alpha \rangle$ is in $\mathcal{V}_{\mathbf{R},\mathbf{L}}$. (Hint: if $\mathbf{A} \in \mathcal{V}_{\mathbf{R},\mathbf{L}}$, consider the free **R**-module **F** generated by the set A and consider the smallest submodule **M** such that the map $a \mapsto a$ is a homomorphism $\mathbf{A} \to \mathbf{F}/\mathbf{M}$ for the operations F_r and G_α.) Moreover, in some cases, e.g., $\mathbf{R} = \mathbb{Z}$ and $\mathbf{L} = 2\mathbb{Z}$, there exists such an **A** which is not isomorphic to any $\langle M, F_r, G_\alpha \rangle$.

8. Conclude from the previous exercise that the correspondence

$$\begin{aligned}
F_r(x,y,z) &\mapsto rx - ry + z \\
G_\alpha(x) &\mapsto (1+\alpha)x
\end{aligned}$$

establishes an equivalence of $\mathcal{V}_{\mathbf{R},\mathbf{L}}$ with the reduct of the variety of all **R**-modules to the linear expressions $\sum r_i x_i$ with $\sum (r_i - 1) \in L$.

9. Now assume that the ideal L has a finite orthogonal $\mathbf{R}$-basis of idempotents $e_1,\ldots,e_k$. (This means that

$$e_ie_j = \begin{cases} e_i & \text{if } i=j \\ 0 & \text{otherwise,} \end{cases}$$

and that $\mathbf{L} = \mathbf{R}e_1 + \cdots + \mathbf{R}e_k$.) Prove that for every $\mathbf{A} \in \mathcal{V}_{\mathbf{R},\mathbf{L}}$ there exists $a \in A$ with $G_\alpha(a) = a$ for all $\alpha \in L$. Conclude that every $\mathbf{A} \in \mathcal{V}_{\mathbf{R},\mathbf{L}}$ is isomorphic to $\langle M, F_r, G_\alpha \rangle$ (defined in Exercise 7) for some $\mathbf{R}$-module $\mathbf{M}$.

10. We now let $\mathbf{R}$ be the ring of $n \times n$ matrices over a skew field $\mathbf{K}$, and $\mathbf{L}$ its left ideal consisting of all matrices whose first m columns are zero. Prove that $\mathcal{V}_{\mathbf{R},\mathbf{L}}$ is categorical in power κ unless $\kappa = |k| \geq \aleph_0$. Prove, moreover, that if $\mathbf{K}$ is finite, then m, n and $|k|$ can be recovered from the equivalence type (see §4.12) of $\mathcal{V}_{\mathbf{R},\mathbf{L}}$. Specifically, this information can be recovered from the sequence $\langle \omega_i \rangle$ where ω_i is the size of the i-generated free algebra in $\mathcal{V}_{\mathbf{R},\mathbf{L}}$. (By the result of S. Givant 1979 and E. A. Palyutin 1975, these varieties (with $\mathbf{K}$ finite) are the only affine varieties which are categorical in all powers.)

check with walter if these are the right refs.

11. (Aliev 1966; Baldwin and Lachlan 1973) Let the variety $\mathcal{V}$ be defined by the following axioms in one binary operation:

$$xx \approx x \qquad\qquad xy \approx yx$$
$$x(yx) \approx y \approx (xy)x \qquad (ux)(yv) \approx (uy)(xv)$$

Prove that $\mathcal{V}$ is equivalent to $\mathcal{V}_{\mathbf{R},\mathbf{L}}$ for some $\mathbf{R}$ and $\mathbf{L}$. [For a different approach to this variety, see Exercise 3.12.10.]

12. Prove that if $\mathcal{V} = \mathcal{V}_{\mathbf{R},\mathbf{L}}$ for some $\mathbf{R}$, $\mathbf{L}$, then the varietal power $\mathcal{V}^{[k]}$ is equivalent to $\mathcal{V}_{\mathbf{R}',\mathbf{L}'}$ for some $\mathbf{R}'$ and $\mathbf{L}'$. Give an explicit description of $\mathbf{R}'$ and $\mathbf{L}'$. (One example of the varietal power $\mathcal{V}^{[k]}$ was discussed in Exercises 4.38.18 and 19. For the precise definition of $\mathcal{V}^{[k]}$ one may consult Garcia and Taylor 1984, pages 22–24.)

Check to see if McKenzie will have defined the varietal power at this point.

13. For each varietal property P considered in this chapter, we may ask whether either of the following implications holds, where $\mathcal{V}^{[k]}$ denotes the varietal k^{th} power.

$$\mathrm{P}(\mathcal{V}) \to \mathrm{P}(\mathcal{V}^{[k]})$$
$$\mathrm{P}(\mathcal{V}^{[k]}) \to \mathrm{P}(\mathcal{V})$$

Investigate to what extent these hold. [For some information, see (Garcia and Taylor 1984, pages 22–24).]

exer:pfofCor

14. Prove Corollary 10.58.

exer:qp

15. If **A** is a finite quasi-primal algebra with no one-element subalgebras, then $\boldsymbol{V}(\mathbf{A})$ is semidegenerate. Write out the terms for condition (v) of Theorem 10.59 in the special case that $\mathbf{A} = \langle A, T, f \rangle$, where T is the ternary discriminator and $f : A \to A$ has no fixed point.

er:sdandperm

16. If $\mathcal{V}$ is semidegenerate and congruence permutable (such as, e.g., the variety of Exercise 15), then (v) of Theorem 10.59 is satisfied with $n = 1$.

17. Quackenbush 1982 If $\mathcal{V}$ has constants $a_1, \ldots, a_k$ and a ternary operation m, obeying the equations

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x$$

and

$$m(\cdots m(m(x,a_1,y),a_2,y),\ldots,a_k,y) \approx y,$$

then $\mathcal{V}$ is semidegenerate. (Hint: get terms for condition (v) of Theorem 10.59.) Incidentally, these equations hold on any algebra with exactly k elements having a ternary majority function, if we treat each element as a constant.

18. Quackenbush 1982 If condition (v) of Theorem 10.59 holds for $\mathcal{V}$, then for all $\mathbf{A} \in \mathcal{V}$, $1_{\mathbf{A}} \in \mathrm{Con} A$ is the join of n principal congruences. Prove that this condition fails for $\mathcal{V} = \boldsymbol{HSP}\mathbf{A}$, with $\mathbf{A} = \langle A, m, a_1, \ldots, a_k \rangle$ where $A = \{a_1, \ldots, a_k\}$, $k \geq 2n+2$, m is a majority function, and if x, y, $z \in A$ are distinct, then $m(x,y,z) = a_1$. (Hint: first establish that **CON A** is the same as the congruence lattice of the meet semilattice with universe A diagrammed in Figure 10.20. Thus, by Exercise 16, some semidegenerate varieties are not permutable. Moreover, it is now evident that semidegeneracy is not definable by a strong Maltsev condition.

Walter had a_0.

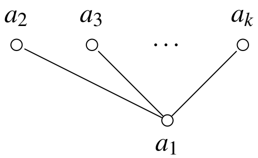

Figure 10.20:

fig:abelFig1

19. Quackenbush 1982 Write a Maltsev condition for the following property of a variety $\mathcal{V}$: every compact congruence on every algebra in $\mathcal{V}$ is the join of n principal congruences. Prove that E. T. Schmidt's variety of k-Boolean algebras given in Example 10.3 satisfies this condition for $n = k$ but not for $n = k-1$.

20. Csákány 1964 Consider the following property of a variety $\mathcal{V}$: if $\mathbf{A} \in \mathcal{V}$, $B_1, \ldots, B_n$ are subuniverses of $\mathbf{A}$, and F is an n-ary operation of $\mathbf{A}$, then $\{F(b_1, \ldots, b_n) : b_1 \in B_1, \ldots, b_n \in B_n\}$ is also a subuniverse. Prove that $\mathcal{V}$ satisfies this condition if and only if for every n and for all n-ary operations F and G of $\mathcal{V}$, there exists n-ary terms $\sigma_1, \ldots, \sigma_n$ such that $\mathcal{V}$ satisfies the following equations.

$$F(G(x_{11}, \ldots, x_{1n}), \ldots, G(x_{n1}, \ldots, x_{nn})) \\ \approx G(\sigma_1(x_{11}, \ldots, x_{1n}), \ldots, \sigma_n(x_{n1}, \ldots, x_{nn}))$$

exer:abeldist

21. No nontrivial Abelian variety is congruence distributive. This is a easy corollary of Theorem 10.56 but here we suggest trying a direct proof: apply the Abelian implication (10.8.1) to the Jónsson identities.

This and the next exercise came from Davey's note to walter and me.

22. Let $\mathcal{V}$ be a variety with a constant 0. Show that $\mathcal{V}$ is semidegenerate if and only if for some $n \geq 1$ there are binary terms $t_1, \ldots, t_n$ and constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $\mathcal{V}$ satisfies

$$\begin{aligned} 0 &\approx t_1(x, a_1) \\ t_1(x, b_1) &\approx t_2(x, a_2) \\ &\vdots \\ t_n(x, b_n) &\approx x. \end{aligned}$$

23. Let $\mathbf{A} = \langle \{0, 1\}, m, ' \rangle$ be the algebra where m is the majority function and $'$ is complementation. Show that the variety generated by $\mathbf{A}$ is semidegenerate but has no constant and no unary term which satisfies $t(x) \approx t(y)$.

Sections Yet to Come

10.9 Notes and Perspectives

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