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ALGEBRAS OF DERIVED DIMENSION ZERO

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We prove that a finite-dimensional algebra over an algebraically closed field is of derived dimension 0 if and only if it is an iterated tilted algebra of Dynkin type.

Key Words: Derived dimension; Iterated tilted algebra; Krull–Schmidt category; Trivial extension algebra.

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1. INTRODUCTION

1.1.

A dimension for a triangulated category has been introduced in Rouquier (to appear), which gives a new invariant for algebras and algebraic varieties under derived equivalences. For related topics see also Bondal and Van den Bergh (2003) and Happel (1988, p. 70).

Let $\mathscr{C}$ be a triangulated category with shift functor [1], $\mathscr{I}$ and $\mathscr{J}$ full subcategories of $\mathscr{C}$. Denote by $\langle\mathscr{I}\rangle$ the smallest full subcategory of $\mathscr{C}$ containing $\mathscr{I}$ and closed under isomorphisms, finite direct sums, direct summands, and shifts. Any object of $\langle\mathscr{I}\rangle$ is isomorphic to a direct summand of a finite direct sum $\bigoplus_i I_i[n_i]$ with each $I_i \in \mathscr{I}$ and $n_i \in \mathbb{Z}$. Define $\mathscr{I} \star \mathscr{J}$ to be the full subcategory of $\mathscr{C}$ consisting of the objects M, for which there is a distinguished triangle $I \longrightarrow M \longrightarrow J \longrightarrow I[1]$ with $I \in \mathscr{I}$ and $J \in \mathscr{J}$. Now define $\langle\mathscr{I}\rangle_0 := \{0\}$, and $\langle\mathscr{I}\rangle_n := \langle\langle\mathscr{I}\rangle_{n-1} \star \langle\mathscr{I}\rangle\rangle$ for $n \geq 1$. Then $\langle\mathscr{I}\rangle_1 = \langle\mathscr{I}\rangle$, and $\langle\mathscr{I}\rangle_n = \langle\langle\mathscr{I}\rangle \star \cdots \star \langle\mathscr{I}\rangle\rangle$, by the associativity of $\star$ (see Bondal and Van den Bergh, 2003). Note that $\langle\mathscr{I}\rangle_\infty := \bigcup_{n=0}^{\infty}\langle\mathscr{I}\rangle_n$ is the smallest thick triangulated subcategory of $\mathscr{C}$ containing $\mathscr{I}$.

By definition, the *dimension* of $\mathscr{C}$, denoted by $\dim(\mathscr{C})$, is the minimal integer $d \geq 0$ such that there exists an object $M \in \mathscr{C}$ with $\mathscr{C} = \langle M\rangle_{d+1}$, or ∞ when there is no such an object M. See Rouquier (to appear).

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Let A be a finite-dimensional algebra over a field k. Denote by A-mod the category of finite-dimensional left A-modules, and by $D^b(A\text{-mod})$ the bounded derived category. Define the *derived dimension* of A, denoted by der.dim(A), to be the dimension of the triangulated category $D^b(A\text{-mod})$. By Rouquier (to appear) and Krause and Kussin (2006), one has

$$\text{der.dim}(A) \leq \min\{l(A), \text{gl.dim}(A), \text{rep.dim}(A)\},$$

where $l(A)$ is the smallest integer $l \geq 0$ such that $\text{rad}^{l+1}(A) = 0$, gl.dim(A) and rep.dim(A) are the global dimension and the representation dimension of A (for the definition of rep.dim(A) see Auslander, 1971), respectively. In particular, we have der.dim$(A) < \infty$.

Our main result is

Theorem. *Let A be a finite-dimensional algebra over an algebraically closed field k. Then* der.dim$(A) = 0$ *if and only if A is an iterated tilted algebra of Dynkin type.*

1.2.

Let us fix some notation. For an additive category $\mathcal{A}$, denote by $C^*(\mathcal{A})$ the category of complexes of $\mathcal{A}$, where $* \in \{-, +, b\}$ means bounded-above, bounded-below, and bounded, respectively; and by $C(\mathcal{A})$ the category of unbounded complexes. Denote by $K^*(\mathcal{A})$ the corresponding homotopy category. If $\mathcal{A}$ is abelian, we have derived category $D^*(\mathcal{A})$.

For a finite-dimensional algebra A, denote by A-mod, A-proj, and A-inj the category of finite-dimensional left A-modules, projective A-modules and injective A-modules, respectively.

For triangulated categories and derived categories we refer to Verdier (1977), Hartshorne (1966), and Happel (1988); for representation theory of algebras we refer to Auslander et al. (1995) and Ringel (1984); and for tilting theory we refer to Ringel (1984) and Happel (1988), in particular, for iterated tilted algebras we refer to Happel (1988, p. 171).

2. PROOF OF THEOREM

Before giving the proof of theorem, we make some preparations.

2.1.

Let $A = \bigoplus_{j\geq 0} A_{(j)}$ be a finite-dimensional positively-graded algebra over k, and A-gr the category of finite-dimensional left $\mathbb{Z}$-graded A-modules with morphisms of degree zero. An object in A-gr is written as $M = \bigoplus_{j\in\mathbb{Z}} M_{(j)}$. For each $i \in \mathbb{Z}$, we have the degree-shift functor (i): $A\text{-gr} \longrightarrow A\text{-gr}$, defined by $M(i)_{(j)} = M_{(i+j)}$, $\forall j \in \mathbb{Z}$. Let $U: A\text{-gr} \longrightarrow A\text{-mod}$ be the degree-forgetful functor. Then $U(M(i)) = U(M)$, $\forall i \in \mathbb{Z}$. Clearly, A-gr is a Hom-finite abelian category, and hence by Remark A.2 in Appendix, it is Krull–Schmidt. An indecomposable in A-gr is called a *gr-indecomposable module*. The category A-gr has projective covers and injective hulls. Assume that $\{e_1, e_2, \ldots, e_n\}$ is a set of orthogonal primitive idempotents of $A_{(0)}$, such that $\{P_i := Ae_i = \bigoplus_{j\geq 0} A_{(j)}e_i \mid 1 \leq i \leq n\}$ is a complete set of pairwise nonisomorphic indecomposable projective A-modules. Then P_i (resp., $I_i := D(e_iA) = \bigoplus_{j\leq 0} D(e_iA_{(-j)})$) is a projective (resp., injective) object in A-gr. One deduces that $\{P_i(j) \mid 1 \leq i \leq n, j \in \mathbb{Z}\}$ is a complete set of pairwise nonisomorphic indecomposable projective objects in A-gr, and $\{I_i(j) \mid 1 \leq i \leq n, j \in \mathbb{Z}\}$ is a complete set of pairwise nonisomorphic indecomposable injective objects in A-gr.

Let $0 \neq M \in A$-gr. Define $t(M) := \max\{i \in \mathbb{Z} \mid M_{(i)} \neq 0\}$ and $b(M) := \min\{i \in \mathbb{Z} \mid M_{(i)} \neq 0\}$. For a graded A-module $M = \bigoplus_{i\in\mathbb{Z}} M_{(i)} \neq 0$, set $\text{top}(M) := M_{(t(M))}$ and $\text{bot}(M) := M_{(b(M))}$, both of which are viewed as $A_{(0)}$-modules. Denote by Ω^n (resp., $\Omega^n_{A_{(0)}}$) the nth syzygy functor on A-gr (resp., $A_{(0)}$-mod), $n \geq 1$. Similarly, we have Ω^{-n} and $\Omega^{-n}_{A_{(0)}}$.

We need the following observation.

Lemma 2.1. *Let M be a nonzero, nonprojective, and noninjective graded A-module. With notation above, we have*:

(i) *Either* $b(\Omega(M)) = b(M)$ *and* $\text{bot}(\Omega(M)) = \Omega_{A_{(0)}}(\text{bot}(M))$, *or* $b(\Omega(M)) > b(M)$;

(i)′ *Either* $t(\Omega^{-1}(M)) = t(M)$ *and* $\text{top}(\Omega^{-1}(M)) = \Omega^{-1}_{A_{(0)}}(\text{top}(M))$, *or* $t(\Omega^{-1}(M)) < t(M)$.

Proof. We only justify (i). Note that $\text{rad}(A) = \text{rad}(A_{(0)}) \oplus A_{(1)} \oplus \cdots$, and that for a graded A-module M, the projective cover P of $M/\text{rad}(A)M$ in A-mod is graded. It follows that it gives the projective cover of M in A-gr. Since A is positively-graded, it follows that $b(P) = b(M)$, and that $\text{bot}(P)$ is the projective cover of $\text{bot}(M)$ as $A_{(0)}$-modules. If $\text{bot}(P) = \text{bot}(M)$, then $b(\Omega(M)) > b(M)$. Otherwise, $b(\Omega(M)) = b(M)$ and $\text{bot}(\Omega(M)) = \Omega_{A_{(0)}}(\text{bot}(M))$. □

2.2.

Let $A = \bigoplus_{j\geq 0} A_{(j)}$ be a finite-dimensional positively-graded algebra over k. The category A-gr is said to be *locally representation-finite*, provided that for each $i \in \mathbb{Z}$, the set

$$\{[M] \mid M \text{ is gr-indecomposable such that } M_{(i)} \neq 0\}$$

is finite, where $[M]$ denotes the isoclass in A-gr of the graded module M. By degree-shifts, one sees that A-gr is locally representation-finite if and only if the set

$$\{[M] \mid M \text{ is gr-indecomposable such that } M_{(0)} \neq 0\}$$

is finite, if and only if A-gr has only finitely many indecomposable objects up to degree-shifts.

If A is in addition self-injective, then A-gr is a Frobenius category. In fact, we already know that A-gr has enough projective objects and injective objects, and each indecomposable projective object is of the form $P_i(j)$; since A is self-injective, it follows that P_i is injective in A-mod, so is $P_i(j)$ in A-gr; similarly, each $I_i(j)$ is a projective object in A-gr.

Note that the stable category $A\text{-}\underline{\mathrm{gr}}$ is triangulated (see Happel, 1988, Chap. 1, Sec. 2), with shift functor induced by $\overline{\Omega^{-1}}$.

Proposition 2.2. *Let $A = \bigoplus_{i\geq 0} A_{(i)}$ be a finite-dimensional positively-graded algebra which is self-injective. Assume that* $\dim(A\text{-}\underline{\mathrm{gr}}) = 0$ *and* $\mathrm{gl.dim}(A_{(0)}) < \infty$. *Then* A-gr *is locally representation-finite.*

Proof. Since $\dim(A\text{-}\underline{\mathrm{gr}}) = 0$, it follows that $A\text{-}\underline{\mathrm{gr}} = \langle X \rangle$ for some graded module X. Without loss of generality, we may assume that $X = \bigoplus_{l=1}^{r} M^l$, where M^l's are pairwise nonisomorphic nonprojective gr-indecomposable modules. It follows that every gr-indecomposable A-module is in the set $\{\Omega^i(M^l), P_j(i) \mid i \in \mathbb{Z}, 1 \leq l \leq r, 1 \leq j \leq n\}$. Therefore, it suffices to prove that for each $1 \leq l \leq r$, the set

$$\{j \in \mathbb{Z} \mid \Omega^j(M^l)_{(0)} \neq 0\}$$

is finite.

For this, assume that $\mathrm{gl.dim}(A_{(0)}) = N$, $b(M^l) = j_0$, and $t(M^l) = i_0$. Since $\mathrm{gl.dim}(A_{(0)}) < \infty$, it follows from Lemma 2.1(i) that if $b(\Omega(M)) = b(M)$, then $\mathrm{p.d}(\mathrm{bot}(\Omega(M))) = \mathrm{p.d}(\mathrm{bot}(M)) - 1$ as $A_{(0)}$-modules, and otherwise $b(\Omega(M)) > b(M)$. By using Lemma 2.1(i) repeatedly we have

$$\text{if } j \geq \max\{1, -j_0 N\}, \qquad \text{then } b(\Omega^j(M^l)) > 0.$$

Dually, if $j \geq \max\{1, i_0 N\}$, then $t(\Omega^{-j}(M^l)) < 0$. Note that $b(\Omega^j(M^l)) > 0$ (resp., $t(\Omega^{-j}(M^l)) < 0$) implies that $\Omega^j(M^l)_{(0)} = 0$ (resp., $\Omega^{-j}(M^l)_{(0)} = 0$). It follows that the set considered above is finite. □

2.3.

Let us recall some related notion in Bongartz and Gabriel (1982) and Gabriel (1981). Let A and $\{e_1, e_2, \ldots, e_n\}$ be the same as in 2.1, and $\mathbf{M}$ the full subcategory of A-gr consisting of objects $\{P_j(i) \mid 1 \leq j \leq n, i \in \mathbb{Z}\}$. Then $\mathbf{M}$ is locally finite-dimensional in the sense of Bongartz and Gabriel (1982). One may identify A-gr with $\mathrm{mod}(\mathbf{M})$ such that a graded A-module M is identified with a contravariant functor sending $P_j(i)$ to $e_j M_{(-i)}$. Now it is direct to see that A-gr is locally representation-finite if and only if the category $\mathbf{M}$ is locally representation-finite in the sense of Bongartz and Gabriel (1982, p. 337).

Let us follow Gabriel (1981, pp. 85–93). Let G be the group $\mathbb{Z}$. Then G acts freely on $\mathbf{M}$ by degree-shifts. Moreover, the orbit category $\mathbf{M}/G$ can be identified with the full subcategory of A-mod consisting of $\{P_j \mid 1 \leq j \leq n\}$. Hence we may identify $\mathrm{mod}(\mathbf{M}/G)$ with A-mod. With these two identifications, the push-down functor $F_\lambda : \mathrm{mod}(\mathbf{M}) \longrightarrow \mathrm{mod}(\mathbf{M}/G)$ is nothing but the degree-forgetful functor $U : A\text{-gr} \longrightarrow A\text{-mod}$. The following is just a restatement of Theorem d) in 3.6 of Gabriel (1981).

Lemma 2.3. *Let k be algebraically closed, and A be a finite-dimensional positively-graded k-algebra. Assume that* A-gr *is locally representation-finite. Then the degree-forgetful functor U is dense, and hence A is of finite representation type.*

2.4. Proof of Theorem

If A is an iterated tilted algebra of Dynkin type, then by Theorem 2.10 in Happel (1988, p. 109), we have a triangle-equivalence $D^b(A\text{-mod}) \simeq D^b(kQ\text{-mod})$ for some Dynkin quiver Q. Note that kQ is of finite representation type, and that $D^b(kQ\text{-mod}) = \langle M[0] \rangle$, where M is the direct sum of all the (finitely many) indecomposable kQ-modules. It follows that $\text{der.dim}(A) = \text{der.dim}(kQ) = 0$.

Conversely, if $\dim D^b(A\text{-mod}) = 0$, it follows from the fact that $D^b(A\text{-mod})$ is Krull–Schmidt (see, e.g., Theorem B.2 in Appendix) that $D^b(A\text{-mod})$ has only finitely many indecomposable objects up to shifts. Since $K^b(A\text{-proj})$ is a thick subcategory of $D^b(A\text{-mod})$, it follows that $K^b(A\text{-proj})$ has finitely many indecomposable objects up to shifts. Consequently, $\text{s.gl.dim}(A) < \infty$ (for the definition of $\text{s.gl.dim}(A)$ see B.3 in Appendix).

By Theorem 4.9 in Happel (1988, p. 88), and Lemma 2.4 in Happel (1988, p. 64), we have an exact embedding

$$F : D^b(A\text{-mod}) \longrightarrow T(A)\text{-}\underline{\text{gr}},$$

where $T(A) = A \oplus DA$ is the trivial extension algebra of A, which is graded with $\deg A = 0$ and $\deg DA = 1$. Since $\text{gl.dim}\, A \leq \text{s.gl.dim}(A) - 1 < \infty$ (see Corollary B.3 in Appendix), it follows from Theorem 4.9 in Happel (1988) that the embedding F is an equivalence. Now by applying Proposition 2.2 to the graded algebra $T(A)$ we know that $T(A)$-gr is locally representation-finite. It follows from Lemma 2.3 that $T(A)$ is of finite representation type, and then the assertion follows from a theorem of Assem et al. (1984), which says the trivial extension algebra $T(A)$ is of finite representation type if and only if A is an iterated tilted algebra of Dynkin type (see also Theorem 2.1 in Happel, 1988, p. 199, and Hughes and Waschbüsch, 1983).

□

APPENDIX

This appendix includes an exposition on some material we used. They are well-known, however their proofs seem to be scattered in various literature.

A. Krull–Schmidt Categories

This part includes a review of Krull–Schmidt categories.

A.1.

An additive category $\mathscr{C}$ is *Krull–Schmidt* if any object X has a decomposition $X = X_1 \oplus \cdots \oplus X_n$, such that each X_i is indecomposable with local endomorphism ring (see Ringel, 1984, p. 52).

Directly by definition, a factor category (see Auslander et al., 1995, p. 101) of a Krull–Schmidt category is Krull–Schmidt.

Let $\mathscr{C}$ be an additive category. An idempotent $e = e^2 \in \text{End}_{\mathscr{C}}(X)$ *splits*, if there are morphisms $u : X \longrightarrow Y$ and $v : Y \longrightarrow X$ such that $e = vu$ and $\text{Id}_Y = uv$. In this case, u (resp., v) is the cokernel (resp., kernel) of $\text{Id}_X - e$; and $\text{End}_{\mathscr{C}}(Y) \simeq e\text{End}_{\mathscr{C}}(X)e$

by sending $f \in \mathrm{End}_{\mathscr{C}}(Y)$ to vfu. If in addition $\mathrm{Id}_X - e$ splits via $X \xrightarrow{u'} Y' \xrightarrow{v'} X$, then $\binom{u}{u'} : X \simeq Y \oplus Y'$. One can prove directly that an idempotent e splits if and only if the cokernel of $\mathrm{Id}_X - e$ exists, if and only if the kernel of $\mathrm{Id}_X - e$ exists. It follows that if $\mathscr{C}$ has cokernels (or kernels) then each idempotent in $\mathscr{C}$ splits; and that if each idempotent in $\mathscr{C}$ splits, then each idempotent in a full subcategory $\mathscr{D}$ splits if and only if $\mathscr{D}$ is closed under direct summands.

A ring R is *semiperfect* if $R/\mathrm{rad}(R)$ is semisimple and any idempotent in $R/\mathrm{rad}(R)$ can be lifted to R, where $\mathrm{rad}(R)$ is the Jacobson radical.

Theorem A.1. *An additive category $\mathscr{C}$ is Krull–Schmidt if and only if any idempotent in $\mathscr{C}$ splits, and $\mathrm{End}_{\mathscr{C}}(X)$ is semiperfect for any $X \in \mathscr{C}$.*

In this case, any object has a unique (up to order) direct decomposition into indecomposables.

Proof. For $X \in \mathscr{C}$, denote by $\mathrm{add}\, X$ the full subcategory of the direct summands of finite direct sums of copies of X, and set $R := \mathrm{End}_{\mathscr{C}}(X)^{op}$. Let R-proj denote the category of finitely-generated projective left R-modules. Consider the fully-faithful functor

$$\Phi_X := \mathrm{Hom}_{\mathscr{C}}(X, -) : \mathrm{add}\, X \longrightarrow R\text{-proj}.$$

Assume that $\mathscr{C}$ is Krull–Schmidt. Then $X = X_1 \oplus \cdots \oplus X_n$ with each X_i indecomposable and $\mathrm{End}_{\mathscr{C}}(X_i)$ local. Set $P_i := \Phi_X(X_i)$. Then ${}_R R = P_1 \oplus \cdots \oplus P_n$ with $\mathrm{End}_R(P_i) \simeq \mathrm{End}_{\mathscr{C}}(X_i)$ local. Thus R is semiperfect by Theorem 27.6(b) in Anderson and Fuller (1974), and so is $\mathrm{End}_{\mathscr{C}}(X) = R^{op}$. Note that every object $P \in R$-proj is a direct sum of finitely many P_i's: in fact, note that $\{S_i := P_i/\mathrm{rad}(P_i)\}_{1\le i\le n}$ is the set of pairwise nonisomorphic simple R-modules and that the projection $P \longrightarrow P/\mathrm{rad}(P) = \bigoplus_i S_i^{m_i}$ is a projective cover, thus $P \simeq \bigoplus_i P_i^{m_i}$. It follows that P is essentially contained in the image of Φ_X, and hence Φ_X is an equivalence. Consider R-Mod, the category of left R-modules. Since R-Mod is abelian, it follows that any idempotent in R-Mod splits. Since R-proj is a full subcategory of R-Mod closed under direct summands, it follows that any idempotent in R-proj splits. So each idempotent in $\mathrm{add}(X)$ splits. This proves that any idempotent in $\mathscr{C}$ splits.

Conversely, assume that each idempotent in $\mathscr{C}$ splits and $R^{op} = \mathrm{End}_{\mathscr{C}}(X)$ is semiperfect for each X. Then again by Theorem 27.6(b) in Anderson and Fuller (1974), we have $R = Re_1 \oplus \cdots \oplus Re_n$ where each e_i is idempotent such that e_iRe_i is local. Since $1 = e_1 + \cdots + e_n$ and e_i splits in $\mathscr{C}$ via, say $X \xrightarrow{u_i} Y_i \xrightarrow{v_i} X$, it follows that $X \simeq Y_1 \oplus \cdots \oplus Y_n$ via the morphism $(u_1, \cdots, u_n)^t$ with inverse $(v_1, \ldots, v_n)$. Note that $\mathrm{End}_{\mathscr{C}}(Y_i) \simeq e_i\mathrm{End}_{\mathscr{C}}(X)e_i = (e_iRe_i)^{op}$ is local. This proves that $\mathscr{C}$ is Krull–Schmidt.

For the last statement, it suffices to show the uniqueness of decomposition in $\mathrm{add}\, X$ for each X. This follows from the fact that Φ_X is an equivalence, since the uniqueness of decomposition in R-proj is well known by Azumaya's theorem (see, e.g., Theorem 12.6(2) in Anderson and Fuller, 1974). This completes the proof. $\square$

A.2.

Let k be a field. An additive category $\mathscr{C}$ is a Hom-finite k-category if $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is finite-dimensional k-space for any $X, Y \in \mathscr{C}$, or equivalently, $\mathrm{End}_{\mathscr{C}}(X)$ is a finite-dimensional k-algebra for any object X.

Corollary A.2. *Let $\mathscr{C}$ be a Hom-finite k-category. Then the following are equivalent*:

(i) *$\mathscr{C}$ is Krull–Schmidt*;
(ii) *Each idempotent in $\mathscr{C}$ splits*;
(iii) *For any indecomposable $X \in \mathscr{C}$, $\mathrm{End}_{\mathscr{C}}(X)$ has no non-trivial idempotents.*

Remark A.2. By Corollary A.2(ii), a Hom-finite abelian k-category is Krull–Schmidt.

In particular, the category of coherent sheaves on a complete variety is Krull–Schmidt (see Atiyah, 1956, Theorem 2(i)).

B. Homotopically-Minimal Complexes

In this part, A is a finite-dimensional algebra over a field k.

B.1.

A complex $P^\bullet = (P^n, d^n) \in C(A\text{-proj})$ is called *homotopically-minimal* provided that a chain map $\phi^\bullet : P^\bullet \longrightarrow P^\bullet$ is an isomorphism if and only if it is an isomorphism in $K(A\text{-proj})$ (see Krause, 2005).

Applying Lemma B.1 and Proposition B.2 in Krause (2005), and duality, we have the following proposition.

Proposition B.1 (Krause, 2005). *Let $P^\bullet = (P^n, d^n) \in C(A\text{-proj})$. The following statements are equivalent*:

(i) *The complex $P^\bullet$ is homotopically-minimal*;
(ii) *Each differential d^n factors through* $\mathrm{rad}(P^{n+1})$;
(iii) *The complex $P^\bullet$ has no nonzero direct summands in $C(A\text{-proj})$ which are null-homotopic.*

Moreover, in $C(A\text{-proj})$ every complex $P^\bullet$ has a decomposition $P^\bullet = P'^\bullet \oplus P''^\bullet$ such that $P'^\bullet$ is homotopically-minimal and $P''^\bullet$ is null-homotopic.

B.2.

For $P^\bullet \in C(A\text{-proj})$, consider the ideal of $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)$:

$$\mathrm{Htp}(P^\bullet) = \{\phi^\bullet : P^\bullet \longrightarrow P^\bullet \mid \phi^\bullet \text{ is homotopic to zero}\}.$$

Lemma B.2. *Assume $\mathrm{rad}^l(A) = 0$. Let $P^\bullet$ be homotopically-minimal. Then* $(\mathrm{Htp}(P^\bullet))^l = 0$.

Proof. Let $\phi^\bullet \in \mathrm{Htp}(P^\bullet)$ with homotopy $\{h^n\}$. Then $\phi^n = d^{n-1}h^n + h^{n+1}d^n$. Since by assumption both d^{n-1} and d^n factor through radicals, it follows that ϕ^n factors through $\mathrm{rad}\, P^n$. Therefore, for $k \geq 1$ morphisms in $(\mathrm{Htp}(P^\bullet))^k$ factor through the kth radicals. So the assertion follows from $\mathrm{rad}^l(A) = 0$. □

Denote by $C^{-,b}(A\text{-proj})$ the category of bounded above complexes of projective modules with finitely many nonzero cohomologies, and by $K^{-,b}(A\text{-proj})$ its homotopy category. It is well known that there is a triangle-equivalence $\mathbf{p}: D^b(A\text{-mod}) \simeq K^{-,b}(A\text{-proj})$.

The following result can be deduced from Corollary 2.10 in Balmer and Schlichting (2001). See also Burban and Drozd (2004).

Theorem B.2. *The bounded derived category $D^b(A\text{-mod})$ is Krull–Schmidt.*

Proof. Clearly, $D^b(A\text{-mod})$ is Hom-finite. By Corollary A.2 it suffices to show that $\mathrm{End}_{D^b(A\text{-mod})}(X^\bullet)$ has no nontrivial idempotents, for any indecomposable $X^\bullet$.

By Proposition B.1 we may assume that $P^\bullet := \mathbf{p}X^\bullet$ is homotopically-minimal. Since $P^\bullet$ is indecomposable in $K^{-,b}(A\text{-proj})$, it follows from Proposition B.1(iii) that $P^\bullet$ is indecomposable in $C(A\text{-proj})$. Since idempotents in $C(A\text{-proj})$ split, it follows that $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)$ has no nontrivial idempotents. Note that

$$\mathrm{End}_{D^b(A\text{-mod})}(X^\bullet) = \mathrm{End}_{K^{-,b}(A\text{-proj})}(P^\bullet) = \mathrm{End}_{C(A\text{-proj})}(P^\bullet)/\mathrm{Htp}(P^\bullet).$$

Since by Lemma B.2 $\mathrm{Htp}(P^\bullet)$ is a nilpotent ideal, it follows that any idempotent in the quotient algebra $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)/\mathrm{Htp}(P^\bullet)$ lifts to $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)$. Therefore, $\mathrm{End}_{C(A\text{-proj})}(P^\bullet)/\mathrm{Htp}(P^\bullet)$ has no nontrivial idempotents. □

B.3.

For $X^\bullet = (X^n, d^n)$ in $C^b(A\text{-mod})$, define the *width* $w(X^\bullet)$ of $X^\bullet$ to be the cardinality of $\{n \in \mathbb{Z} \mid X^n \neq 0\}$. The strong global dimension $\mathrm{s.gl.dim}(A)$ of A is defined by (see Skowronski, 1987)

$$\mathrm{s.gl.dim}(A) := \sup\{w(X^\bullet) \mid X^\bullet \text{ is indecomposable in } C^b(A\text{-proj})\}.$$

By Proposition B.1 an indecomposable $X^\bullet$ in $C^b(A\text{-proj})$ is either homotopically-minimal, or null-homotopic (thus it is of the form $\cdots \longrightarrow 0 \longrightarrow P \xrightarrow{\mathrm{Id}} P \longrightarrow 0 \longrightarrow \cdots$, for some indecomposable projective A-module P). So we have

$$\mathrm{s.gl.dim}(A) = \sup\{2, w(P^\bullet) \mid P^\bullet \text{ is homotopically-minimal and indecomposable in } C^b(A\text{-proj})\}.$$

Let M be an indecomposable A-module with minimal projective resolution $P^\bullet \xrightarrow{\varepsilon} M$. Denote by $\tau^{\geq -m}P^\bullet$ the brutal truncation of $P^\bullet$, $m \geq 1$. By Proposition B.1(ii) $\tau^{\geq -m}P^\bullet$ is homotopically-minimal.

If $\tau^{\geq -m}P^\bullet = P'^\bullet \oplus Q^\bullet$ in $C^b(A\text{-proj})$ with $P^\bullet = (P^n, d^n)$, $P'^\bullet = (P'^n, \delta^n)$, and $Q^\bullet = (Q^n, \partial^n)$, then both $P'^\bullet$ and $Q^\bullet$ are homotopically-minimal. Assume that $P'^0 \neq 0$, and set $t_0 := \max\{t \in \mathbb{Z} \mid Q^t \neq 0\}$. Then $-m \leq t_0 \leq 0$. Since M is indecomposable and both $P'^\bullet$ and $Q^\bullet$ are homotopically-minimal, it follows that $t_0 \neq 0$, and hence $Q^{t_0} \subseteq \operatorname{Ker} d^{t_0} \subseteq \operatorname{rad}(P^{t_0}) = \operatorname{rad}(P'^{t_0} \oplus Q^{t_0})$, a contradiction. This proves the following lemma.

Lemma B.3. *The complex $\tau^{\geq -m}P^\bullet$ is homotopically-minimal and indecomposable in* $C^b(A\text{-proj})$.

As a consequence we have the following corollary.

Corollary B.3 (Skowronski, 1987, p. 541)**.** *Let A be a finite-dimensional algebra. Then*

$$\text{s.gl.dim}(A) \geq \max(2, 1 + \text{gl.dim}(A)).$$

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