# Algebras of integrals of motion for the Hamilton-Jacobi and Klein-Gordon-Fock equations in spacetime with a four-parameter groups of motions in the presence of an external electromagnetic field 

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## 1 Introduction

There are currently two main methods of exact integration of the classical and quantum equations of a test particle motion (the Hamilton-Jacobi, Klein-Gordon-Fock, and Dirac - Fock equations) in flat and curved spacetime, including when the external electromagnetic field is present. Both methods are based on the algebra of integrals of motion, each of which is linear or quadratic in momenta.

Symmetry operators of commutative algebra form a complete set of integrals of motion. A necessary condition for this set's existence is the presence of a complete set of Killing vector and tensor fields in the space. In this case, a complete separation of variables takes place.

The complete separation of variables is based on the theory of Stackel spaces. Spaces in which the free Hamilton-Jacobi equation is integrated by the complete separation of variables are called Stackel spaces. The coefficients at the highest powers of the products of momenta included in the first integrals of motion are vector (if the integral is linear in momenta) or tensor (for integrals of motion quadratic in momenta) Killing fields. It is also true for the HamiltonJacobi and Klein-Gordon-Fock equations in an electromagnetic field. Thus, the symmetry of the classical and quantum equations of motion is associated with the symmetry of space. Methods of complete separation of variables for classical and quantum equations of motion of charged test particles in the presence of external fields are constructed within the framework
of the theory of Stackel spaces. The methods have been successfully used to solve classification problems in general relativity since 1966. (see, for example, [1]-8] and the links indicated therein).

Otherwise, the non-commutative integration method is applied. The non-commutative integration method was developed for quantum one-particle equations of motion (Klein-GordonFock, Dirac-Fock, etc.) In this case, the set is formed by the operators of the non-commutative group of space motions. Since these operators do not commute between themselves, they cannot be directly used to construct a basic solution. The original algebra is supplemented with linear differential operators acting in the space of functions depending on coordinates and parameters to get around this problem. Extended operators are selected in such a way that they mutually commute. Using them, one can find a basic solution.

The problem of non-commutative integration attracted the attention of researchers much later, although the complete algebraic classification of spaces that admit groups of motions required for its statement was constructed by A.Z. Petrov back in the sixties of the last century (see [9]). The classification served as the basis for constructing a method for the exact integration of linear partial differential equations using non-commutative algebras of first-order integrals of motion (see [10] - [15]).

The method made it possible to significantly expand the sets of external fields and metrics of spacetime, in which basic solutions of the quantum equations of motion for a charged test particle, can be found. It became an incentive for the construction the set of admissible electromagnetic fields as a necessary stage in the application of a non-commutative integration procedure (electromagnetic fields in which the Hamilton-Jacobi and (or) Klein-Gordon-Fock equations for a charged test particle admit algebra of integrals of motion linear in momenta are called admissible).

In particular, in [11, [13] a classification of all potentials of admissible electromagnetic fields for ((metrics of spacetime)) admitting a transitive action of four-parameter movement groups $G_{4}$ is constructed. In [16],[?] a similar classification was carried out for homogeneous spaces with a three-parameter groups of motions $\quad G_{3}$, acting transitively on non null hyperspace of spacetime (see [18] too).

In present work, the following is done.

1. For the case of an $n$-dimensional Riemannian space with an arbitrary signature $V_{n}$, it is proved that, in an admissible electromagnetic field, the algebras of integrals of motion of the Hamilton-Jacobi equation and the symmetry operators of the Klein-Gordon-Fock equation for a charged particle coincide, and they coincide with the algebras of symmetry operators for the free Hamilton-Jacobi and Klein-Gordon-Fock equations.
2. In the $n$-dimensional Riemannian space $\quad V_{n}$, conditions that must be satisfied by admissible electromagnetic fields are obtained, and the compatibility of the equations systems following from these conditions is proved for the case when the group of motions $G_{r}$ acts transitively on the subspace $\quad V_{r} \quad\left(r \leq n, \quad \operatorname{rank}\left(G_{r}\right)=r\right)$.
3. The classification of admissible electromagnetic fields in spacetime with four-parameter groups of motions $\quad G_{4}, \quad$ is completed for the case when $\operatorname{rank}\left(G_{4}\right)<4$.

Note that additional conditions on metrics and potentials (for example, the Einstein equations) are not used.

## 2 Conditions for the existence of the algebra of integrals of motion.

### 2.1 Designations.

In the next two sections, we consider the $n$-dimensional Riemannian space $V_{n}$ (with an arbitrary signature), on the $r$-dimensional subspace $\quad V_{r}$ of which the $R$-parameter groups of motions $\quad G_{R} \quad\left(\operatorname{rank}\left(G_{R}\right)=r \leq R\right) \quad$ of the space $\quad V_{n}$ acts transitively. This means that any two points of the subspace $V_{r}$ can be transformed into each other by some transformation of the group $\quad G_{R}$. In this case, the subspace $\quad V_{r}$ is called the subspace of transitivity (or the hypersurface of transitivity if $r=n-1$ ).

The following index notations are used. The coordinate indices of the canonical local coordinate system variables $\left[u^{i}\right]$ of the space $V_{n}$ will be denoted by small Latin letters: $i, j, l=1, \ldots n$. By analogy with Stackel spaces, all the variables of the canonical coordinate system are divided into two sets. Variables of the local coordinate system of the subspace $V_{r}$, on which the group of motions $G_{R}$ acts transitively, are called ignored and are supplied with small letters of the Greek alphabet: $\alpha, \beta, \gamma$ varying within $1, \ldots r$. Ignored variables enter the metric tensor only through specific functions.

The rest of the variables are called essential (or non-ignored). Coordinate indices of essential variables are denoted by the letters $p, q=1, \ldots r$. The essential variables enter the metric tensor and the components of the potential of an admissible electromagnet field through arbitrary functions, each of which depends only on these variables.

If $R>r$, the group indices will be denoted by capital Latin letters: $A, B, C, D=$ $1, \ldots R$. The Independent operators $\quad X_{A}$ of the group $G_{R}$ are numbered, the same way as the local coordinates of the subspace $\quad V_{r}, \quad$ in small Greek letters. The other operators of the group $G_{R}$, are linear combinations of the independent operators $X_{\alpha}$ (with variable coefficients) and are numbered by the indices: $a, b, c, d=(r+1), \ldots, R$. The coordinate indices of the essential variables of the local canonical coordinate system are denoted by the following letters: $\quad p, q=r+1, \ldots, n$. In the metric tensor and in the components of the admissible electromagnetic field potential, essential variables enter through arbitrary functions, each of which depends only on these variables. The repeated superscripts and subscripts are summed. The subject of our consideration is the conditions imposed on the external electromagnetic field, under which there are algebras of integrals of motion linear in momenta of the classical and quantum equations of motion for a charged scalar test particle.

### 2.2 Derivation of the conditions for the existence of an admissible electromagnetic field.

For a charged test particle in an external electromagnetic field with potential $A_{i}$, the Hamilton-Jacobi equation has the form:

$$
\begin{gather*}
g^{i j} P_{i} P_{j}=\lambda .  \tag{2.1}\\
P_{i}=p_{i}+A_{i}, \quad p_{i}=\varphi_{, i}=\partial \varphi / \partial u^{i}, \quad \lambda=\text { const. }
\end{gather*}
$$

Let us consider the Klein-Gordon-Fock equation too:

$$
\hat{H} \varphi=\left(g^{i j} \hat{P}_{i} \hat{P}_{j}\right) \varphi=\lambda \varphi, \quad \hat{P}_{j}=-\imath \hat{\nabla}_{i}+A_{i} .
$$

$\hat{\nabla}_{i}$ - operator of the covariant derivative, with metric-compatible connectivity, corresponding to the operator of the partial derivative - $\hat{\partial}_{i}=\imath \hat{p}_{i} \quad$ with respect to the coordinate $\quad u_{i} ; \quad \varphi$ is a field of a scalar particle with mass $m=\sqrt{|\lambda|}$. For both equations, integrals of motion linear in momenta have the same form:

$$
\begin{equation*}
\hat{Y}_{A}=\xi_{A}^{i} P_{i}+\gamma_{A} . \tag{2.2}
\end{equation*}
$$

It was proved in the articles [11] and [16], that the conditions for the existence of an admissible electromagnetic field for the Klein-Gordon-Fock equation have exactly the same form as for the Hamilton-Jacobi equation. Therefore, we will consider only the Hamilton-Jacobi equation. All the results obtained in this case are also valid for the Klein - Gordon-Fock equation.

The equation 2.1] admits the integral of motion 2.2 if and only if the functions $\hat{H}=g^{i j} P_{i} P_{j}$ and $\quad \hat{Y}_{A}$ commute with respect to the Poisson brackets:

$$
\begin{gather*}
{\left[\hat{H}, \hat{Y}_{A}\right]_{\mathcal{P}}=\left(g^{i l} \xi_{A, l}^{j}+g^{j l} \xi_{A, l}^{i}-g_{, l}^{i j} \xi_{A}^{l}\right) p_{i} p_{j}+2 g^{i l}\left(\xi_{A}^{j} F_{i j}+\gamma_{A, i}\right) p_{l}=0}  \tag{2.3}\\
\left(F_{i j}=A_{j, i}-A_{i, j}\right)
\end{gather*}
$$

The ratios 2.3 must be satisfied for any momentum value. Equating to zero the coefficients in front of the $p_{i}$ powers, one obtains:

$$
\begin{gather*}
g^{i l} \xi_{A, l}^{j}+g^{j l} \xi_{A, l}^{i}-g_{, l}^{i j} \xi_{A}^{l}=0,  \tag{2.4}\\
\gamma_{A, i}=\xi_{A}^{j} F_{j i} . \tag{2.5}
\end{gather*}
$$

It follows from the equations 2.4 that the functions $\xi_{A}^{j}$ are the components of the Killing vector. These functions form a groups of motions $G_{R}$ of $\operatorname{rank}\left(G_{R}\right)=r$. Unlike the free Hamilton-Jacobi equation, equation 2.1, generally speaking, has no integrals of motion. The system of equations 2.5 specifies the conditions under which integrals of motion of the form 2.2 exist. This system was first developed by [11]. If the integrals of motion 2.2 form an algebra with respect to Poisson brackets with the same structure constants as for the algebra of the group operators, the following condition is satisfied:

$$
\begin{equation*}
\left[\hat{X}_{A}, \hat{X}_{B}\right]_{\mathcal{P}}=C_{A B}^{D} \hat{X}_{D} \tag{2.6}
\end{equation*}
$$

In [16] it is stated that in an admissible electromagnetic field, the integrals of motion have the form:

$$
\begin{equation*}
\hat{X}_{A}=\xi_{A}^{i} p_{i} . \tag{2.7}
\end{equation*}
$$

Let us prove this statement in the following theorem.

Theorem 1 An admissible electromagnetic field does not deform the algebra of integrals of motion for the Hamilton-Jacobi and Klein-Gordon-Fock equations. The algebra operators for a free test particle and for a charged one have the form 2.7.

## Proof.

As $\operatorname{rank}\left(G_{r}\right)=\operatorname{rank}\left\|\xi_{A}^{i}\right\|=r, \quad$ the transitivity subspace for the group motions $G_{R}$ is the $r$-dimensional space $\quad V_{r}$. We represent the dependent operators of the group as follows:

$$
\begin{equation*}
\mathbf{X}_{a}=\omega_{a}^{\alpha} \mathbf{X}_{\alpha} \tag{2.8}
\end{equation*}
$$

Here $\quad \omega_{a}^{\alpha} \quad$ functions expressed in terms of the components of the Killing vectors $\quad \xi_{A}^{i}$ as follows:

$$
\begin{equation*}
\omega_{a}^{\alpha}=\lambda_{\beta}^{\alpha} \xi_{a}^{\beta}, \quad \lambda_{\beta}^{\alpha} \xi_{\gamma}^{\beta}=\delta_{\gamma}^{\alpha}, \quad \xi_{A}^{i}=\delta_{\beta}^{i} \xi_{A}^{\beta} . \tag{2.9}
\end{equation*}
$$

Transform the functions $\gamma_{\alpha}$ as follows:

$$
\gamma_{\alpha}=\xi_{\alpha}^{i}\left(\omega_{i}-A_{i}\right) .
$$

From here it follows:

$$
\begin{equation*}
\hat{X}_{\alpha}=\xi_{\alpha}^{i}\left(p_{i}+\omega_{i}\right) . \tag{2.10}
\end{equation*}
$$

Let us show that from the conditions 2.6 it follows:

$$
\begin{equation*}
\gamma_{\alpha}=\xi_{\alpha}^{i} A_{i}=0 \rightarrow \omega_{\alpha}=0 \tag{2.11}
\end{equation*}
$$

Substitute 2.10 into 2.6. As a result, we get:

$$
\begin{equation*}
\left[Y_{\alpha}, Y_{\beta}\right]_{\mathcal{P}}=C_{\alpha \beta}^{\gamma} \xi_{\gamma}^{i} p_{i}+\omega_{\beta \mid \alpha}-\omega_{\alpha \mid \beta}=C_{\alpha \beta}^{\gamma} \xi_{\gamma}^{i}\left(p_{i}+\omega_{i}\right) \quad\left(\left.\right|_{A}=\xi_{A}^{i} \partial_{i} . \quad \omega_{\alpha}=\xi_{\alpha}^{i} \omega_{i}\right) \tag{2.12}
\end{equation*}
$$

Thus, the functions $\omega_{\alpha}$ obey the equations:

$$
\omega_{\beta \mid \alpha}-\omega_{\alpha \mid \beta}=C_{\alpha \beta}^{\gamma} \omega_{\gamma} \quad \rightarrow \quad \omega_{i, j}=\omega_{j, i} \quad \rightarrow \quad \omega_{i}=\omega_{, i} .
$$

By the gradient transformation of the potential, the function $\quad \omega_{i}$ can be turned to 0 . The theorem is proved.

Thereby, the condition for the admissible electromagnetic field existence has the form:

$$
\left(A_{j} \xi_{A}^{j}\right)_{, i}=\xi_{A}^{j} F_{i j} .
$$

Let us introduce a nonholonomic frame associated with the group $G_{R}$ :

$$
\hat{\sigma}_{j}=\sigma_{j}^{i} \hat{\partial}_{i}=\delta_{j}^{p} \hat{\partial}_{p}+\delta_{j}^{\alpha} \xi_{\alpha}^{\beta} \hat{\partial}_{\beta}, \quad \hat{e}^{j}=e_{i}^{j} \hat{d u}{ }^{i}=\delta_{p}^{j} \hat{d u}^{p}+\delta_{\alpha}^{j} \lambda_{\beta}^{\alpha} \hat{d u}{ }^{\beta},
$$

non-holonomic components of the vector potential:

$$
\mathbf{A}_{i}=\sigma_{i}^{j} A_{j}
$$

and also the potential projections onto the vector fields $\quad \xi_{A}^{i}: \quad \mathbb{A}_{A}=\xi_{A}^{i} A_{i}$. Then the equation 2.5 takes the following form:

$$
\begin{equation*}
\mathbb{A}_{A, i}=\xi_{A}^{j} F_{i j} \tag{2.13}
\end{equation*}
$$

Consider two cases separately: 1. $i=\alpha \quad$ and $\quad 2 . i=p$.

1. $i=\alpha$. The system (2.13) can be reduced to the form:

$$
\mathbb{A}_{A \mid B}=\xi_{A}^{j} \xi_{B}^{i}\left(A_{j, i}-A_{i, j}\right)=\mathbb{A}_{A \mid B}-\mathbb{A}_{B \mid A}+C_{A B}^{D} \mathbb{A}_{D}
$$

From here it follows:

$$
\begin{equation*}
\mathbb{A}_{B \mid A}=C_{A B}^{D} \mathbb{A}_{D} \tag{2.14}
\end{equation*}
$$

2. $\quad i=p$. The system (2.13) can be reduced to the following form:

$$
\left(\xi_{B}^{j} A_{j}\right)_{, p}=\xi_{B}^{j}\left(A_{j, p}-A_{p, j}\right)=\left(\xi_{B}^{j} A_{j}\right)_{, p}-A_{p \mid B}-\xi_{B, p}^{j} A_{j} .
$$

From here it follows:

$$
\begin{equation*}
A_{p \mid A}=-\xi_{A, p}^{i} A_{i}, \tag{2.15}
\end{equation*}
$$

The Systems 2.14, 2.15 need to be examined for compatibility. Let us consider separately the cases when the order of the group $G_{R}$ coincides with the dimension of the transitivity subspace $\quad V_{r} \quad(\mathrm{R}=\mathrm{r})$, and when $\quad R>r$.

### 2.3 Compatibility conditions for the case, when the order of the group $G_{R}$ coincides with the dimension of the transitivity subspace $V_{r}$.

Let us prove the compatibility of the system (2.14), (2.15). To do this, show that

$$
\begin{equation*}
\mathbf{A}_{\beta \mid \alpha \gamma}-\mathbf{A}_{\beta \mid \gamma \alpha}=C_{\alpha \beta}^{\sigma} \mathbf{A}_{\sigma \mid \gamma}-C_{\gamma \beta}^{\sigma} \mathbf{A}_{\sigma \mid \alpha} . \tag{2.16}
\end{equation*}
$$

Since

$$
\begin{gathered}
\mathbf{A}_{\beta \mid \alpha \gamma}-\mathbf{A}_{\beta \mid \gamma \alpha}=C_{\gamma \alpha}^{\sigma} \mathbf{A}_{\alpha \mid \sigma}, \\
C_{\alpha \beta}^{\sigma} \mathbf{A}_{\sigma \mid \gamma}-C_{\gamma \beta}^{\sigma} \mathbf{A}_{\sigma \mid \alpha}=\left(C_{\alpha \beta}^{\sigma} C_{\gamma \sigma}^{\rho}-C_{\gamma \beta}^{\sigma} C_{\alpha \sigma}^{\rho}\right) \mathbf{A}_{\rho},
\end{gathered}
$$

the system (2.16) is reduced to the following form:

$$
\left(C_{\sigma \rho}^{\gamma} C_{\beta \gamma}^{\alpha}+C_{\rho \beta}^{\gamma} C_{\sigma \gamma}^{\alpha}+C_{\beta \sigma}^{\gamma} C_{\rho \gamma}^{\alpha}\right) \mathbf{A}_{\alpha}=0,
$$

which is fulfilled by the Bianchi identities.
To prove the compatibility of the (2.15) system, it is necessary to show that

$$
\begin{equation*}
A_{p \mid \beta \alpha}-A_{p \mid \alpha \beta}=C_{\alpha \beta}^{\gamma} A_{p \mid \gamma}=\left(\xi_{\alpha, p}^{\sigma} \lambda_{\sigma}^{\gamma} \mathbf{A}_{\gamma}\right)_{\mid \beta}-\left(\xi_{\beta, p}^{\sigma} \lambda_{\sigma}^{\gamma} \mathbf{A}_{\gamma}\right)_{\mid \alpha} . \tag{2.17}
\end{equation*}
$$

Using the systems of equations (2.14), (2.15), as well as the consequences of the Killing equations:

$$
\xi_{\alpha, p \mid \beta}^{\gamma}-\xi_{\beta, p \mid \alpha}^{\gamma}=\xi_{\alpha, p}^{\nu} \xi_{\beta, \nu}^{\gamma}-\xi_{\beta, p}^{\nu} \xi_{\alpha, \nu}^{\gamma}+C_{\beta \alpha}^{\nu} \xi_{\nu, p}^{\gamma} .
$$

we bring the conditions (2.17) to the form:

$$
\begin{equation*}
\xi_{\beta, p}^{\sigma}\left(\xi_{\alpha, \sigma}^{\rho} \lambda_{\rho}^{\gamma}+\lambda_{\sigma \mid \alpha}^{\gamma}+\lambda_{\sigma}^{\rho} C_{\alpha \rho}^{\gamma}\right) \mathbf{A}_{\gamma}=\xi_{\beta, p}^{\sigma}\left(\xi_{\beta, \sigma}^{\rho} \lambda_{\rho}^{\gamma}+\lambda_{\sigma \mid \beta}^{\gamma}+\lambda_{\sigma}^{\rho} C_{\beta \rho}^{\gamma}\right) \mathbf{A}_{\gamma} \tag{2.18}
\end{equation*}
$$

It is easy to show that

$$
\xi_{\alpha, \sigma}^{\rho} \lambda_{\rho}^{\gamma}+\lambda_{\sigma \mid \alpha}^{\gamma}+\lambda_{\sigma}^{\rho} C_{\alpha \rho}^{\gamma}=0 .
$$

The compatibility of the system of equations (2.15) is proved.

### 2.4 Compatibility conditions for the case when the order of the group $G_{R}$ is greater than the dimension of the transitivity subspace $V_{r}$.

When studying the compatibility conditions for the systems of equations (2.14) (2.15), one cannot use the results obtained in the previous section, since in this case the systems are overflowing, because the functions $\mathbb{A}_{B}$ are linearly dependent with coefficients expressed in terms of the functions $\xi_{B}^{A}$. Since the order of the group is greater than the dimension of the transitivity subspace, $(R-r)$ operators of the group are linear combinations of $r$ of the basic operators with variable coefficients. Without loss of generality, we can assume that $\operatorname{rank}\left\|\xi_{\alpha}^{i}\right\|=r \quad \rightarrow \quad$ the first $r$ of the $\hat{X}_{\alpha}$ operators are basic. In this case $\operatorname{det}\left\|\xi_{\beta}^{\alpha}\right\| \neq 0$, and the inverse matrix $\quad\left\|\lambda_{\beta}^{\alpha}\right\|$ exists:

$$
\xi_{\gamma}^{\alpha} \lambda_{\beta}^{\gamma}=\delta_{\beta}^{\alpha} .
$$

Then:

$$
\begin{equation*}
\hat{X}_{a}=\xi_{a}^{\beta} \lambda_{\beta}^{\alpha} \hat{X}_{\alpha} \quad \rightarrow \quad \mathbb{A}_{a}=\xi_{a}^{\beta} \lambda_{\beta}^{\alpha} \mathbf{A}_{\beta} \quad \rightarrow \quad \mathbb{A}_{a}=\omega_{a}^{\alpha} \mathbf{A}_{\alpha} . \tag{2.19}
\end{equation*}
$$

Introduce the functions:

$$
\tilde{C}_{A B}^{\gamma}=C_{A B}^{\gamma}+\omega_{a}^{\gamma} C_{A B}^{a}
$$

Then from the commutation relations:

$$
\left[\hat{X}_{A}, \hat{X}_{B}\right]_{P}=C_{A B}^{D} \hat{X}_{D},
$$

one obtains the following identities:

$$
\begin{equation*}
\omega_{a \mid \alpha}^{\gamma}=\tilde{C}_{\alpha a}^{\gamma}+\omega_{a}^{\beta} \tilde{C}_{\beta \alpha}^{\gamma}, \quad \tilde{C}_{a b}^{\gamma}=\omega_{a}^{\alpha} \tilde{C}_{\alpha b}^{\gamma}-\omega_{b}^{\alpha} \tilde{C}_{\alpha a}^{\gamma} . \tag{2.20}
\end{equation*}
$$

The conditions (2.6) are reduced to the following systems of equations:

$$
\begin{gather*}
\mathbb{A}_{A \mid B}=\tilde{C}_{B A}^{\gamma} \mathbf{A}_{\gamma}  \tag{2.21}\\
A_{p \mid A}=-\xi_{A, p}^{\beta} A_{\beta}=-\xi_{A, p}^{\beta} \lambda_{\beta}^{\alpha} \mathbf{A}_{\alpha} . \tag{2.22}
\end{gather*}
$$

Let us prove that the functions $\quad \mathbf{A}_{\alpha}$ form a system of linearly dependent functions with coefficients that are rational functions of the components $\quad \xi_{\alpha}^{i}$ and structural constants. Indeed, the systems of equations (2.21), (2.21), depending on the values of the indices, can be broken down as follows:

$$
\begin{gather*}
\mathbf{A}_{\beta \mid \alpha}=\tilde{C}_{\alpha \beta}^{\gamma} \mathbf{A}_{\gamma} ;  \tag{2.23}\\
\mathbf{A}_{\beta \mid a}=\tilde{C}_{a \beta}^{\gamma} \mathbf{A}_{\gamma} ;  \tag{2.24}\\
\mathbf{A}_{b \mid A}=\tilde{C}_{A b}^{\gamma} \mathbf{A}_{\gamma} ;  \tag{2.25}\\
\mathbf{A}_{p \mid a}=\omega_{a}^{\alpha} \mathbf{A}_{p, \alpha}=-\xi_{a, p}^{\beta} \lambda_{\beta}^{\gamma} \mathbf{A}_{\gamma} ;  \tag{2.26}\\
\mathbf{A}_{p \mid \alpha}=-\xi_{\alpha, p}^{\beta} \gamma_{\beta}^{\gamma} \mathbf{A}_{\gamma} . \tag{2.27}
\end{gather*}
$$

From (2.24) it follows:

$$
\begin{equation*}
\left(\tilde{C}_{\alpha a}^{\gamma}+\omega_{a}^{\beta} \tilde{C}_{\beta \gamma}^{\gamma}\right) \mathbf{A}_{\gamma}=0 \tag{2.28}
\end{equation*}
$$

Taking into account (2.20) one obtains:

$$
\begin{equation*}
\omega_{a \mid \alpha}^{\gamma} \mathbf{A}_{\gamma}=0 . \tag{2.29}
\end{equation*}
$$

Since

$$
\xi_{a}^{\alpha}=\omega_{a}^{\beta} \xi_{\beta}^{\alpha},
$$

from (2.26), (2.27) it follows:

$$
\begin{equation*}
\omega_{a \mid p}^{\gamma} \mathbf{A}_{\gamma}=0 \rightarrow \omega_{a, i}^{\gamma} \mathbf{A}_{\gamma}=0 . \tag{2.30}
\end{equation*}
$$

Obviously, $\quad \omega_{a, i}^{\gamma} \neq 0, \quad$ otherwise $\quad \omega_{a}^{\gamma}=$ const. and $\quad G_{R}$ is reduced to $G_{r}$. The statement is proven true.

The system (2.25) is satisfied identically due to the conditions (2.20). Consider the (2.27) system. The compatibility conditions are as follows:

$$
\begin{equation*}
\mathbf{A}_{p \mid \alpha \gamma}-\mathbf{A}_{p \mid \gamma \alpha}=\left(\xi_{\gamma, p}^{\beta} A_{\beta}\right)_{\mid \alpha}-\left(\xi_{\alpha, p}^{\beta} A_{\beta}\right)_{\mid \gamma} . \tag{2.31}
\end{equation*}
$$

Using the condition (2.23), as well as the consequence from the Killing equations:

$$
\left(\xi_{\alpha \mid \beta}^{\gamma}-\xi_{\beta \mid \alpha}^{\gamma}\right)_{, p}=\omega_{a \mid p}^{\nu} \xi_{\nu}^{\gamma} \tilde{C}_{\alpha \beta}^{a}+\xi_{\nu, p}^{\gamma} \tilde{C}_{\alpha \beta}^{\gamma}
$$

we get:

$$
\begin{equation*}
\omega_{a}^{\gamma} \lambda_{\nu}^{\mu}\left(\xi_{\beta, p}^{\nu} C_{\alpha \mu}^{a}-\xi_{\alpha, p}^{\nu} C_{\beta \mu}^{a}\right) \mathbf{A}_{\gamma}=0 . \tag{2.32}
\end{equation*}
$$

Let us supplement this system with equations (2.11), (2.12):

$$
\begin{equation*}
\omega_{a, i}^{\gamma} \mathbf{A}_{\gamma}=0 . \tag{2.33}
\end{equation*}
$$

Let $K$ be the number of independent equations in the system (2.32), (2.33), and the numbers $\mu$ range from 1 to $K$. Let us randomly enumerate all pairs (in (2.33)) and all triples (in (2.32)) of subscripts in the coefficients before $\mathbf{A}_{\alpha}$ and represent these coefficients as elements of the matrix $\hat{W}(0)=\left\|W(0)_{\mu}^{\alpha}\right\| . \quad$ Thus, the functions $\quad \mathbf{A}_{\alpha} \quad$ satisfy the system of equations:

$$
\begin{equation*}
W(0)_{\mu}^{\alpha} \mathbf{A}_{\alpha}=0 . \tag{2.34}
\end{equation*}
$$

Denote:

$$
\tilde{r}=r-\operatorname{Rank}\|\hat{W}(0)\| \quad(r-\tilde{r} \neq 0),
$$

In the matrix $\hat{W}(0)$ choose an arbitrary minor $\hat{Z} \quad$ of $\operatorname{rank}=\tilde{r}$. The indices numbering the rows of this minor are denoted by the letters $\tilde{a}, \tilde{b}, \tilde{c}$. The rest of the minor columns will be denoted by the letters $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. These letters will also be used to number the columns that form the non-singular square matrix $\hat{\Omega}$ of $\operatorname{rank}=\hat{r} \quad$ in this minor.

Then the solution of the system of equations (2.23) can be represented as

$$
\begin{equation*}
\mathbf{A}_{\tilde{a}}=\theta_{\tilde{a}}^{\tilde{\alpha}} \mathbf{A}_{\tilde{\alpha}} \tag{2.35}
\end{equation*}
$$

where

$$
\begin{equation*}
\theta_{\tilde{a}}^{\tilde{\alpha}}=-\|\hat{\Omega}\|_{\tilde{a}}^{\tilde{b}} Z_{\tilde{b}}^{\hat{\alpha}} . \tag{2.36}
\end{equation*}
$$

The functions (2.36) convert the systems of equations (2.32) and (2.33) into identities:

$$
\omega_{a \mid i}^{\tilde{\gamma}}+\omega_{a \mid i}^{\tilde{a}} \mid \theta_{\tilde{\boldsymbol{q}}}^{\tilde{\alpha}}=0 \quad\left(\omega_{a}^{\tilde{\gamma}}+\omega_{a}^{\tilde{a}} \theta_{\tilde{\tilde{a}}}^{\tilde{\gamma}}\right) \lambda_{\nu}^{\mu}\left(\xi_{\beta, p}^{\nu} C_{\alpha \mu}^{a}-\xi_{\alpha, p}^{\nu} C_{\beta \mu}^{a}\right)=0 .
$$

Let's go back to the system of equations (2.23). Substitute expressions (2.34) into it. As a result, we get the following subsystems:

$$
\begin{align*}
& \mathbf{A}_{\tilde{\alpha} \mid \alpha}=\tilde{C}_{\alpha \tilde{\alpha}}^{\gamma} \mathbf{A}_{\gamma}=\left(\tilde{C}_{\alpha \tilde{\alpha}}^{\tilde{\gamma}}+\theta_{\tilde{b}}^{\tilde{\gamma}} \tilde{C}_{\alpha \tilde{\alpha}}^{\tilde{b}}\right) \mathbf{A}_{\tilde{\gamma}} .  \tag{2.37}\\
& \mathbf{A}_{\tilde{a} \mid \alpha}=\tilde{C}_{\alpha \tilde{a}}^{\gamma} \mathbf{A}_{\gamma}=\left(\tilde{C}_{\alpha \tilde{a}}^{\tilde{\tilde{a}}}+\theta_{\tilde{b}}^{\tilde{\tilde{b}}} \tilde{C}_{\alpha \tilde{a}}^{\tilde{b}}\right) \mathbf{A}_{\tilde{\gamma}} . \tag{2.38}
\end{align*}
$$

Using (2.34) and (2.37), the system of equations (2.38) can be represented as a system of algebraic equations:

$$
\begin{equation*}
\left(\theta_{\tilde{a} \mid \alpha}^{\tilde{\gamma}}+\theta_{\tilde{a}}^{\tilde{\beta}}\left(\tilde{C}_{\alpha \tilde{\beta}}^{\tilde{\gamma}}+\theta_{\tilde{b}}^{\tilde{\gamma}} \tilde{C}_{\alpha \tilde{\beta}}^{\tilde{b}}\right)-\left(\tilde{C}_{\alpha \tilde{a}}^{\tilde{\gamma}}+\theta_{\tilde{b}}^{\tilde{\gamma}} \tilde{C}_{\alpha \tilde{a}}^{\tilde{b}}\right)\right) \mathbf{A}_{\tilde{\gamma}}=0 \tag{2.39}
\end{equation*}
$$

Combining (2.39) and the compatibility conditions for the systems of equations, (2.37)(2.27), one obtains new compatibility conditions in the form similar to the conditions (2.34):

$$
W(1)_{\mu_{1}}^{\tilde{\alpha}} \mathbf{A}_{\tilde{\alpha}}=0
$$

The index $\mu_{1}$ is constructed in the same way as the index in the matrix $W(0)_{\mu}^{\alpha}$.
If $W(1)_{\mu_{1}}^{\tilde{\alpha}}=0$, the solution has the form (2.35) and the systems of equations (2.13), (2.15) are compatible.

If $W(1)_{\mu_{1}}^{\tilde{\alpha}} \neq 0, \quad$ this procedure should be repeated until at step $(\varkappa)$ the condition is met $W(\varkappa)_{\mu_{\varkappa}}^{\tilde{\alpha}_{\varkappa}}=0$, and the remaining system of differential equations:

$$
\begin{equation*}
\mathbf{A}_{\tilde{a}_{\varkappa} \mid \tilde{\alpha}}=\tilde{C}_{\tilde{\alpha}\left(\tilde{a}_{\varkappa}\right)}^{\tilde{\sigma}_{\varkappa}} \mathbf{A}_{\gamma_{\varkappa}}, \quad \mathbf{A}_{\tilde{\alpha}}=\vartheta_{\tilde{\alpha}}^{\tilde{\alpha}_{\varkappa}} \mathbf{A}_{\left(\tilde{a}_{\varkappa)}\right)} \tag{2.40}
\end{equation*}
$$

will be compatible. Otherwise, the only solution to the system (2.23) will be - $A_{i}=0$. Note that for a specific space on which the group $G_{R}$ acts, $\omega_{a}^{\alpha}$, $\theta_{\tilde{a}}^{\tilde{\gamma}}, \vartheta_{\alpha}^{\left(\tilde{\alpha}_{N}\right)}$, are the given functions. Therefore, checking the compatibility conditions does not imply the solution to any equations.

## 3 Admissible electromagnetic fields for the groups of motions $G_{4}$ acting on transitivity subspaces of a spacetime manifold

The complete classification of admissible electromagnetic field potential is an obvious continuation of the classification problem, solved by A.Z. Petrov ([9). Despite its obvious importance, the first papers ([11, [12], [16], [18]) on this topic appeared quite recently. This section continues the research begun in these articles. The potentials of all admissible electromagnetic fields are found for the case when the group of motions $G_{4}$ acts nontransitively on the space $V_{4}$. The results obtained in the previous section are used.

Let us clarify the notation of the indices used in this section. As is known, every fourparameter Lie group $G_{4}$ has a three-parameter subgroup $G_{3}$ (see [9]). Therefore, as the basic operators $\quad \hat{X}_{\alpha}$ of the group $G_{4}$, one can choose the operators of the subgroup $\quad G_{3}$. The indices introduced earlier in this section vary within the following limits: $i, j=0, \ldots 3$;
$\alpha, \beta, \gamma,=1, \ldots, 3 ; \quad A, B=1, \ldots 4 ; \quad a=4 ; \quad$ The following algorithm is used for constructing the potentials of an admissible electromagnetic field.

1. Using the known Killing vectors $\xi_{A}^{\alpha}$ of the group $G_{4}$ there can be found the matrix $\quad\left\|\lambda_{\beta}^{\alpha}\right\| \quad$ and the functions $\quad \omega^{\alpha}=-\lambda_{\beta}^{\alpha} \xi_{4}^{\beta}$.
2. The functions $\quad \tilde{C}_{A B}^{\alpha}$ are calculated and the matrix

$$
\left\|W_{\alpha}^{\gamma}(0)\right\|=\left\|\tilde{C}_{\alpha 4}^{\gamma}+\omega^{\beta} \tilde{C}_{\beta \alpha}^{\gamma}\right\|
$$

is constructed using the (2.20) formula.
3. The solution to the system of equations is found

$$
\begin{equation*}
W_{\alpha}^{\gamma}(0) \mathbf{A}_{\gamma}=0 \tag{3.1}
\end{equation*}
$$

in the following form:

$$
\mathbf{A}_{\tilde{a}}=\theta_{\tilde{a}}^{\tilde{\alpha}} \mathbf{A}_{\tilde{\alpha}} . \quad \operatorname{rank}\left\|\mid \theta_{\tilde{a}}^{\tilde{\alpha}}\right\|=(3-\tilde{r}) .
$$

4. The component $\quad \mathbf{A}_{\mathbf{4}}$ is calculated by the formula:

$$
\mathbf{A}_{a}=\mathbf{A}_{4}=\omega^{\alpha} \mathbf{A}_{\alpha}
$$

the system (2.11) compatibility is checked and the solutions to the remaining equations

$$
\begin{equation*}
\mathbf{A}_{\tilde{\alpha} \mid \tilde{\beta}}=\tilde{C}_{\tilde{\beta} \tilde{\alpha}}^{\tilde{\gamma}} \mathbf{A}_{\tilde{\gamma}} \tag{3.2}
\end{equation*}
$$

are found.
As already noted, group operators, spacetime metrics and canonical coordinate systems for all groups of motions $G_{R}$ were found in [9]. Below, for ease of use, these formulas and structural constants are given for each considered group. Moreover in every subsection the results of calculation are presented in the following order. Matrices $\hat{\lambda}, \hat{W}$, are given as well as nonholonomic $\quad \mathbf{A}_{A}$ and holonomic $\quad A_{i}$ components of the potentials of the admissible electromagnetic field. The functions denoted by the letter $a$ with a single lower index depend only on the variable $u^{0}, \quad\left(e_{0}\right)^{2}=1$

The transitivity hypersurface of spacetime $\quad V_{4}$ in the canonical coordinate system is given by the equation:

$$
\phi\left(u^{i}\right)=u^{0}=\text { const } .
$$

If $g^{i j} \phi_{, i} \phi_{, j} \neq 0$, the surface is called non-isotropic (non-null) and is denoted by $V_{3}$. Otherwise the hypersurface is called isotropic (null) and is denoted by $V_{3}^{*}$.

Each case when the group $G_{4}$ acts transitively on the hypersurfaces $V_{3}$ and $V_{3}^{*}$, also when the mentioned above subgroups $\quad G_{3}$ act transitively on $V_{2}$ and $V_{2}^{*}$ (they are the subspaces of the hypersurface $\left.\quad V_{3}\right)$ is considered separately.

The notation $G_{R}(N)$ means that this group has an order $R$ and the group structure is of the type $N$ according to the Bianchi classification. [19]

### 3.1 The groups $\boldsymbol{G}_{4}$ act transitively on a non null subspace $\boldsymbol{V}_{3}$

When the group $\quad G_{4} \quad$ acts transitively on a non null subspace $\quad V_{r}, \quad r \leq 3 \quad$ it follows from [9] that $\left(\xi_{A}^{\alpha}\right)_{, 0}=0$. According to (2.27) this means: $A_{0}=A_{0}\left(u^{0}\right)$, which is equivalent to

$$
A_{0}=0
$$

### 3.1.1 Group $G_{4}(I)$

Metric, in which this group acts, has the form:

$$
d s^{2}=2 a_{1} d u^{1} d u^{3}+a_{2}\left(d u^{2}+u^{1} d u^{3}\right)^{2}+e_{0}\left(d u^{0}\right)^{2}
$$

Let us present operators of the group:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=u^{2} p_{1}-p_{3}, \quad X_{4}=u^{2} p_{2}-u^{3} p_{3}
$$

and structural constants:

$$
C_{A 1}^{\gamma}=0, \quad C_{23}^{\gamma}=\delta_{1}^{\gamma}, \quad C_{24}^{\gamma}=\delta_{2}^{\gamma}, \quad C_{34}^{\gamma}=-\delta_{3}^{\gamma} .
$$

Matrix $\quad \hat{\lambda}, \quad \hat{W}(0)$ and functions $\quad \omega^{\alpha}$, which defined by the formulas (2.9), (2.34), have the form:

$$
\begin{gathered}
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
u^{2} & 0 & -1
\end{array}\right), \\
\omega^{\alpha}=-u^{2} u^{3} \delta_{1}^{\alpha}+u^{2} \delta_{2}^{\alpha}+u^{3} \delta_{3}^{\alpha}
\end{gathered}
$$

Matrix:

$$
\left\|W_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
0 & 0 & 0 \\
u^{3} & -1 & 0 \\
-u^{2} & 0 & 1
\end{array}\right)
$$

From the system of equations (3.1) it follows:

$$
\begin{equation*}
\mathbf{A}_{\mathbf{2}}=u^{3} \mathbf{A}_{\mathbf{1}}, \quad \mathbf{A}_{\mathbf{3}}=u^{2} \mathbf{A}_{\mathbf{1}} \tag{3.3}
\end{equation*}
$$

Substitute the resulting expressions into (3.2). As a result, we get:

$$
\begin{gathered}
\mathbf{A}_{1,2}=\mathbf{A}_{2,2}=\mathbf{A}_{\alpha, 1}=\mathbf{A}_{1,3}=\mathbf{A}_{3,3}=0 \\
\mathbf{A}_{3,2}=\mathbf{A}_{1}, \quad \mathbf{A}_{2,3}=\mathbf{A}_{1}
\end{gathered}
$$

The solution is:

$$
\mathbf{A}_{1}=a_{0}, \quad \mathbf{A}_{2}=a_{0} u^{3}, \quad \mathbf{A}_{3}=a_{0} u^{2}
$$

The components of the potential of the admissible electromagnetic field are found by the formula

$$
\begin{equation*}
A_{\alpha}=\mathbf{A}_{\beta} \lambda_{\alpha}^{\beta}, \quad A_{0}=0 \tag{3.4}
\end{equation*}
$$

and have the form:

$$
A_{0}=0, \quad A_{1}=a_{0}, \quad A_{2}=a_{0} u^{3}, \quad A_{3}=0
$$

### 3.1.2 Group $G_{4}(I I I)$.

Metric:

$$
d s^{2}=a_{1}\left(\left(d u^{1}\right)^{2}+\left(d u^{3}\right)^{2}\right)+a_{2}\left(d u^{2}+u^{1} d u^{3}\right)^{2}+e_{0}\left(d u^{0}\right)^{2} .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=u^{2} p_{1}+p_{3}, \quad X_{4}=\frac{\left(u^{2}\right)^{2}+\left(u^{3}\right)^{2}}{2} p_{1}+u^{3} p_{2}+u^{2} p_{3}
$$

Structural constants:

$$
C_{23}^{\gamma}=\delta_{1}^{\gamma}, \quad C_{24}^{\gamma}=\delta_{3}^{\gamma}, \quad C_{34}^{\gamma}=-\delta_{2}^{\gamma} .
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-u^{2} & 0 & 1
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=\frac{\left(u^{2}\right)^{2}+\left(u^{2}\right)^{2}}{2} \delta_{1}^{\alpha}+u^{3} \delta_{2}^{\alpha}+u^{2} \delta_{3}^{\alpha}
$$

Matrix $W_{\beta}^{\alpha}$ :

$$
\left\|W_{\beta}^{\alpha}\right\|=-\left(\begin{array}{ccc}
0 & 0 & 0 \\
-u^{2} & 0 & 1 \\
u^{3} & 1 & 0
\end{array}\right)
$$

From the system of equations (3.1) it follows:

$$
\begin{equation*}
\mathbf{A}_{\mathbf{2}}=-u^{3} \mathbf{A}_{\mathbf{1}}, \quad \mathbf{A}_{\mathbf{3}}=u^{2} \mathbf{A}_{\mathbf{1}} . \tag{3.5}
\end{equation*}
$$

Substitute the resulting expressions into (3.2). As a result, we get:

$$
\begin{gathered}
\mathbf{A}_{1, \alpha}=0, \quad \mathbf{A}_{2,2}=\mathbf{A}_{2,1}=\mathbf{A}_{3,1}=\mathbf{A}_{3,3}=0 \\
\mathbf{A}_{3,2}=\mathbf{A}_{1}, \quad \mathbf{A}_{2,3}=-\mathbf{A}_{1}
\end{gathered}
$$

The solution is:

$$
\mathbf{A}_{1}=a_{0}, \quad \mathbf{A}_{2}=-a_{0} u^{3}, \quad \mathbf{A}_{3}=a_{0} u^{2}
$$

The components of the potential of the admissible electromagnetic field are found by the formula (3.4) and have the form:

$$
A_{0}=0, \quad A_{1}=a_{0}, \quad A_{2}=-a_{0} u^{3}, \quad A_{3}=0
$$

### 3.1.3 Group $G_{4}(I V)$

Metric:

$$
d s^{2}=a_{1}\left(d u^{1}\right)^{2}+2 a_{2} \exp u^{1} d u^{2} d u^{3}+e_{0}\left(d u^{0}\right)^{2} .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=u^{1} p_{1}+p_{3}, \quad X_{4}=u^{2} p_{2}+p_{3} .
$$

Structural constants:

$$
C_{13}^{\gamma}=\delta_{1}^{\gamma}, \quad C_{24}^{\gamma}=\delta_{2}^{\gamma} .
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-u^{2} & 0 & 1
\end{array}\right),
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=-u^{1} \delta_{1}^{\alpha}+\delta_{3}^{\alpha}
$$

Matrix $W_{\beta}^{\alpha}$ :

$$
W_{\beta}^{\alpha}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
u^{1} & 0 & 0
\end{array}\right) .
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{\mathbf{1}}=\mathbf{A}_{\mathbf{2}}=0
$$

From the system (3.2). it follows:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=a_{0}
$$

The components of the potential of the admissible electromagnetic field are found by the formula (3.4) and have the form:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=a_{0} .
$$

### 3.1.4 Group $G_{4}(V)$

Metric:

$$
d s^{2}=a_{1}\left(d u^{1}\right)^{2}+a_{2}\left(\left(d u^{2}\right)^{2}+\left(d u^{3}\right)^{2}\right) \exp 2 u^{1}+e_{0}\left(d u^{0}\right)^{2} .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=u^{1} p_{1}+u^{2} p_{2}-p_{3}, \quad X_{4}=-u^{2} p_{1}+u^{1} p_{2} .
$$

Structural constants:

$$
C_{13}^{\gamma}=C_{42}^{\gamma}=\delta_{1}^{\gamma}, \quad C_{23}^{\gamma}=C_{14}^{\gamma}=\delta_{2}^{\gamma} .
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
u^{1} & u^{2} & -1
\end{array}\right),
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=-u^{2} \delta_{1}^{\alpha}+u^{1} \delta_{2}^{\alpha}
$$

Matrix $W_{\beta}^{\alpha}$ :

$$
W_{\beta}^{\alpha}=\left(\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & 0 \\
u^{2} & -u^{1} & 0
\end{array}\right) .
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{\mathbf{1}}=\mathbf{A}_{\mathbf{2}}=0
$$

Substitute into the system (3.2). As a result, we get:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=a_{0}
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=a_{0}
$$

### 3.1.5 Group $G_{4}(V I I)$

Metric:

$$
d s^{2}=4 a_{1} d u^{1}\left(d u^{2}-u^{2} d u^{3}\right)+a_{2}\left(d u^{2}-u^{2} d u^{3}\right)^{2}-a_{1}\left(d u^{3}\right)^{2}+e_{0}\left(d u^{0}\right)^{2} .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2} \exp u^{3}, \quad X_{3}=p_{3}, \quad X_{4}=\left(p_{1}-\left(u^{2}\right)^{2} p_{2}-2 u^{2} p_{3}\right) \exp -u^{3} .
$$

Structural constants:

$$
C_{1 \alpha}^{\gamma}=0, \quad C_{32}^{\gamma}=\delta_{2}^{\gamma}, \quad C_{42}^{\gamma}=2 \delta_{3}^{\gamma}, \quad C_{43}^{A}=\delta_{4}^{A} \quad \rightarrow \quad \tilde{C}_{43}^{\gamma}=\omega^{\gamma}
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \exp -u^{3} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=\left(\delta_{1}^{\alpha}-\left(u^{2}\right)^{2} \exp \left(-u^{3}\right) \delta_{2}^{\alpha}-2 u^{2} \delta_{3}^{\alpha}\right) \exp -u^{3} .
$$

Matrix $W_{\beta}^{\alpha}$ :

$$
W_{\beta}^{\alpha}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & u^{2} \exp \left(-u^{3}\right) & 1 \\
\exp \left(-u^{3}\right) & -2\left(u^{2} \exp \left(-u^{3}\right)\right)^{2} & -2 u^{2} \exp \left(-u^{3}\right)
\end{array}\right) .
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{\alpha, 1}=0, \quad \mathbf{A}_{\alpha}=-u^{2} \exp \left(-u^{3}\right) \mathbf{A}_{\mathbf{2}} .
$$

Substitute this into the system (3.2). As a result, we get:

$$
\mathbf{A}_{\alpha, 1}=0, \quad \mathbf{A}_{\alpha, 2}=-\delta_{\alpha 3} \exp \left(-u^{3}\right) \mathbf{A}_{2} \quad \mathbf{A}_{\alpha, 3}=\mathbf{A}_{2} \delta_{\alpha 2}
$$

From here it follows:

$$
\mathbf{A}_{1}=0, \quad \mathbf{A}_{2}=a_{0} \exp u^{3}, \quad \mathbf{A}_{3}=-a_{0} u^{2}
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=0, \quad A_{2}=a_{0}, \quad A_{3}=-a_{0} u^{2} .
$$

### 3.1.6 Group $G_{4}(I V)$

Metric:

$$
d s^{2}=a_{1}\left(\left(d u^{1}\right)^{2}+\sin ^{2} u^{1}\left(d u^{2}\right)^{2}\right)+a_{2}\left(\cos u^{1} d u^{2}+d u^{3}\right)^{2}+e_{0}\left(d u^{0}\right)^{2} .
$$

Group operators:

$$
\begin{aligned}
& X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=-\frac{\sin u^{3} \sin u^{1}}{\cos u^{3}} p^{1}+\frac{\sin u^{1}}{\sin u^{3}} p_{2}+\cos u^{1} p_{3}, \quad X_{4}=\frac{\partial X_{3}}{\partial u^{2}} . \\
& X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=-\frac{\cos u^{3} \sin u^{1}}{\sin u^{3}} p^{1}+\frac{\varepsilon \sin u^{1}}{\sin u^{3}} p_{2}+\cos u^{1} p_{3}, \quad X_{4}=\frac{\partial X_{3}}{\partial u^{2}} .
\end{aligned}
$$

Structural constants:

$$
C_{13}^{A}=\delta_{4}^{A} \rightarrow \tilde{C}_{13}^{\gamma}=\omega^{\gamma}, \quad C_{41}^{A}=\delta_{3}^{A}, \quad C_{34}^{A}=\delta_{1}^{A} .
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{\sin u^{1} \cos u^{3}}{\cos u^{1} \sin u^{3}} & -\frac{\sin u^{1}}{\cos u^{1} \sin u^{3}} & \frac{1}{\cos u^{1}}
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=-\frac{\delta_{1}^{\alpha} \cos u^{3}}{\cos u^{1} \sin u^{3}}+\frac{\delta_{2}^{\alpha}}{\cos u^{1} \sin u^{3}}-\frac{\delta_{3}^{\alpha} \sin u^{1}}{\cos u^{1}}
$$

Elements of the matrix $W_{\beta}^{\alpha}$ :

$$
W_{\gamma}^{\alpha}=\delta_{\alpha 1}\left(\delta_{3}^{\gamma}+\omega^{3} \omega^{\gamma}\right)-\delta_{\alpha 3}\left(\delta_{1}^{\gamma}+\omega^{1} \omega^{\gamma}\right) .
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{3}=-\mathbf{A}_{\alpha} \omega^{\alpha} \omega^{3}, \quad \mathbf{A}_{1}=-\mathbf{A}_{\alpha} \omega^{\alpha} \omega^{2}
$$

Denote:

$$
\mathbf{A}=-\mathbf{A}_{\alpha} \omega^{\alpha} \quad \rightarrow \quad \mathbf{A}_{3}=\mathbf{A} \omega^{3}, \quad \mathbf{A}_{1}=\mathbf{A} \omega^{2}
$$

From here it follows:

$$
\begin{equation*}
\mathbf{A}=-\mathbf{A}_{2} \sin u^{3} \cos u^{1} \quad \mathbf{A}_{1}=\mathbf{A}_{2} \cos u^{3}, \quad \mathbf{A}_{3}=\mathbf{A}_{2} \sin u^{1} \sin u^{3} \tag{3.6}
\end{equation*}
$$

From the system (3.2) we get

$$
\mathbf{A}_{2, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{2}=a_{0} .
$$

Thus, the components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{3}=0, \quad A_{2}=a_{0}, \quad A_{1}=a_{0} \cos u^{3}
$$

### 3.2 The subgroups $G_{3}$ of the group $G_{4}$ act transitively on subspce $V_{2}$ of $V_{3}$

### 3.2.1 $\quad G_{3}$ is a subgroup of the group $G_{4}(I)$

. Metric:

$$
d s^{2}=a_{1}\left(u^{3}\right)^{\frac{1}{c-1}}\left(2 d u^{1} d u^{3}+\left(d u^{2}\right)^{2}\right)+a_{2}\left(\frac{d u^{3}}{u^{3}}\right)^{2}+e_{0}\left(d u^{0}\right)^{2} .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=2 c u^{1} p_{1}+u^{2} p_{2}+2(1-c) u^{3} p_{3}, \quad X_{4}=u^{2} p_{1}-u^{3} p_{2} .
$$

Structural constants:

$$
C_{13}^{A}=2 c \delta_{1}^{A} \quad C_{23}^{A}=\delta_{2}^{A}, \quad C_{24}^{A}=\delta_{1}^{A} C_{34}^{A}=(1-2 c) \delta_{4}^{A} \quad \rightarrow \quad \tilde{C}_{34}^{\alpha}=(1-2 c)\left(u^{3} \delta_{2}^{\alpha}-u^{2} \delta_{1}^{\alpha}\right) .
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{c u^{1}}{(c-1) u^{3}} & \frac{u^{2}}{2(c-1) u^{3}} & \frac{1}{2(1-c) u^{3}} .
\end{array}\right),
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=u^{2} \delta_{1}^{\alpha}-u^{3} \delta_{2}^{\alpha} .
$$

Matrix $\hat{W}$ :

$$
W_{\alpha}^{\gamma}=-\delta_{1}^{\gamma}\left(\delta_{\alpha 2}+u^{2} \delta_{\alpha 3}\right)+2(1-c) \delta_{2}^{\gamma} \delta_{\alpha 3} .
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{1}=\mathbf{A}_{2}=0
$$

Taking this into account the (2.38) will take the form:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=2(1-c) a_{0}
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{3}=A_{2}=0, \quad A_{1}=\frac{a_{0}}{u^{3}}
$$

### 3.2.2 $\quad G_{3}$ is a subgroup of the group $G_{4}(I I)$

. Metric:

$$
d s^{2}=a_{1}\left(2 d u^{1} d u^{3}+\left(d u^{2}\right)^{2}\right) \exp -2 u^{3}+a_{2}\left(u^{3}\right)^{2}+e_{0}\left(d u^{0}\right)^{2} .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=2 u^{1} p_{1}+u^{2} p_{2}+p_{3}, \quad X_{4}=u^{2} p_{1}-u^{3} p_{2} .
$$

Structural constants:

$$
\left.C_{13}^{A}=2 \delta_{1}^{A}, \quad C_{23}^{A}=\delta_{2}^{A}, \quad C_{24}^{A}=\delta_{1}^{A}, \quad C_{43}^{A}=\delta_{2}^{A}+\delta_{4}^{A} \quad \rightarrow \quad \tilde{C}_{34}^{\alpha}=\left(u^{3}-1\right)\right) \delta_{2}^{\alpha}-u^{2} \delta_{1}^{\alpha} .
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 u^{1} & -u^{2} & 1
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=u^{2} \delta_{1}^{\alpha}-u^{3} \delta_{2}^{\alpha}
$$

Matrix $\hat{W}$ :

$$
W_{\alpha}^{\gamma}=\delta_{2}^{\gamma}\left(-\delta_{\alpha 2}+\delta_{\alpha 3}\right)-u_{2} \delta_{1}^{\gamma} \delta_{\alpha 3} .
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{1}=\mathbf{A}_{2}=0
$$

Taking this into account the (2.38) will take the form:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=a_{0}
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=a_{0} .
$$

### 3.2.3 $G_{3}$ is a subgroup of the group $G_{4}(I I I)$

Metric:

$$
\begin{gathered}
d s^{2}=R_{3}^{-1}\left(a_{1} S\left(2 d u^{1} d u^{3}+\left(d u^{2}\right)^{2}\right)+a_{2}\left(u^{3}\right)^{2}\right)+e_{0}\left(d u^{0}\right)^{2}, \\
R_{3}=\left(u_{3}\right)^{2}+2 u^{3} \sin c+1, \quad S=\exp \left((\sin 2 c) \arctan \frac{u^{3}+\sin c}{\cos c}\right) .
\end{gathered}
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=\frac{\left(4 u^{1} \sin c+\left(u^{2}\right)^{2}\right) p_{1}}{2}-u^{2} u^{3} p_{2}-R_{3} p_{3}, \quad X_{4}=u^{2} p_{1}-u^{3} p_{2}
$$

Structural constants:
$C_{13}^{A}=2 \delta_{1}^{A} \sin c, \quad C_{23}^{A}=\delta_{1}^{A}, \quad C_{24}^{A}=\delta_{1}^{A}, \quad C_{34}^{A}=\delta_{2}^{A}-\delta_{4}^{A} \sin c \quad \rightarrow \quad \tilde{C}_{34}^{\alpha}=\delta_{2}^{\alpha}-2\left(\delta_{1}^{\alpha} u^{2}-\delta_{2}^{\alpha} u^{3}\right) \sin c$.
Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{\left(4 u^{1} \sin c+\left(u^{2}\right)^{2}\right)}{2 R_{3}} & \frac{u^{2} u^{3}}{R_{3}} & -\frac{1}{R_{3}}
\end{array}\right) .
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=u^{2} \delta_{1}^{\alpha}-u^{3} \delta_{2}^{\alpha}
$$

Matrix $\hat{W}$ :

$$
W_{\alpha}^{\gamma}=\delta_{1}^{\gamma}\left(-\delta_{\alpha_{2}}+u^{2} u^{3} \delta_{\alpha 3}\right)-R_{3} \delta_{2}^{\gamma} \delta_{\alpha 3} .
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{1}=\mathbf{A}_{3}=0
$$

Taking this into account the (2.38) will take the form:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=-a_{0} .
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=\frac{a_{0}}{R_{3}}
$$

### 3.2.4 $G_{3}$ are subgroup of the groups $G_{4}(V I I)-G_{4}(V I I I)$

If we introduce the functions:

$$
\operatorname{st}\left(u^{\nu}\right), \quad \operatorname{ct}\left(u^{\nu}\right), \quad \operatorname{tn}\left(u^{\nu}\right)=\frac{\operatorname{st}\left(u^{\nu}\right)}{\operatorname{ct}\left(u^{\nu}\right)}, \quad \operatorname{ctn}\left(u^{\nu}\right)=\operatorname{tn}^{-1}\left(u^{\nu}\right)
$$

such that

$$
\left(s t\left(u^{\nu}\right)\right)_{, \nu}=\operatorname{ct}\left(u^{\nu}\right), \quad\left(c t\left(u^{\nu}\right)\right)_{, \nu}=-e_{\nu} s t\left(u^{\nu}\right) \quad\left(c t\left(u^{\nu}\right)\right)^{2}+e_{\nu}\left(s t\left(u^{\nu}\right)\right)^{2}=\left(e_{\nu}\right)^{2}=1,
$$

then expressions for metrics of spaces admitting these groups and operators of these groups can be combined.

Metrics:

$$
d s^{2}=a_{1}\left(e\left(d u^{1}\right)^{2} c t\left(u^{2}\right)+\left(d u^{2}\right)^{2}\right)+a_{2}\left(u^{3}\right)^{2}+e_{0}\left(d u^{0}\right)^{2} \quad e=e_{1} e_{2} .
$$

Group operators:

$$
X_{1}=p_{1} e_{2} \operatorname{tn}\left(u^{2}\right) s t\left(u^{1}\right)+\operatorname{ct}\left(u^{1}\right) p^{2}, \quad X_{2}=-e_{1}\left(X_{1}\right)_{1}, \quad X_{3}=p_{3}, \quad X_{4}=p_{1} .
$$

Structural constants:

$$
C_{1}^{A}=-e \delta_{4}^{A}, \quad C_{14}^{A}=e_{1} \delta_{2}^{A}, \quad C_{24}^{A}=-\delta_{1}^{A} \quad \rightarrow \quad \tilde{C}_{12}^{\alpha}=\operatorname{ctn}\left(u^{2}\right)\left(-\operatorname{st}\left(u^{1}\right) \delta_{1}^{\alpha}+\operatorname{ct}\left(u^{1}\right) \delta_{2}^{\alpha}\right)
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
\operatorname{ctn}\left(u^{2}\right) s t\left(u^{1}\right) e & -\operatorname{ctn}\left(u^{2}\right) c t\left(u^{1}\right) e & 0 \\
c t\left(u^{1}\right) & \operatorname{st}\left(u^{1}\right) e^{1} & 0 \\
0 & 0 & 1
\end{array}\right) .
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=\operatorname{ctn}\left(u^{2}\right)\left(e \delta_{1}^{\alpha} \operatorname{st}\left(u^{1}\right)-\operatorname{ct}\left(u^{1}\right) \delta_{2}^{\alpha}\right)+\delta_{3}^{\alpha} .
$$

Matrix $\hat{W}$ :

$$
\left\|W_{\alpha}^{\gamma}\right\|=\left(\begin{array}{ccc}
e \omega^{1} \omega^{2} & e\left(\omega^{2}\right)^{2} & 0 \\
-1-e\left(\omega^{1}\right)^{2} & -e \omega^{1} \omega^{2} & 0 \\
0 & 0 & 0
\end{array}\right)
$$

From the system of equations (3.1) it follows:

$$
\mathbf{A}_{1}=\mathbf{A}_{2}=0 .
$$

Taking this into account the (2.38) will take the form:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=-a_{0} .
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=a_{0} .
$$

### 3.3 The groups $G_{4}$ act transitively on a null subspace $V_{3}^{*}$

### 3.3.1 Group $G_{4}(I)$

Metric:

$$
\left.d s^{2}=a_{1} \exp \left(-2 u^{3}\right)\left(2 d u^{1} d u^{0}+\left(d u^{2}\right)^{2}\right)+a_{2}\left(u^{3}\right)^{2}\right)+e_{0}\left(d u^{0}\right)^{2},
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=2 u^{1} p_{1}+u^{2} p_{2}+p_{3}, \quad X_{4}=u^{2} p_{1}-u^{0} p_{2} .
$$

(( Only for this case the condition $\quad\left(\xi_{A}^{\alpha}\right)_{0} \neq 0$ arises.)) Structural constants:

$$
C_{13}^{A}=2 \delta_{1}^{A}, \quad C_{23}^{A}=\delta_{2}^{A}, \quad C_{24}^{A}=\delta_{1}^{A}, \quad C_{43}^{A}=2 \delta_{4}^{A} \quad \rightarrow \quad \tilde{C}_{43}^{\alpha}=2 \omega^{\alpha}=2\left(u^{2} \delta_{1}^{\alpha}-u^{0} \delta_{2}^{\alpha}\right)
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 u^{1} & -u^{2} & 1
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=u^{2} \delta_{1}^{\alpha}-u^{3} \delta_{2}^{\alpha} .
$$

Matrix $\hat{W}$ :

$$
W_{i}^{\gamma}=-\delta_{2}^{\gamma} \delta_{i 0}-\delta_{1}^{\gamma} \delta_{i 2}+u^{0} \delta_{2}^{\gamma} \delta_{i 3} .
$$

From the system of equations $\quad W_{i}^{\gamma} \mathbf{A}_{\gamma}=0 \quad$ it follows:

$$
\mathbf{A}_{1}=\mathbf{A}_{2}=0
$$

Taking this into account the (2.38) will take the form:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=-a_{0} .
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=a_{0} .
$$

### 3.3.2 Group $G_{4}$

Metric:

$$
d s^{2}=4 d u^{3} d u^{0}+a \exp \left(-2 u^{3}\right)\left(\left(d u^{1}\right)^{2}+\left(d u^{2}\right)^{2}\right),
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=u^{1} p_{1}+u^{2} p_{2}+p_{3}, \quad X_{4}=u^{1} p_{2}-u^{2} p_{1} .
$$

Structural constants:

$$
C_{13}^{A}=-\delta_{1}^{A}, \quad C_{23}^{A}=\delta_{2}^{A}, \quad C_{14}^{A}=\delta_{2}^{A}, \quad C_{24}^{A}=-\delta_{1}^{A},
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 u^{1} & -u^{2} & 1
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=u^{1} \delta_{2}^{\alpha}-u^{2} \delta_{1}^{\alpha} .
$$

Matrix $\hat{W}$ :

$$
W_{\alpha}^{\gamma}=-\delta_{2}^{\gamma} \delta_{\alpha 1}+\delta_{1}^{\gamma} \delta_{\alpha 2}+\left(u^{2} \delta_{1}^{\gamma}+u^{1} \delta_{2}^{\gamma}\right) \delta_{\alpha 3} .
$$

From the system of equations $\quad W_{i}^{\gamma} \mathbf{A}_{\gamma}=0$ it follows:

$$
\mathbf{A}_{1}=\mathbf{A}_{2}=0
$$

Taking this into account the (2.38) will take the form:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=-a_{0} .
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=a_{0} .
$$

### 3.3.3 Group $G_{4}(V)$

Metric:

$$
d s^{2}=2 d u^{3} d u^{0}+a\left(\exp \left(-2 u^{3}\right)\left(d u^{2}\right)^{2}+\left(d u^{2}\right)^{2}\right)+\varepsilon \exp \left(-u^{3}\right) d u^{0} d u^{2} \quad \varepsilon=0,1 .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=u^{2} p_{2}+p_{3}, \quad X_{4}=-\varepsilon p_{1} \exp -u^{3}+\left(\left(u^{2}\right)^{2}+\exp -2 u^{3}\right) p_{2}+2 u^{2} p_{3} .
$$

Structural constants:

$$
C_{23}^{A}=\delta_{2}^{A}, \quad C_{24}^{A}=2 \delta_{3}^{A}, \quad C_{34}^{A}=2 u^{2} \delta_{3}^{A}-\delta_{2}^{A}\left(\left(u^{2}\right)^{2}+\exp 2 u^{2}\right),
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -u^{2} & 1
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=-\delta_{1}^{\alpha} \exp u^{3}-\left(\left(u^{2}\right)^{2}+\exp -2 u^{3}\right) \delta_{2}^{\alpha}+2 u^{2} \delta_{3}^{\alpha} .
$$

Matrix $\hat{W}$ :

$$
W_{\beta}^{\gamma}=-2\left(\delta_{3}^{\gamma}+u^{2} \delta_{2}^{\gamma}\right) \delta_{\beta 2}-\left(2 u^{2} \delta_{3}^{\gamma}-\varepsilon \delta_{1}^{\gamma} \exp u^{3}\right) \delta_{\beta 3} .
$$

From the system of equations $\quad W_{i}^{\gamma} \mathbf{A}_{\gamma}=0 \quad$ it follows:

$$
\begin{equation*}
\mathbf{A}_{3}+u^{2} \mathbf{A}_{2}=0, \quad 2 u^{2} \mathbf{A}_{3}-\varepsilon \mathbf{A}_{1}=0 \tag{3.7}
\end{equation*}
$$

Consider separately the variants $\quad \varepsilon=0$ and $\varepsilon=1$.
A) $\varepsilon=0$. In this case, the system (2.38) will take the form:

$$
\mathbf{A}_{3, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{3}=-a_{0} .
$$

The components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{1}=A_{2}=0, \quad A_{3}=a_{0} .
$$

B) $\varepsilon=1$. From (3.7) it follows:

$$
\begin{equation*}
\mathbf{A}_{2}=-\frac{\mathbf{A}_{1} \exp u^{3}}{2\left(u^{2}\right)^{2}}, \quad \mathbf{A}_{3}=\frac{\mathbf{A}_{1}}{u^{2}} \tag{3.8}
\end{equation*}
$$

From the system (2.38) it follows:

$$
\mathbf{A}_{1}=a_{0}, \quad \mathbf{A}_{2,2},
$$

which contradicts the relations (3.8). Thus, in this case, there is no permissible electromagnetic field.

### 3.3.4 Group $G_{4}(V I I I)$

Metric:

$$
d s^{2}=a_{1}\left(\left(d u^{1}\right)^{2}+\sin ^{2} u^{1}\left(d u^{2}\right)^{2}\right)+2 a_{2} \varepsilon \cos u^{1} d u^{1} d u^{0}+e_{0}\left(d u^{0}\right)^{2} \quad \varepsilon=0,1 .
$$

Group operators:

$$
X_{1}=p_{1}, \quad X_{2}=p_{2}, \quad X_{3}=-\frac{\cos u^{3} \sin u^{1}}{\sin u^{3}} p^{1}+\frac{\varepsilon \sin u^{1}}{\sin u^{3}} p_{2}+\cos u^{1} p_{3}, \quad X_{4}=\frac{\partial X_{3}}{\partial u^{2}} .
$$

Structural constants:

$$
C_{13}^{A}=\delta_{4}^{A} \rightarrow \tilde{C}_{13}^{\gamma}=\omega^{\gamma}, \quad C_{41}^{A}=\delta_{3}^{A}, \quad C_{34}^{A}=\delta_{1}^{A} .
$$

Matrix $\hat{\lambda}$ :

$$
\left\|\lambda_{\beta}^{\alpha}\right\|=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{\sin u^{1} \cos u^{3}}{\cos u^{1} \sin u^{3}} & -\frac{\varepsilon \sin u^{1}}{\cos u^{1} \sin u^{3}} & \frac{1}{\cos u^{1}}
\end{array}\right)
$$

Functions $\omega^{\alpha}$ :

$$
\omega^{\alpha}=-\frac{\delta_{1}^{\alpha} \cos u^{3}}{\cos u^{1} \sin u^{3}}+\frac{\varepsilon \delta_{2}^{\alpha}}{\cos u^{1} \sin u^{3}}-\frac{\delta_{3}^{\alpha} \sin u^{1}}{\cos u^{1}} .
$$

Elements of the matrix $W_{\beta}^{\alpha}$ :

$$
W_{\gamma}^{\alpha}=\delta_{\alpha 1}\left(\delta_{3}^{\gamma}+\omega^{3} \omega^{\gamma}\right)-\delta_{\alpha 3}\left(\delta_{1}^{\gamma}+\omega^{1} \omega^{\gamma}\right) .
$$

These formulas differ from the formulas given in the variant 6 (when the group $G_{4}(V I I I)$ acts on a non-isotropic hypersurface $\quad V_{3}$ ), by the presence of the $\varepsilon$ quantity. Repeating the calculations performed in variant 6 , we obtain the solution of the system $W_{\alpha}^{\gamma} \mathbf{A}_{\gamma}=0$ in the following form:

$$
\begin{equation*}
\mathbf{A}=-\varepsilon \mathbf{A}_{2} \sin u^{3} \cos u^{1} \quad \mathbf{A}_{1}=\varepsilon \mathbf{A}_{2} \cos u^{3}, \quad \mathbf{A}_{3}=\varepsilon \mathbf{A}_{2} \sin u^{1} \sin u^{3} . \tag{3.9}
\end{equation*}
$$

If $\varepsilon=1, \quad$ From the system (3.2) we get

$$
\mathbf{A}_{2, \alpha}=0 \quad \rightarrow \quad \mathbf{A}_{2}=a_{0},
$$

And the components of the potential of the admissible electromagnetic field are as follows:

$$
A_{0}=A_{3}=0, \quad A_{2}=a_{0}, \quad A_{1}=a_{0} \cos u^{3} .
$$

If $\varepsilon=0, \quad$ from (3.2) and (3.9) it follows:

$$
A_{i}=0 .
$$

## 4 Conclusion

The classification of admissible electromagnetic fields carried out in this article should be considered as a stage in the general program of research into the problem of integrating the classical and quantum equations of motion of a test particle in external fields of different nature in spaces with symmetry due to the sets of Killing fields. This program is effectively implemented for the Stackel spaces. Complete or partial classifications of the Stackel and special Stackel spaces of the electrovacuum are obtained in the General Theory of Relativity, as well as in the scalar-tensor theory.

The possibility of applying the theory of symmetry to the construction of cosmological models in the theory of gravity, including the Brans-Dicke theory, is also being studied (see, for example, [20] - 21]). As is known, from a physical point of view, the Robertson-Walker space is the most important special case of the Stackel space.

We also note the activity on the study of spaces belonging to the intersection of Stackel sets and homogeneous spaces (see [22]-[25]). The solution of the classification problem considered in the program allows us to extend this activity to solving problems in the presence of admissible electromagnetic fields.

As a rule, in the given examples, the classification problem was considered within the framework of a specific theory of gravity with the involvement of the gravitational field equations. This greatly simplifies the classification since it imposes additional serious restrictions on the potential and metric. In [26]-28] the classification of Stackel spaces and admissible external electromagnetic fields, in which the Hamilton-Jacobi equation for a charged test particle allows complete separation of variables, without involving the field equations, was first carried out. At the same time, the same classification problem for the Klein - Gordon - Fock equation is far from being solved.

A fundamentally different situation takes place in the implementation of the classification of admissible electromagnetic fields in spaces admitting groups of motions, since in these spaces the algebras of integrals of motion are already known, due to the theorem proved in this paper. To solve the problem, it remains to integrate the systems of equations (2.13) for each space with a Lorentzian signature that admits a group of motions. This classification problem is expected to be completely solved in the near future.

At the next stage of research, within the framework of the general program, one can consider the squared Dirac-Fock equations. In [30, [29] a method for integrating the Dirac-Fock equation was proposed, which allows one to reduce the solution of some bispinor equations to the problem of integrating linear scalar equations of the second order. It is supposed to investigate these scalar equations for the existence of admissible electromagnetic fields.

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