Algebras of integrals of motion for the Hamilton-Jacobi and Klein-Gordon-Fock equations in spacetime with a four-parameter groups of motions in the presence of an external electromagnetic field

V.V. Obukhov

Institute of Scietific Research and Development, Tomsk State Pedagogical University (TSPU), 634061 Tomsk, Russia

Laboratory for Theoretical Cosmology, International Centre of Gravity and Cosmos, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk

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1 Introduction

There are currently two main methods of exact integration of the classical and quantum equations of a test particle motion (the Hamilton—Jacobi, Klein—Gordon—Fock, and Dirac—Fock equations) in flat and curved spacetime, including when the external electromagnetic field is present. Both methods are based on the algebra of integrals of motion, each of which is linear or quadratic in momenta.

Symmetry operators of commutative algebra form a complete set of integrals of motion. A necessary condition for this set's existence is the presence of a complete set of Killing vector and tensor fields in the space. In this case, a complete separation of variables takes place.

The complete separation of variables is based on the theory of Stackel spaces. Spaces in which the free Hamilton-Jacobi equation is integrated by the complete separation of variables are called Stackel spaces. The coefficients at the highest powers of the products of momenta included in the first integrals of motion are vector (if the integral is linear in momenta) or tensor (for integrals of motion quadratic in momenta) Killing fields. It is also true for the Hamilton-Jacobi and Klein-Gordon-Fock equations in an electromagnetic field. Thus, the symmetry of the classical and quantum equations of motion is associated with the symmetry of space. Methods of complete separation of variables for classical and quantum equations of motion of charged test particles in the presence of external fields are constructed within the framework of the theory of Stackel spaces. The methods have been successfully used to solve classification problems in general relativity since 1966. (see, for example, [1]-[8] and the links indicated therein).

Otherwise, the non-commutative integration method is applied. The non-commutative integration method was developed for quantum one-particle equations of motion (Klein-Gordon-Fock, Dirac-Fock, etc.) In this case, the set is formed by the operators of the non-commutative group of space motions. Since these operators do not commute between themselves, they cannot be directly used to construct a basic solution. The original algebra is supplemented with linear differential operators acting in the space of functions depending on coordinates and parameters to get around this problem. Extended operators are selected in such a way that they mutually commute. Using them, one can find a basic solution.

The problem of non-commutative integration attracted the attention of researchers much later, although the complete algebraic classification of spaces that admit groups of motions required for its statement was constructed by A.Z. Petrov back in the sixties of the last century (see [9]). The classification served as the basis for constructing a method for the exact integration of linear partial differential equations using non-commutative algebras of first-order integrals of motion (see [10] – [15]).

The method made it possible to significantly expand the sets of external fields and metrics of spacetime, in which basic solutions of the quantum equations of motion for a charged test particle, can be found. It became an incentive for the construction the set of admissible electromagnetic fields as a necessary stage in the application of a non-commutative integration procedure (electromagnetic fields in which the Hamilton—Jacobi and (or) Klein—Gordon—Fock equations for a charged test particle admit algebra of integrals of motion linear in momenta are called admissible).

In particular, in [11], [13] a classification of all potentials of admissible electromagnetic fields for ((metrics of spacetime)) admitting a transitive action of four-parameter movement groups G_4 is constructed. In [16], [?] a similar classification was carried out for homogeneous spaces with a three-parameter groups of motions G_3 , acting transitively on non null hyperspace of spacetime (see [18] too).

In present work, the following is done.

1. For the case of an n -dimensional Riemannian space with an arbitrary signature V_n , it is proved that, in an admissible electromagnetic field, the algebras of integrals of motion of the Hamilton-Jacobi equation and the symmetry operators of the Klein-Gordon-Fock equation for a charged particle coincide, and they coincide with the algebras of symmetry operators for the free Hamilton-Jacobi and Klein-Gordon-Fock equations.

2. In the *n*-dimensional Riemannian space V_n , conditions that must be satisfied by admissible electromagnetic fields are obtained, and the compatibility of the equations systems following from these conditions is proved for the case when the group of motions G_r acts transitively on the subspace V_r $(r \leq n, rank(G_r) = r)$.

3. The classification of admissible electromagnetic fields in spacetime with four-parameter groups of motions G_4 , is completed for the case when $rank(G_4) < 4$.

Note that additional conditions on metrics and potentials (for example, the Einstein equations) are not used.

2 Conditions for the existence of the algebra of integrals of motion.

2.1 Designations.

In the next two sections, we consider the n - dimensional Riemannian space V_n (with an arbitrary signature), on the r-dimensional subspace V_r of which the R-parameter groups of motions G_R ($rank(G_R) = r \leq R$) of the space V_n acts transitively. This means that any two points of the subspace V_r can be transformed into each other by some transformation of the group G_R . In this case, the subspace V_r is called the subspace of transitivity (or the hypersurface of transitivity if r = n - 1).

The following index notations are used. The coordinate indices of the canonical local coordinate system variables $[u^i]$ of the space V_n will be denoted by small Latin letters: i, j, l = 1, ...n. By analogy with Stackel spaces, all the variables of the canonical coordinate system are divided into two sets. Variables of the local coordinate system of the subspace V_r , on which the group of motions G_R acts transitively, are called ignored and are supplied with small letters of the Greek alphabet: α, β, γ varying within $1, \ldots r$. Ignored variables enter the metric tensor only through specific functions.

The rest of the variables are called essential (or non-ignored). Coordinate indices of essential variables are denoted by the letters $p, q = 1, \ldots r$. The essential variables enter the metric tensor and the components of the potential of an admissible electromagnet field through arbitrary functions, each of which depends only on these variables.

If R > r, the group indices will be denoted by capital Latin letters: A, B, C, D =The Independent operators X_A of the group G_R are numbered, the same $1,\ldots R.$ way as the local coordinates of the subspace V_r , in small Greek letters. The other operators of the group G_R , are linear combinations of the independent operators X_{α} (with variable coefficients) and are numbered by the indices: $a, b, c, d = (r+1), \ldots, R$. The coordinate indices of the essential variables of the local canonical coordinate system are denoted by $p, q = r + 1, \ldots, n$. In the metric tensor and in the components of the the following letters: admissible electromagnetic field potential, essential variables enter through arbitrary functions, each of which depends only on these variables. The repeated superscripts and subscripts are summed. The subject of our consideration is the conditions imposed on the external electromagnetic field, under which there are algebras of integrals of motion linear in momenta of the classical and quantum equations of motion for a charged scalar test particle.

2.2 Derivation of the conditions for the existence of an admissible electromagnetic field.

For a charged test particle in an external electromagnetic field with potential A_i , the Hamilton—Jacobi equation has the form:

$$g^{ij}P_iP_j = \lambda. \tag{2.1}$$

$$P_i = p_i + A_i, \quad p_i = \varphi_{,i} = \partial \varphi / \partial u^i, \quad \lambda = const.$$

Let us consider the Klein—Gordon—Fock equation too:

$$\hat{H}\varphi = (g^{ij}\hat{P}_i\hat{P}_j)\varphi = \lambda\varphi, \quad \hat{P}_j = -i\hat{\nabla}_i + A_i.$$

 $\hat{\nabla}_i$ — operator of the covariant derivative, with metric-compatible connectivity, corresponding to the operator of the partial derivative — $\hat{\partial}_i = i\hat{p}_i$ with respect to the coordinate u_i ; φ is a field of a scalar particle with mass $m = \sqrt{|\lambda|}$. For both equations, integrals of motion linear in momenta have the same form:

$$\hat{Y}_A = \xi_A^i P_i + \gamma_A. \tag{2.2}$$

It was proved in the articles [11] and [16], that the conditions for the existence of an admissible electromagnetic field for the Klein-Gordon-Fock equation have exactly the same form as for the Hamilton—Jacobi equation. Therefore, we will consider only the Hamilton—Jacobi equation. All the results obtained in this case are also valid for the Klein—Gordon—Fock equation.

The equation 2.1 admits the integral of motion 2.2 if and only if the functions $\hat{H} = g^{ij}P_iP_j$ and \hat{Y}_A commute with respect to the Poisson brackets:

$$[\hat{H}, \hat{Y}_{A}]_{\mathcal{P}} = (g^{il}\xi^{j}_{A,l} + g^{jl}\xi^{i}_{A,l} - g^{ij}_{,l}\xi^{l}_{A})p_{i}p_{j} + 2g^{il}(\xi^{j}_{A}F_{ij} + \gamma_{A,i})p_{l} = 0$$

$$(F_{ij} = A_{j,i} - A_{i,j}).$$

$$(2.3)$$

The ratios 2.3 must be satisfied for any momentum value. Equating to zero the coefficients in front of the p_i powers, one obtains:

$$g^{il}\xi^{j}_{A,l} + g^{jl}\xi^{i}_{A,l} - g^{ij}_{,l}\xi^{l}_{A} = 0, (2.4)$$

$$\gamma_{A,i} = \xi^{\jmath}_A F_{ji}. \tag{2.5}$$

It follows from the equations 2.4 that the functions ξ_A^j are the components of the Killing vector. These functions form a groups of motions G_R of $rank(G_R) = r$. Unlike the free Hamilton-Jacobi equation, equation 2.1, generally speaking, has no integrals of motion. The system of equations 2.5 specifies the conditions under which integrals of motion of the form 2.2 exist. This system was first developed by [11]. If the integrals of motion 2.2 form an algebra with respect to Poisson brackets with the same structure constants as for the algebra of the group operators, the following condition is satisfied:

$$[\hat{X}_A, \hat{X}_B]_{\mathcal{P}} = C^D_{AB} \hat{X}_D, \qquad (2.6)$$

In [16] it is stated that in an admissible electromagnetic field, the integrals of motion have the form:

$$\hat{X}_A = \xi^i_A p_i. \tag{2.7}$$

Let us prove this statement in the following theorem.

Theorem 1 An admissible electromagnetic field does not deform the algebra of integrals of motion for the Hamilton-Jacobi and Klein—Gordon—Fock equations. The algebra operators for a free test particle and for a charged one have the form 2.7.

Proof.

As $rank(G_r) = rank||\xi_A^i|| = r$, the transitivity subspace for the group motions G_R is the *r*-dimensional space V_r . We represent the dependent operators of the group as follows:

$$\mathbf{X}_a = \omega_a^{\alpha} \mathbf{X}_{\alpha}, \tag{2.8}$$

Here ω_a^{α} functions expressed in terms of the components of the Killing vectors ξ_A^i as follows:

$$\omega_a^{\alpha} = \lambda_{\beta}^{\alpha} \xi_a^{\beta}, \quad \lambda_{\beta}^{\alpha} \xi_{\gamma}^{\beta} = \delta_{\gamma}^{\alpha}, \quad \xi_A^i = \delta_{\beta}^i \xi_A^{\beta}. \tag{2.9}$$

Transform the functions γ_{α} as follows:

$$\gamma_{\alpha} = \xi^i_{\alpha}(\omega_i - A_i)$$

From here it follows:

$$\hat{X}_{\alpha} = \xi^i_{\alpha}(p_i + \omega_i). \tag{2.10}$$

Let us show that from the conditions 2.6 it follows:

$$\gamma_{\alpha} = \xi^i_{\alpha} A_i = 0 \to \omega_{\alpha} = 0, \qquad (2.11)$$

Substitute 2.10 into 2.6. As a result, we get:

$$[Y_{\alpha}, Y_{\beta}]_{\mathcal{P}} = C^{\gamma}_{\alpha\beta}\xi^{i}_{\gamma}p_{i} + \omega_{\beta|\alpha} - \omega_{\alpha|\beta} = C^{\gamma}_{\alpha\beta}\xi^{i}_{\gamma}(p_{i} + \omega_{i}) \quad (|_{A} = \xi^{i}_{A}\partial_{i}. \quad \omega_{\alpha} = \xi^{i}_{\alpha}\omega_{i}). \tag{2.12}$$

Thus, the functions ω_{α} obey the equations:

$$\omega_{\beta|\alpha} - \omega_{\alpha|\beta} = C^{\gamma}_{\alpha\beta}\omega_{\gamma} \quad \rightarrow \quad \omega_{i,j} = \omega_{j,i} \quad \rightarrow \quad \omega_i = \omega_{,i}.$$

By the gradient transformation of the potential, the function ω_i can be turned to 0. The theorem is proved.

Thereby, the condition for the admissible electromagnetic field existence has the form:

$$(A_j \xi_A^j)_{,i} = \xi_A^j F_{ij}.$$

Let us introduce a nonholonomic frame associated with the group G_R :

$$\hat{\sigma}_j = \sigma_j^i \hat{\partial}_i = \delta_j^p \hat{\partial}_p + \delta_j^\alpha \xi_\alpha^\beta \hat{\partial}_\beta, \quad \hat{e}^j = e_i^j \hat{d}\hat{u}^i = \delta_p^j \hat{d}\hat{u}^p + \delta_\alpha^j \lambda_\beta^\alpha \hat{d}\hat{u}^\beta,$$

non-holonomic components of the vector potential:

$$\mathbf{A}_i = \sigma_i^{j} A_j,$$

and also the potential projections onto the vector fields ξ_A^i : $\mathbb{A}_A = \xi_A^i A_i$. Then the equation 2.5 takes the following form:

$$\mathbb{A}_{A,i} = \xi^j_A F_{ij}.\tag{2.13}$$

Consider two cases separately: 1. $i = \alpha$ and 2. i = p.

1. $i = \alpha$. The system (2.13) can be reduced to the form:

$$\mathbb{A}_{A|B} = \xi_A^j \xi_B^i (A_{j,i} - A_{i,j}) = \mathbb{A}_{A|B} - \mathbb{A}_{B|A} + C_{AB}^D \mathbb{A}_D.$$

From here it follows:

$$\mathbb{A}_{B|A} = C^D_{AB} \mathbb{A}_D. \tag{2.14}$$

2. i = p. The system (2.13) can be reduced to the following form:

$$(\xi_B^j A_j)_{,p} = \xi_B^j (A_{j,p} - A_{p,j}) = (\xi_B^j A_j)_{,p} - A_{p|B} - \xi_{B,p}^j A_j.$$

From here it follows:

$$A_{p|A} = -\xi^i_{A,p} A_i, \qquad (2.15)$$

The Systems 2.14, 2.15 need to be examined for compatibility. Let us consider separately the cases when the order of the group G_R coincides with the dimension of the transitivity subspace V_r (R=r), and when R > r.

2.3 Compatibility conditions for the case, when the order of the group G_R coincides with the dimension of the transitivity subspace V_r .

Let us prove the compatibility of the system (2.14), (2.15). To do this, show that

$$\mathbf{A}_{\beta|\alpha\gamma} - \mathbf{A}_{\beta|\gamma\alpha} = C^{\sigma}_{\alpha\beta} \mathbf{A}_{\sigma|\gamma} - C^{\sigma}_{\gamma\beta} \mathbf{A}_{\sigma|\alpha}.$$
(2.16)

Since

$$\mathbf{A}_{\beta|\alpha\gamma} - \mathbf{A}_{\beta|\gamma\alpha} = C^{\sigma}_{\gamma\alpha} \mathbf{A}_{\alpha|\sigma},$$
$$C^{\sigma}_{\alpha\beta} \mathbf{A}_{\sigma|\gamma} - C^{\sigma}_{\gamma\beta} \mathbf{A}_{\sigma|\alpha} = (C^{\sigma}_{\alpha\beta} C^{\rho}_{\gamma\sigma} - C^{\sigma}_{\gamma\beta} C^{\rho}_{\alpha\sigma}) \mathbf{A}_{\rho},$$

the system (2.16) is reduced to the following form:

$$(C^{\gamma}_{\sigma\rho}C^{\alpha}_{\beta\gamma} + C^{\gamma}_{\rho\beta}C^{\alpha}_{\sigma\gamma} + C^{\gamma}_{\beta\sigma}C^{\alpha}_{\rho\gamma})\mathbf{A}_{\alpha} = 0,$$

which is fulfilled by the Bianchi identities.

To prove the compatibility of the (2.15) system, it is necessary to show that

$$A_{p|\beta\alpha} - A_{p|\alpha\beta} = C^{\gamma}_{\alpha\beta} A_{p|\gamma} = (\xi^{\sigma}_{\alpha,p} \lambda^{\gamma}_{\sigma} \mathbf{A}_{\gamma})_{|\beta} - (\xi^{\sigma}_{\beta,p} \lambda^{\gamma}_{\sigma} \mathbf{A}_{\gamma})_{|\alpha}.$$
(2.17)

Using the systems of equations (2.14), (2.15), as well as the consequences of the Killing equations:

$$\xi^{\gamma}_{\alpha,p|\beta} - \xi^{\gamma}_{\beta,p|\alpha} = \xi^{\nu}_{\alpha,p}\xi^{\gamma}_{\beta,\nu} - \xi^{\nu}_{\beta,p}\xi^{\gamma}_{\alpha,\nu} + C^{\nu}_{\beta\alpha}\xi^{\gamma}_{\nu,p}$$

we bring the conditions (2.17) to the form:

$$\xi^{\sigma}_{\beta,p}(\xi^{\rho}_{\alpha,\sigma}\lambda^{\gamma}_{\rho}+\lambda^{\gamma}_{\sigma\mid\alpha}+\lambda^{\rho}_{\sigma}C^{\gamma}_{\alpha\rho})\mathbf{A}_{\gamma}=\xi^{\sigma}_{\beta,p}(\xi^{\rho}_{\beta,\sigma}\lambda^{\gamma}_{\rho}+\lambda^{\gamma}_{\sigma\mid\beta}+\lambda^{\rho}_{\sigma}C^{\gamma}_{\beta\rho})\mathbf{A}_{\gamma}$$
(2.18)

It is easy to show that

$$\xi^{\rho}_{\alpha,\sigma}\lambda^{\gamma}_{\rho} + \lambda^{\gamma}_{\sigma|\alpha} + \lambda^{\rho}_{\sigma}C^{\gamma}_{\alpha\rho} = 0.$$

The compatibility of the system of equations (2.15) is proved.

2.4 Compatibility conditions for the case when the order of the group G_R is greater than the dimension of the transitivity subspace V_r .

When studying the compatibility conditions for the systems of equations (2.14) (2.15), one cannot use the results obtained in the previous section, since in this case the systems are overflowing, because the functions A_B are linearly dependent with coefficients expressed in terms of the functions ξ_B^A . Since the order of the group is greater than the dimension of the transitivity subspace, (R - r) operators of the group are linear combinations of r of the basic operators with variable coefficients. Without loss of generality, we can assume that $rank||\xi_{\alpha}^i|| = r \rightarrow$ the first r of the \hat{X}_{α} operators are basic. In this case $det||\xi_{\beta}^{\alpha}|| \neq 0$, and the inverse matrix $||\lambda_{\beta}^{\alpha}||$ exists:

$$\xi^{\alpha}_{\gamma}\lambda^{\gamma}_{\beta} = \delta^{\alpha}_{\beta}$$

Then:

$$\hat{X}_a = \xi^\beta_a \lambda^\alpha_\beta \hat{X}_\alpha \quad \to \quad \mathbb{A}_a = \xi^\beta_a \lambda^\alpha_\beta \mathbf{A}_\beta \quad \to \quad \mathbb{A}_a = \omega^\alpha_a \mathbf{A}_\alpha. \tag{2.19}$$

Introduce the functions:

$$\tilde{C}^{\gamma}_{AB} = C^{\gamma}_{AB} + \omega^{\gamma}_a C^a_{AB}.$$

Then from the commutation relations:

$$[\hat{X}_A, \hat{X}_B]_P = C^D_{AB} \hat{X}_D,$$

one obtains the following identities:

$$\omega_{a|\alpha}^{\gamma} = \tilde{C}_{\alpha a}^{\gamma} + \omega_{a}^{\beta} \tilde{C}_{\beta \alpha}^{\gamma}, \quad \tilde{C}_{ab}^{\gamma} = \omega_{a}^{\alpha} \tilde{C}_{\alpha b}^{\gamma} - \omega_{b}^{\alpha} \tilde{C}_{\alpha a}^{\gamma}.$$
(2.20)

The conditions (2.6) are reduced to the following systems of equations:

$$\mathbb{A}_{A|B} = \tilde{C}^{\gamma}_{BA} \mathbf{A}_{\gamma} \tag{2.21}$$

$$A_{p|A} = -\xi^{\beta}_{A,p} A_{\beta} = -\xi^{\beta}_{A,p} \lambda^{\alpha}_{\beta} \mathbf{A}_{\alpha}.$$
(2.22)

Let us prove that the functions \mathbf{A}_{α} form a system of linearly dependent functions with coefficients that are rational functions of the components ξ_{α}^{i} and structural constants. Indeed, the systems of equations (2.21), (2.21), depending on the values of the indices, can be broken down as follows:

$$\mathbf{A}_{\beta|\alpha} = \tilde{C}^{\gamma}_{\alpha\beta} \mathbf{A}_{\gamma}; \tag{2.23}$$

$$\mathbf{A}_{\beta|a} = \tilde{C}^{\gamma}_{a\beta} \mathbf{A}_{\gamma}; \tag{2.24}$$

$$\mathbf{A}_{b|A} = \tilde{C}^{\gamma}_{Ab} \mathbf{A}_{\gamma}; \tag{2.25}$$

$$\mathbf{A}_{p|a} = \omega_a^{\alpha} \mathbf{A}_{p,\alpha} = -\xi_{a,p}^{\beta} \lambda_{\beta}^{\gamma} \mathbf{A}_{\gamma}; \qquad (2.26)$$

$$\mathbf{A}_{p|\alpha} = -\xi^{\beta}_{\alpha,p} \lambda^{\gamma}_{\beta} \mathbf{A}_{\gamma}. \tag{2.27}$$

From (2.24) it follows:

$$(\tilde{C}^{\gamma}_{\alpha a} + \omega^{\beta}_{a} \tilde{C}^{\gamma}_{\beta \gamma}) \mathbf{A}_{\gamma} = 0, \qquad (2.28)$$

Taking into account (2.20) one obtains:

$$\omega_{a|\alpha}^{\gamma} \mathbf{A}_{\gamma} = 0. \tag{2.29}$$

Since

$$\xi_a^{\alpha} = \omega_a^{\beta} \xi_{\beta}^{\alpha}$$

from (2.26), (2.27) it follows:

$$\omega_{a|p}^{\gamma} \mathbf{A}_{\gamma} = 0 \to \omega_{a,i}^{\gamma} \mathbf{A}_{\gamma} = 0.$$
(2.30)

Obviously, $\omega_{a,i}^{\gamma} \neq 0$, otherwise $\omega_a^{\gamma} = const.$ and G_R is reduced to G_r . The statement is proven true.

The system (2.25) is satisfied identically due to the conditions (2.20). Consider the (2.27) system. The compatibility conditions are as follows:

$$\mathbf{A}_{p|\alpha\gamma} - \mathbf{A}_{p|\gamma\alpha} = (\xi^{\beta}_{\gamma,p} A_{\beta})_{|\alpha} - (\xi^{\beta}_{\alpha,p} A_{\beta})_{|\gamma}.$$
(2.31)

Using the condition (2.23), as well as the consequence from the Killing equations:

$$(\xi^{\gamma}_{\alpha|\beta} - \xi^{\gamma}_{\beta|\alpha})_{,p} = \omega^{\nu}_{a|p}\xi^{\gamma}_{\nu}\tilde{C}^{a}_{\alpha\beta} + \xi^{\gamma}_{\nu,p}\tilde{C}^{\gamma}_{\alpha\beta}$$

we get:

$$\omega_a^{\gamma} \lambda_{\nu}^{\mu} (\xi_{\beta,p}^{\nu} C_{\alpha\mu}^a - \xi_{\alpha,p}^{\nu} C_{\beta\mu}^a) \mathbf{A}_{\gamma} = 0.$$
(2.32)

Let us supplement this system with equations (2.11), (2.12):

$$\omega_{a,i}^{\gamma} \mathbf{A}_{\gamma} = 0. \tag{2.33}$$

Let K be the number of independent equations in the system (2.32), (2.33), and the numbers μ range from 1 to K. Let us randomly enumerate all pairs (in (2.33)) and all triples (in (2.32)) of subscripts in the coefficients before \mathbf{A}_{α} and represent these coefficients as elements of the matrix $\hat{W}(0) = ||W(0)^{\alpha}_{\mu}||$. Thus, the functions \mathbf{A}_{α} satisfy the system of equations:

$$W(0)^{\alpha}_{\mu}\mathbf{A}_{\alpha} = 0. \tag{2.34}$$

Denote:

$$\tilde{r} = r - Rank ||\hat{W}(0)|| \quad (r - \tilde{r} \neq 0),$$

In the matrix $\hat{W}(0)$ choose an arbitrary minor \hat{Z} of $rank = \tilde{r}$. The indices numbering the rows of this minor are denoted by the letters $\tilde{a}, \tilde{b}, \tilde{c}$. The rest of the minor columns will be denoted by the letters $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. These letters will also be used to number the columns that form the non-singular square matrix $\hat{\Omega}$ of $rank = \hat{r}$ in this minor.

Then the solution of the system of equations (2.23) can be represented as

$$\mathbf{A}_{\tilde{a}} = \theta_{\tilde{a}}^{\tilde{\alpha}} \mathbf{A}_{\tilde{\alpha}}, \tag{2.35}$$

where

$$\theta_{\tilde{a}}^{\tilde{\alpha}} = -||\hat{\Omega}||_{\tilde{a}}^{\tilde{b}} Z_{\tilde{b}}^{\hat{\alpha}}.$$
(2.36)

The functions (2.36) convert the systems of equations (2.32) and (2.33) into identities:

$$\omega_{a|i}^{\tilde{\gamma}} + \omega_{a|i}^{\tilde{a}} \theta_{\tilde{a}}^{\tilde{\gamma}} = 0 \qquad (\omega_a^{\tilde{\gamma}} + \omega_a^{\tilde{a}} \theta_{\tilde{a}}^{\tilde{\gamma}}) \lambda_{\nu}^{\mu} (\xi_{\beta,p}^{\nu} C_{\alpha\mu}^{a} - \xi_{\alpha,p}^{\nu} C_{\beta\mu}^{a}) = 0$$

Let's go back to the system of equations (2.23). Substitute expressions (2.34) into it. As a result, we get the following subsystems:

$$\mathbf{A}_{\tilde{\alpha}|\alpha} = \tilde{C}^{\gamma}_{\alpha\tilde{\alpha}} \mathbf{A}_{\gamma} = (\tilde{C}^{\tilde{\gamma}}_{\alpha\tilde{\alpha}} + \theta^{\tilde{\gamma}}_{\tilde{b}} \tilde{C}^{\tilde{b}}_{\alpha\tilde{\alpha}}) \mathbf{A}_{\tilde{\gamma}}.$$
(2.37)

$$\mathbf{A}_{\tilde{a}|\alpha} = \tilde{C}^{\gamma}_{\alpha\tilde{a}} \mathbf{A}_{\gamma} = (\tilde{C}^{\tilde{\gamma}}_{\alpha\tilde{a}} + \theta^{\tilde{\gamma}}_{\tilde{b}} \tilde{C}^{\tilde{b}}_{\alpha\tilde{a}}) \mathbf{A}_{\tilde{\gamma}}.$$
(2.38)

Using (2.34) and (2.37), the system of equations (2.38) can be represented as a system of algebraic equations:

$$(\theta^{\tilde{\gamma}}_{\tilde{a}|\alpha} + \theta^{\tilde{\beta}}_{\tilde{a}}(\tilde{C}^{\tilde{\gamma}}_{\alpha\tilde{\beta}} + \theta^{\tilde{\gamma}}_{\tilde{b}}\tilde{C}^{\tilde{b}}_{\alpha\tilde{\beta}}) - (\tilde{C}^{\tilde{\gamma}}_{\alpha\tilde{a}} + \theta^{\tilde{\gamma}}_{\tilde{b}}\tilde{C}^{\tilde{b}}_{\alpha\tilde{a}}))\mathbf{A}_{\tilde{\gamma}} = 0$$
(2.39)

Combining (2.39) and the compatibility conditions for the systems of equations, (2.37)–(2.27), one obtains new compatibility conditions in the form similar to the conditions (2.34):

$$W(1)^{\tilde{\alpha}}_{\mu_1} \mathbf{A}_{\tilde{\alpha}} = 0.$$

The index μ_1 is constructed in the same way as the index in the matrix $W(0)^{\alpha}_{\mu}$.

If $W(1)_{\mu_1}^{\tilde{\alpha}} = 0$, the solution has the form (2.35) and the systems of equations (2.13), (2.15) are compatible.

If $W(1)_{\mu_1}^{\tilde{\alpha}} \neq 0$, this procedure should be repeated until at step (\varkappa) the condition is met $W(\varkappa)_{\mu_{\varkappa}}^{\tilde{\alpha}_{\varkappa}} = 0$, and the remaining system of differential equations:

$$\mathbf{A}_{\tilde{a}_{\varkappa}|\tilde{\alpha}} = \tilde{C}_{\tilde{\alpha}(\tilde{a}_{\varkappa})}^{\tilde{\gamma}_{\varkappa}} \mathbf{A}_{\gamma_{\varkappa}}, \quad \mathbf{A}_{\tilde{\alpha}} = \vartheta_{\tilde{\alpha}}^{\tilde{\alpha}_{\varkappa}} \mathbf{A}_{(\tilde{a}_{\varkappa})}$$
(2.40)

will be compatible. Otherwise, the only solution to the system (2.23) will be - $A_i = 0$. Note that for a specific space on which the group G_R acts, ω_a^{α} , $\theta_{\tilde{a}}^{\tilde{\gamma}}$, $\vartheta_{\alpha}^{(\tilde{\alpha}_N)}$, are the given functions. Therefore, checking the compatibility conditions does not imply the solution to any equations.

3 Admissible electromagnetic fields for the groups of motions G_4 acting on transitivity subspaces of a spacetime manifold

The complete classification of admissible electromagnetic field potential is an obvious continuation of the classification problem, solved by A.Z. Petrov ([9]). Despite its obvious importance, the first papers ([11], [12], [16], [18]) on this topic appeared quite recently. This section continues the research begun in these articles. The potentials of all admissible electromagnetic fields are found for the case when the group of motions G_4 acts nontransitively on the space V_4 . The results obtained in the previous section are used.

Let us clarify the notation of the indices used in this section. As is known, every fourparameter Lie group G_4 has a three-parameter subgroup G_3 (see [9]). Therefore, as the basic operators \hat{X}_{α} of the group G_4 , one can choose the operators of the subgroup G_3 . The indices introduced earlier in this section vary within the following limits: $i, j = 0, \ldots 3$; $\alpha, \beta, \gamma, = 1, \dots, 3;$ $A, B = 1, \dots, 4;$ a = 4; The following algorithm is used for constructing the potentials of an admissible electromagnetic field.

1. Using the known Killing vectors ξ_A^{α} of the group G_4 there can be found the matrix $||\lambda_{\beta}^{\alpha}||$ and the functions $\omega^{\alpha} = -\lambda_{\beta}^{\alpha}\xi_4^{\beta}$.

2. The functions \tilde{C}^{α}_{AB} are calculated and the matrix

$$||W^{\gamma}_{\alpha}(0)|| = ||\tilde{C}^{\gamma}_{\alpha 4} + \omega^{\beta}\tilde{C}^{\gamma}_{\beta\alpha}||$$

is constructed using the (2.20) formula.

3. The solution to the system of equations is found

$$W^{\gamma}_{\alpha}(0)\mathbf{A}_{\gamma} = 0 \tag{3.1}$$

in the following form:

$$\mathbf{A}_{\tilde{a}} = \theta_{\tilde{a}}^{\tilde{\alpha}} \mathbf{A}_{\tilde{\alpha}}. \quad rank ||\theta_{\tilde{a}}^{\tilde{\alpha}}|| = (3 - \tilde{r}).$$

4. The component A_4 is calculated by the formula:

$$\mathbf{A}_a = \mathbf{A}_4 = \omega^{\alpha} \mathbf{A}_{\alpha},$$

the system (2.11) compatibility is checked and the solutions to the remaining equations

$$\mathbf{A}_{\tilde{\alpha}|\tilde{\beta}} = \tilde{C}^{\tilde{\gamma}}_{\tilde{\beta}\tilde{\alpha}} \mathbf{A}_{\tilde{\gamma}}.$$
(3.2)

are found.

As already noted, group operators, spacetime metrics and canonical coordinate systems for all groups of motions G_R were found in [9]. Below, for ease of use, these formulas and structural constants are given for each considered group. Moreover in every subsection the results of calculation are presented in the following order. Matrices $\hat{\lambda}$, \hat{W} , are given as well as nonholonomic \mathbf{A}_A and holonomic A_i components of the potentials of the admissible electromagnetic field. The functions denoted by the letter a with a single lower index depend only on the variable u^0 , $(e_0)^2 = 1$

The transitivity hypersurface of spacetime V_4 in the canonical coordinate system is given by the equation:

$$\phi(u^i) = u^0 = const.$$

If $g^{ij}\phi_{,i}\phi_{,j} \neq 0$, the surface is called non-isotropic (non-null) and is denoted by V_3 . Otherwise the hypersurface is called isotropic (null) and is denoted by V_3^* .

Each case when the group G_4 acts transitively on the hypersurfaces V_3 and V_3^* , also when the mentioned above subgroups G_3 act transitively on V_2 and V_2^* (they are the subspaces of the hypersurface V_3) is considered separately.

The notation $G_R(N)$ means that this group has an order R and the group structure is of the type N according to the Bianchi classification. [19]

3.1 The groups G_4 act transitively on a non null subspace V_3

When the group G_4 acts transitively on a non null subspace V_r , $r \leq 3$ it follows from [9] that $(\xi^{\alpha}_A)_{,0} = 0$. According to (2.27) this means: $A_0 = A_0(u^0)$, which is equivalent to

$$A_0 = 0.$$

3.1.1 Group $G_4(I)$

Metric, in which this group acts, has the form:

$$ds^{2} = 2a_{1}du^{1}du^{3} + a_{2}(du^{2} + u^{1}du^{3})^{2} + e_{0}(du^{0})^{2}.$$

Let us present operators of the group:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = u^2 p_1 - p_3, \quad X_4 = u^2 p_2 - u^3 p_3,$$

and structural constants:

$$C_{A1}^{\gamma} = 0, \quad C_{23}^{\gamma} = \delta_1^{\gamma}, \quad C_{24}^{\gamma} = \delta_2^{\gamma}, \quad C_{34}^{\gamma} = -\delta_3^{\gamma}.$$

Matrix $\hat{\lambda}$, $\hat{W}(0)$ and functions ω^{α} , which defined by the formulas (2.9), (2.34), have the form:

$$\begin{aligned} ||\lambda_{\beta}^{\alpha}|| &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u^{2} & 0 & -1 \end{pmatrix}, \\ \omega^{\alpha} &= -u^{2}u^{3}\delta_{1}^{\alpha} + u^{2}\delta_{2}^{\alpha} + u^{3}\delta_{3}^{\alpha} \end{aligned}$$

Matrix:

$$||W^{\alpha}_{\beta}|| = \begin{pmatrix} 0 & 0 & 0 \\ u^{3} & -1 & 0 \\ -u^{2} & 0 & 1 \end{pmatrix},$$

From the system of equations (3.1) it follows:

$$\mathbf{A_2} = u^3 \mathbf{A_1}, \quad \mathbf{A_3} = u^2 \mathbf{A_1}. \tag{3.3}$$

Substitute the resulting expressions into (3.2). As a result, we get:

$$A_{1,2} = A_{2,2} = A_{\alpha,1} = A_{1,3} = A_{3,3} = 0;$$

 $A_{3,2} = A_1, \quad A_{2,3} = A_1.$

The solution is:

$$\mathbf{A}_1 = a_0, \quad \mathbf{A}_2 = a_0 u^3, \quad \mathbf{A}_3 = a_0 u^2.$$

The components of the potential of the admissible electromagnetic field are found by the formula

$$A_{\alpha} = \mathbf{A}_{\beta} \lambda_{\alpha}^{\beta}, \quad A_0 = 0 \tag{3.4}$$

and have the form:

$$A_0 = 0,$$
 $A_1 = a_0,$ $A_2 = a_0 u^3,$ $A_3 = 0.$

3.1.2 Group $G_4(III)$.

Metric:

$$ds^{2} = a_{1}((du^{1})^{2} + (du^{3})^{2}) + a_{2}(du^{2} + u^{1}du^{3})^{2} + e_{0}(du^{0})^{2}.$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = u^2 p_1 + p_3, \quad X_4 = \frac{(u^2)^2 + (u^3)^2}{2} p_1 + u^3 p_2 + u^2 p_3.$$

Structural constants:

$$C_{23}^{\gamma} = \delta_1^{\gamma}, \quad C_{24}^{\gamma} = \delta_3^{\gamma}, \quad C_{34}^{\gamma} = -\delta_2^{\gamma}.$$

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u^2 & 0 & 1 \end{pmatrix},$$

Functions ω^{α} :

Matrix $\hat{\lambda}$:

$$\omega^{\alpha} = \frac{(u^2)^2 + (u^2)^2}{2}\delta_1^{\alpha} + u^3\delta_2^{\alpha} + u^2\delta_3^{\alpha}$$

Matrix W^{α}_{β} :

$$||W^{\alpha}_{\beta}|| = -\begin{pmatrix} 0 & 0 & 0 \\ -u^2 & 0 & 1 \\ u^3 & 1 & 0 \end{pmatrix}$$

From the system of equations (3.1) it follows:

$$\mathbf{A_2} = -u^3 \mathbf{A_1}, \quad \mathbf{A_3} = u^2 \mathbf{A_1}. \tag{3.5}$$

Substitute the resulting expressions into (3.2). As a result, we get:

$$\mathbf{A}_{1,\alpha} = 0, \quad \mathbf{A}_{2,2} = \mathbf{A}_{2,1} = \mathbf{A}_{3,1} = \mathbf{A}_{3,3} = 0;$$

 $\mathbf{A}_{3,2} = \mathbf{A}_1, \quad \mathbf{A}_{2,3} = -\mathbf{A}_1.$

The solution is:

$$\mathbf{A}_1 = a_0, \quad \mathbf{A}_2 = -a_0 u^3, \quad \mathbf{A}_3 = a_0 u^2.$$

The components of the potential of the admissible electromagnetic field are found by the formula (3.4) and have the form:

$$A_0 = 0,$$
 $A_1 = a_0,$ $A_2 = -a_0 u^3,$ $A_3 = 0.$

3.1.3 Group $G_4(IV)$

Metric:

$$ds^{2} = a_{1}(du^{1})^{2} + 2a_{2}\exp u^{1}du^{2}du^{3} + e_{0}(du^{0})^{2}.$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = u^1 p_1 + p_3, \quad X_4 = u^2 p_2 + p_3.$$

Structural constants:

$$C_{13}^{\gamma} = \delta_1^{\gamma}, \quad C_{24}^{\gamma} = \delta_2^{\gamma}.$$
$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -u^2 & 0 & 1 \end{pmatrix},$$

 $\omega^{\alpha} = -u^1 \delta_1^{\alpha} + \delta_3^{\alpha}$

Functions ω^{α} :

Matrix $\hat{\lambda}$:

Matrix
$$W^{\alpha}_{\beta}$$
:

$$W^{\alpha}_{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ u^1 & 0 & 0 \end{pmatrix}.$$

From the system of equations (3.1) it follows:

$$\mathbf{A_1} = \mathbf{A_2} = 0.$$

From the system (3.2). it follows:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = a_0.$$

The components of the potential of the admissible electromagnetic field are found by the formula (3.4) and have the form:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = a_0.$$

3.1.4 Group $G_4(V)$

Metric:

$$ds^{2} = a_{1}(du^{1})^{2} + a_{2}((du^{2})^{2} + (du^{3})^{2}) \exp 2u^{1} + e_{0}(du^{0})^{2}.$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = u^1 p_1 + u^2 p_2 - p_3, \quad X_4 = -u^2 p_1 + u^1 p_2.$$

Structural constants:

$$C_{13}^{\gamma} = C_{42}^{\gamma} = \delta_1^{\gamma}, \quad C_{23}^{\gamma} = C_{14}^{\gamma} = \delta_2^{\gamma}.$$

Matrix $\hat{\lambda}$:

$$||\lambda^{\alpha}_{\beta}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u^{1} & u^{2} & -1 \end{pmatrix},$$

Functions ω^{α} :

$$\omega^{\alpha} = -u^2 \delta_1^{\alpha} + u^1 \delta_2^{\alpha}.$$

Matrix W^{α}_{β} :

$$W^{\alpha}_{\beta} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ u^2 & -u^1 & 0 \end{pmatrix}.$$

From the system of equations (3.1) it follows:

$$\mathbf{A_1} = \mathbf{A_2} = 0.$$

Substitute into the system (3.2). As a result, we get:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = a_0.$$

3.1.5 Group $G_4(VII)$

Metric:

$$ds^{2} = 4a_{1}du^{1}(du^{2} - u^{2}du^{3}) + a_{2}(du^{2} - u^{2}du^{3})^{2} - a_{1}(du^{3})^{2} + e_{0}(du^{0})^{2}.$$

Group operators:

 $X_1 = p_1, \quad X_2 = p_2 \exp u^3, \quad X_3 = p_3, \quad X_4 = (p_1 - (u^2)^2 p_2 - 2u^2 p_3) \exp -u^3.$ Structural constants:

$$C_{1\alpha}^{\gamma} = 0, \quad C_{32}^{\gamma} = \delta_2^{\gamma}, \quad C_{42}^{\gamma} = 2\delta_3^{\gamma}, \quad C_{43}^{A} = \delta_4^{A} \quad \to \quad \tilde{C}_{43}^{\gamma} = \omega^{\gamma}.$$

Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & \exp{-u^3} & 0\\ 0 & 0 & 1 \end{pmatrix},$$

Functions ω^{α} :

$$\omega^{\alpha} = (\delta_1^{\alpha} - (u^2)^2 \exp(-u^3)\delta_2^{\alpha} - 2u^2\delta_3^{\alpha}) \exp(-u^3).$$

Matrix W^{α}_{β} :

$$W_{\beta}^{\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u^2 \exp\left(-u^3\right) & 1 \\ \exp\left(-u^3\right) & -2(u^2 \exp\left(-u^3\right))^2 & -2u^2 \exp\left(-u^3\right) \end{pmatrix}.$$

From the system of equations (3.1) it follows:

$$\mathbf{A}_{\alpha,1} = 0, \quad \mathbf{A}_{\alpha} = -u^2 \exp\left(-u^3\right) \mathbf{A}_2$$

Substitute this into the system (3.2). As a result, we get:

$$\mathbf{A}_{\alpha,1} = 0, \quad \mathbf{A}_{\alpha,2} = -\delta_{\alpha 3} \exp\left(-u^3\right) \mathbf{A}_2 \quad \mathbf{A}_{\alpha,3} = \mathbf{A}_2 \delta_{\alpha 2}.$$

From here it follows:

$$\mathbf{A}_1 = 0, \quad \mathbf{A}_2 = a_0 \exp u^3, \quad \mathbf{A}_3 = -a_0 u^2.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = 0, \quad A_2 = a_0, \quad A_3 = -a_0 u^2.$$

3.1.6 Group $G_4(IV)$

Metric:

.

$$ds^{2} = a_{1}((du^{1})^{2} + \sin^{2} u^{1}(du^{2})^{2}) + a_{2}(\cos u^{1}du^{2} + du^{3})^{2} + e_{0}(du^{0})^{2}.$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = -\frac{\sin u^3 \sin u^1}{\cos u^3} p^1 + \frac{\sin u^1}{\sin u^3} p_2 + \cos u^1 p_3, \quad X_4 = \frac{\partial X_3}{\partial u^2}.$$

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = -\frac{\cos u^3 \sin u^1}{\sin u^3} p^1 + \frac{\varepsilon \sin u^1}{\sin u^3} p_2 + \cos u^1 p_3, \quad X_4 = \frac{\partial X_3}{\partial u^2}.$$

Structural constants:

$$C_{13}^A = \delta_4^A \to \tilde{C}_{13}^\gamma = \omega^\gamma, \quad C_{41}^A = \delta_3^A, \quad C_{34}^A = \delta_1^A.$$

Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \frac{\sin u^{1} \cos u^{3}}{\cos u^{1} \sin u^{3}} & -\frac{\sin u^{1}}{\cos u^{1} \sin u^{3}} & \frac{1}{\cos u^{1}} \end{pmatrix},$$

Functions ω^{α} :

$$\omega^{\alpha} = -\frac{\delta_1^{\alpha} \cos u^3}{\cos u^1 \sin u^3} + \frac{\delta_2^{\alpha}}{\cos u^1 \sin u^3} - \frac{\delta_3^{\alpha} \sin u^1}{\cos u^1}.$$

Elements of the matrix W^{α}_{β} :

$$W^{\alpha}_{\gamma} = \delta_{\alpha 1} (\delta^{\gamma}_3 + \omega^3 \omega^{\gamma}) - \delta_{\alpha 3} (\delta^{\gamma}_1 + \omega^1 \omega^{\gamma}).$$

From the system of equations (3.1) it follows:

$$\mathbf{A}_3 = -\mathbf{A}_\alpha \omega^\alpha \omega^3, \quad \mathbf{A}_1 = -\mathbf{A}_\alpha \omega^\alpha \omega^2.$$

Denote:

$$\mathbf{A} = -\mathbf{A}_{\alpha}\omega^{\alpha} \quad \rightarrow \quad \mathbf{A}_{3} = \mathbf{A}\omega^{3}, \quad \mathbf{A}_{1} = \mathbf{A}\omega^{2}.$$

From here it follows:

$$\mathbf{A} = -\mathbf{A}_2 \sin u^3 \cos u^1 \quad \mathbf{A}_1 = \mathbf{A}_2 \cos u^3, \quad \mathbf{A}_3 = \mathbf{A}_2 \sin u^1 \sin u^3.$$
(3.6)

From the system (3.2) we get

$$\mathbf{A}_{2,\alpha} = 0 \quad \to \quad \mathbf{A}_2 = a_0.$$

Thus, the components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_3 = 0, \quad A_2 = a_0, \quad A_1 = a_0 \cos u^3.$$

3.2 The subgroups G_3 of the group G_4 act transitively on subspce V_2 of V_3

3.2.1 G_3 is a subgroup of the group $G_4(I)$

. Metric:

$$ds^{2} = a_{1}(u^{3})^{\frac{1}{c-1}}(2du^{1}du^{3} + (du^{2})^{2}) + a_{2}(\frac{du^{3}}{u^{3}})^{2} + e_{0}(du^{0})^{2}.$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = 2cu^1p_1 + u^2p_2 + 2(1-c)u^3p_3, \quad X_4 = u^2p_1 - u^3p_2.$$

Structural constants:

 $C_{13}^{A} = 2c\delta_{1}^{A} \quad C_{23}^{A} = \delta_{2}^{A}, \quad C_{24}^{A} = \delta_{1}^{A}C_{34}^{A} = (1 - 2c)\delta_{4}^{A} \quad \rightarrow \quad \tilde{C}_{34}^{\alpha} = (1 - 2c)(u^{3}\delta_{2}^{\alpha} - u^{2}\delta_{1}^{\alpha}).$ Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \frac{cu^{1}}{(c-1)u^{3}} & \frac{u^{2}}{2(c-1)u^{3}} & \frac{1}{2(1-c)u^{3}} \end{pmatrix},$$

Functions ω^{α} :

$$\omega^{\alpha} = u^2 \delta_1^{\alpha} - u^3 \delta_2^{\alpha}.$$

Matrix \hat{W} :

$$W_{\alpha}^{\gamma} = -\delta_1^{\gamma} (\delta_{\alpha 2} + u^2 \delta_{\alpha 3}) + 2(1-c)\delta_2^{\gamma} \delta_{\alpha 3}.$$

From the system of equations (3.1) it follows:

$$\mathbf{A}_1 = \mathbf{A}_2 = 0.$$

Taking this into account the (2.38) will take the form:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = 2(1-c)a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_3 = A_2 = 0, \quad A_1 = \frac{a_0}{u^3}$$

3.2.2 G_3 is a subgroup of the group $G_4(II)$

. Metric:

$$ds^{2} = a_{1}(2du^{1}du^{3} + (du^{2})^{2})\exp{-2u^{3}} + a_{2}(u^{3})^{2} + e_{0}(du^{0})^{2}.$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = 2u^1p_1 + u^2p_2 + p_3, \quad X_4 = u^2p_1 - u^3p_2.$$

Structural constants:

$$C_{13}^{A} = 2\delta_{1}^{A}, \quad C_{23}^{A} = \delta_{2}^{A}, \quad C_{24}^{A} = \delta_{1}^{A}, \quad C_{43}^{A} = \delta_{2}^{A} + \delta_{4}^{A} \quad \rightarrow \quad \tilde{C}_{34}^{\alpha} = (u^{3} - 1))\delta_{2}^{\alpha} - u^{2}\delta_{1}^{\alpha}.$$

Matrix $\hat{\lambda}:$

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2u^{1} & -u^{2} & 1 \end{pmatrix}.$$

Functions ω^{α} :

$$\omega^{\alpha} = u^2 \delta_1^{\alpha} - u^3 \delta_2^{\alpha}$$

Matrix \hat{W} :

$$W^{\gamma}_{\alpha} = \delta^{\gamma}_2(-\delta_{\alpha 2} + \delta_{\alpha 3}) - u_2 \delta^{\gamma}_1 \delta_{\alpha 3}.$$

From the system of equations (3.1) it follows:

$$\mathbf{A}_1 = \mathbf{A}_2 = 0.$$

Taking this into account the (2.38) will take the form:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = a_0.$$

3.2.3 G_3 is a subgroup of the group $G_4(III)$

Metric:

$$ds^{2} = R_{3}^{-1}(a_{1}S(2du^{1}du^{3} + (du^{2})^{2}) + a_{2}(u^{3})^{2}) + e_{0}(du^{0})^{2},$$

$$R_{3} = (u_{3})^{2} + 2u^{3}\sin c + 1, \quad S = \exp\left((\sin 2c)\arctan\frac{u^{3} + \sin c}{\cos c}\right).$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = \frac{(4u^1 \sin c + (u^2)^2)p_1}{2} - u^2 u^3 p_2 - R_3 p_3, \quad X_4 = u^2 p_1 - u^3 p_2.$$

Structural constants:

$$\begin{split} C_{13}^A &= 2\delta_1^A \sin c, \quad C_{23}^A = \delta_1^A, \quad C_{24}^A = \delta_1^A, \quad C_{34}^A = \delta_2^A - \delta_4^A \sin c \quad \rightarrow \quad \tilde{C}_{34}^\alpha = \delta_2^\alpha - 2(\delta_1^\alpha u^2 - \delta_2^\alpha u^3) \sin c. \end{split}$$
 Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \frac{(4u^{1}\sin c + (u^{2})^{2})}{2R_{3}} & \frac{u^{2}u^{3}}{R_{3}} & -\frac{1}{R_{3}} \end{pmatrix}.$$

Functions ω^{α} :

$$\omega^{\alpha} = u^2 \delta_1^{\alpha} - u^3 \delta_2^{\alpha}$$

Matrix \hat{W} :

$$W^{\gamma}_{\alpha} = \delta^{\gamma}_1(-\delta_{\alpha_2} + u^2 u^3 \delta_{\alpha_3}) - R_3 \delta^{\gamma}_2 \delta_{\alpha_3}$$

From the system of equations (3.1) it follows:

$$\mathbf{A}_1 = \mathbf{A}_3 = 0.$$

Taking this into account the (2.38) will take the form:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = -a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = \frac{a_0}{R_3}.$$

3.2.4 G_3 are subgroup of the groups $G_4(VII) - G_4(VIII)$

If we introduce the functions:

$$st(u^{\nu}), \quad ct(u^{\nu}), \quad tn(u^{\nu}) = \frac{st(u^{\nu})}{ct(u^{\nu})}, \quad ctn(u^{\nu}) = tn^{-1}(u^{\nu})$$

such that

$$(st(u^{\nu}))_{,\nu} = ct(u^{\nu}), \quad (ct(u^{\nu}))_{,\nu} = -e_{\nu}st(u^{\nu}) \quad (ct(u^{\nu}))^2 + e_{\nu}(st(u^{\nu}))^2 = (e_{\nu})^2 = 1,$$

then expressions for metrics of spaces admitting these groups and operators of these groups can be combined.

Metrics:

$$ds^{2} = a_{1}(e(du^{1})^{2}ct(u^{2}) + (du^{2})^{2}) + a_{2}(u^{3})^{2} + e_{0}(du^{0})^{2} \quad e = e_{1}e_{2}.$$

Group operators:

$$X_1 = p_1 e_2 tn(u^2) st(u^1) + ct(u^1)p^2$$
, $X_2 = -e_1(X_1)_{,1}$, $X_3 = p_3$, $X_4 = p_1$.

Structural constants:

$$C_1^A = -e\delta_4^A, \quad C_{14}^A = e_1\delta_2^A, \quad C_{24}^A = -\delta_1^A \quad \to \quad \tilde{C}_{12}^\alpha = ctn(u^2)(-st(u^1)\delta_1^\alpha + ct(u^1)\delta_2^\alpha)$$

Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} ctn(u^{2})st(u^{1})e & -ctn(u^{2})ct(u^{1})e & 0\\ ct(u^{1}) & st(u^{1})e^{1} & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

Functions ω^{α} :

$$\omega^{\alpha} = ctn(u^2)(e\delta_1^{\alpha}st(u^1) - ct(u^1)\delta_2^{\alpha}) + \delta_3^{\alpha}.$$

Matrix \hat{W} :

$$||W_{\alpha}^{\gamma}|| = \begin{pmatrix} e\omega^{1}\omega^{2} & e(\omega^{2})^{2} & 0\\ -1 - e(\omega^{1})^{2} & -e\omega^{1}\omega^{2} & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

From the system of equations (3.1) it follows:

$$\mathbf{A}_1 = \mathbf{A}_2 = 0.$$

Taking this into account the (2.38) will take the form:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = -a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = a_0.$$

3.3 The groups G_4 act transitively on a null subspace V_3^* 3.3.1 Group $G_4(I)$

Metric:

$$ds^{2} = a_{1} \exp\left(-2u^{3}\right)\left(2du^{1}du^{0} + (du^{2})^{2}\right) + a_{2}(u^{3})^{2}\right) + e_{0}(du^{0})^{2},$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = 2u^1p_1 + u^2p_2 + p_3, \quad X_4 = u^2p_1 - u^0p_2.$$

((Only for this case the condition $(\xi_A^{\alpha})_{,0} \neq 0$ arises.)) Structural constants:

 $C_{13}^{A} = 2\delta_{1}^{A}, \quad C_{23}^{A} = \delta_{2}^{A}, \quad C_{24}^{A} = \delta_{1}^{A}, \quad C_{43}^{A} = 2\delta_{4}^{A} \quad \rightarrow \quad \tilde{C}_{43}^{\alpha} = 2\omega^{\alpha} = 2(u^{2}\delta_{1}^{\alpha} - u^{0}\delta_{2}^{\alpha}).$ Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -2u^{1} & -u^{2} & 1 \end{pmatrix}.$$

Functions ω^{α} :

$$\omega^{\alpha} = u^2 \delta_1^{\alpha} - u^3 \delta_2^{\alpha}$$

Matrix \hat{W} :

$$W_i^{\gamma} = -\delta_2^{\gamma}\delta_{i0} - \delta_1^{\gamma}\delta_{i2} + u^0\delta_2^{\gamma}\delta_{i3}.$$

From the system of equations $W_i^{\gamma} \mathbf{A}_{\gamma} = 0$ it follows:

$$\mathbf{A}_1 = \mathbf{A}_2 = 0.$$

Taking this into account the (2.38) will take the form:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = -a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = a_0.$$

3.3.2 Group G_4

Metric:

.

$$ds^{2} = 4du^{3}du^{0} + a\exp\left(-2u^{3}\right)\left((du^{1})^{2} + (du^{2})^{2}\right)$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = u^1 p_1 + u^2 p_2 + p_3, \quad X_4 = u^1 p_2 - u^2 p_1.$$

Structural constants:

$$C_{13}^A = -\delta_1^A, \quad C_{23}^A = \delta_2^A, \quad C_{14}^A = \delta_2^A, \quad C_{24}^A = -\delta_1^A,$$

Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -2u^{1} & -u^{2} & 1 \end{pmatrix}.$$

Functions ω^{α} :

$$\omega^{\alpha} = u^1 \delta_2^{\alpha} - u^2 \delta_1^{\alpha}.$$

Matrix \hat{W} :

$$W^{\gamma}_{\alpha} = -\delta^{\gamma}_{2}\delta_{\alpha 1} + \delta^{\gamma}_{1}\delta_{\alpha 2} + (u^{2}\delta^{\gamma}_{1} + u^{1}\delta^{\gamma}_{2})\delta_{\alpha 3}.$$

From the system of equations $W_i^{\gamma} \mathbf{A}_{\gamma} = 0$ it follows:

 $\mathbf{A}_1 = \mathbf{A}_2 = 0.$

Taking this into account the (2.38) will take the form:

$$\mathbf{A}_{3,\alpha} = 0 \quad \to \quad \mathbf{A}_3 = -a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = a_0.$$

3.3.3 Group $G_4(V)$

Metric:

$$ds^{2} = 2du^{3}du^{0} + a(\exp(-2u^{3})(du^{2})^{2} + (du^{2})^{2}) + \varepsilon \exp(-u^{3})du^{0}du^{2} \quad \varepsilon = 0, 1$$

Group operators:

 $X_1 = p_1$, $X_2 = p_2$, $X_3 = u^2 p_2 + p_3$, $X_4 = -\varepsilon p_1 \exp -u^3 + ((u^2)^2 + \exp -2u^3)p_2 + 2u^2 p_3$. Structural constants:

$$C_{23}^{A} = \delta_{2}^{A}, \quad C_{24}^{A} = 2\delta_{3}^{A}, \quad C_{34}^{A} = 2u^{2}\delta_{3}^{A} - \delta_{2}^{A}((u^{2})^{2} + \exp 2u^{2}),$$

Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -u^2 & 1 \end{pmatrix}.$$

Functions ω^{α} :

$$\omega^{\alpha} = -\delta_1^{\alpha} \exp u^3 - ((u^2)^2 + \exp -2u^3)\delta_2^{\alpha} + 2u^2\delta_3^{\alpha}.$$

Matrix \hat{W} :

$$W_{\beta}^{\gamma} = -2(\delta_3^{\gamma} + u^2 \delta_2^{\gamma})\delta_{\beta 2} - (2u^2 \delta_3^{\gamma} - \varepsilon \delta_1^{\gamma} \exp u^3)\delta_{\beta 3}.$$

From the system of equations $W_i^{\gamma} \mathbf{A}_{\gamma} = 0$ it follows:

$$\mathbf{A}_3 + u^2 \mathbf{A}_2 = 0, \quad 2u^2 \mathbf{A}_3 - \varepsilon \mathbf{A}_1 = 0.$$
(3.7)

Consider separately the variants $\varepsilon = 0$ and $\varepsilon = 1$.

A) $\varepsilon = 0$. In this case, the system (2.38) will take the form:

$$\mathbf{A}_{3,\alpha} = 0 \quad \rightarrow \quad \mathbf{A}_3 = -a_0.$$

The components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_1 = A_2 = 0, \quad A_3 = a_0.$$

B) $\varepsilon = 1$. From (3.7) it follows:

$$\mathbf{A}_{2} = -\frac{\mathbf{A}_{1} \exp u^{3}}{2(u^{2})^{2}}, \quad \mathbf{A}_{3} = \frac{\mathbf{A}_{1}}{u^{2}}.$$
(3.8)

From the system (2.38) it follows:

$$\mathbf{A}_1 = a_0, \quad \mathbf{A}_{2,2},$$

which contradicts the relations (3.8). Thus, in this case, there is no permissible electromagnetic field.

3.3.4 Group $G_4(VIII)$

Metric:

$$ds^{2} = a_{1}((du^{1})^{2} + \sin^{2} u^{1}(du^{2})^{2}) + 2a_{2}\varepsilon \cos u^{1}du^{1}du^{0} + e_{0}(du^{0})^{2} \quad \varepsilon = 0, 1.$$

Group operators:

$$X_1 = p_1, \quad X_2 = p_2, \quad X_3 = -\frac{\cos u^3 \sin u^1}{\sin u^3} p^1 + \frac{\varepsilon \sin u^1}{\sin u^3} p_2 + \cos u^1 p_3, \quad X_4 = \frac{\partial X_3}{\partial u^2}$$

Structural constants:

$$C_{13}^A = \delta_4^A \to \tilde{C}_{13}^\gamma = \omega^\gamma, \quad C_{41}^A = \delta_3^A, \quad C_{34}^A = \delta_1^A$$

Matrix $\hat{\lambda}$:

$$||\lambda_{\beta}^{\alpha}|| = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ \frac{\sin u^{1} \cos u^{3}}{\cos u^{1} \sin u^{3}} & -\frac{\varepsilon \sin u^{1}}{\cos u^{1} \sin u^{3}} & \frac{1}{\cos u^{1}} \end{pmatrix},$$
$$\omega^{\alpha} = -\frac{\delta_{1}^{\alpha} \cos u^{3}}{\delta_{1}^{\alpha} \cos u^{3}} + \frac{\varepsilon \delta_{2}^{\alpha}}{\varepsilon \delta_{2}^{\alpha}} - \frac{\delta_{3}^{\alpha} \sin u^{1}}{\varepsilon \delta_{3}^{\alpha} \sin u^{1}}$$

Functions ω^{α} :

$$\omega^{\alpha} = -\frac{\delta_1^{\alpha} \cos u^3}{\cos u^1 \sin u^3} + \frac{\varepsilon \delta_2^{\alpha}}{\cos u^1 \sin u^3} - \frac{\delta_3^{\alpha} \sin u^1}{\cos u^1}$$

Elements of the matrix W^{α}_{β} :

$$W_{\gamma}^{\alpha} = \delta_{\alpha 1} (\delta_3^{\gamma} + \omega^3 \omega^{\gamma}) - \delta_{\alpha 3} (\delta_1^{\gamma} + \omega^1 \omega^{\gamma}).$$

These formulas differ from the formulas given in the variant 6 (when the group $G_4(VIII)$ acts on a non-isotropic hypersurface V_3), by the presence of the ε quantity. Repeating the calculations performed in variant 6, we obtain the solution of the system $W^{\gamma}_{\alpha} \mathbf{A}_{\gamma} = 0$ in the following form:

$$\mathbf{A} = -\varepsilon \mathbf{A}_2 \sin u^3 \cos u^1 \quad \mathbf{A}_1 = \varepsilon \mathbf{A}_2 \cos u^3, \quad \mathbf{A}_3 = \varepsilon \mathbf{A}_2 \sin u^1 \sin u^3.$$
(3.9)

If $\varepsilon = 1$, From the system (3.2) we get

$$\mathbf{A}_{2,\alpha} = 0 \quad \to \quad \mathbf{A}_2 = a_0,$$

And the components of the potential of the admissible electromagnetic field are as follows:

$$A_0 = A_3 = 0, \quad A_2 = a_0, \quad A_1 = a_0 \cos u^3.$$

If $\varepsilon = 0$, from (3.2) and (3.9) it follows:

 $A_i = 0.$

4 Conclusion

The classification of admissible electromagnetic fields carried out in this article should be considered as a stage in the general program of research into the problem of integrating the classical and quantum equations of motion of a test particle in external fields of different nature in spaces with symmetry due to the sets of Killing fields. This program is effectively implemented for the Stackel spaces. Complete or partial classifications of the Stackel and special Stackel spaces of the electrovacuum are obtained in the General Theory of Relativity, as well as in the scalar-tensor theory.

The possibility of applying the theory of symmetry to the construction of cosmological models in the theory of gravity, including the Brans—Dicke theory, is also being studied (see, for example, [20] - [21]). As is known, from a physical point of view, the Robertson—Walker space is the most important special case of the Stackel space.

We also note the activity on the study of spaces belonging to the intersection of Stackel sets and homogeneous spaces (see [22]-[25]). The solution of the classification problem considered in the program allows us to extend this activity to solving problems in the presence of admissible electromagnetic fields. As a rule, in the given examples, the classification problem was considered within the framework of a specific theory of gravity with the involvement of the gravitational field equations. This greatly simplifies the classification since it imposes additional serious restrictions on the potential and metric. In [26]-[28] the classification of Stackel spaces and admissible external electromagnetic fields, in which the Hamilton—Jacobi equation for a charged test particle allows complete separation of variables, without involving the field equations, was first carried out. At the same time, the same classification problem for the Klein—Gordon—Fock equation is far from being solved.

A fundamentally different situation takes place in the implementation of the classification of admissible electromagnetic fields in spaces admitting groups of motions, since in these spaces the algebras of integrals of motion are already known, due to the theorem proved in this paper. To solve the problem, it remains to integrate the systems of equations (2.13) for each space with a Lorentzian signature that admits a group of motions. This classification problem is expected to be completely solved in the near future.

At the next stage of research, within the framework of the general program, one can consider the squared Dirac—Fock equations. In [30],[29] a method for integrating the Dirac—Fock equation was proposed, which allows one to reduce the solution of some bispinor equations to the problem of integrating linear scalar equations of the second order. It is supposed to investigate these scalar equations for the existence of admissible electromagnetic fields.

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