ALGEBRAS OF SYMMETRIC HOLOMORPHIC FUNCTIONS ON $\ell_p$

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Abstract. We study the algebra of uniformly continuous holomorphic symmetric functions on the ball of $\ell_p$, investigating in particular the spectrum of such algebras. To do so, we examine the algebra of symmetric polynomials on $\ell_p$ spaces as well as finitely generated symmetric algebras of holomorphic functions. Such symmetric polynomials determine the points in $\ell_p$ up to a permutation.

In recent years, algebras of holomorphic functions on the unit ball of standard complex Banach spaces have been considered by a number of authors and the spectrum of such algebras was studied in [1],[2], [7]. For example, properties of $A_u(B_X)$, the algebra of uniformly continuous holomorphic functions on the ball of a complex Banach space $X$ have been studied by Gamelin, et al. Unfortunately, this analogue of the classical disc algebra $A(D)$ has a very complicated, not well understood, spectrum. If $X^*$ has the approximation property, the spectrum of $A_u(B_X)$ coincides with the closed unit ball of the bidual if, and only if, $X^*$ generates a dense subalgebra in $A_u(B_X)$ [5]. In a very real sense, however, the problem is that $A_u(B_{\ell_2})$ is usually too large, admitting far too many functions. For instance, $\ell_\infty \subset A_u(B_{\ell_2})$ isometrically via the mapping $a = (a_j) \mapsto P_a$, where $P_a(x) \equiv \sum_{j=1}^{\infty} a_j x_j^2$.

This paper addresses this problem, by severely restricting the functions which we admit. Specifically, we limit our attention here to uniformly continuous symmetric holomorphic functions on $B_{\ell_p}$. By a symmetric function on $\ell_p$ we mean a function which is invariant under any reordering of the sequence in $\ell_p$. Symmetric polynomials in finite dimensional spaces can be studied in [9] or [12]; in the infinite dimensional Hilbert space they already appear in [11]. Throughout this note $P_s(\ell_p)$ is the space of symmetric polynomials on a complex space $\ell_p$, $1 \leq p < \infty$. Such polynomials determine, as we prove, the points in $\ell_p$ up to a permutation. We will use the notation $A_{us}(B_{\ell_p})$ for the uniform algebra of symmetric holomorphic functions which are uniformly continuous on the open unit ball $B_{\ell_p}$ of $\ell_p$, and we also study some particular finitely generated subalgebras. The purpose of this paper is to describe such algebras and their spectra, which we identify with certain subsets of $\ell_\infty$ and $\mathbb{C}^m$, respectively, and as a result of this we show that $A_{us}(B_{\ell_p})$ is algebraically and topologically isomorphic to a uniform Banach algebra generated by coordinate projections in $\ell_\infty$. This is done in Section 3, following algebraic preliminaries and a brief examination of the finite dimensional situation in Sections 1 and 2.


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We denote by $\tau_{pw}$ the topology of pointwise convergence in $\ell_\infty$. We follow the usual conventions, denoting by $\mathcal{H}_b(X)$ the Fréchet algebra of $\mathbb{C}$–valued holomorphic functions on a complex Banach space $X$ which are bounded on bounded subsets of $X$, endowed with the topology of uniform convergence on bounded sets. The subalgebra of symmetric functions will be denoted $\mathcal{H}_{bs}(X)$. For any Banach or Fréchet algebra $A$, we put $\mathcal{M}(A)$ for its spectrum, that is the set of all continuous scalar valued homomorphisms. For background on analytic functions on infinite dimensional Banach spaces we refer the reader to [3].

1. The algebra of symmetric polynomials

Let $X$ be a Banach space and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on $X$. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. A sequence $(G_i)_i$ of polynomials is called an algebraic basis of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is $q \in \mathcal{P}(\mathbb{C}^n)$ for some $n$ such that $P(x) = q(G_1(x), \ldots, G_n(x))$, in other words, if $G$ is the mapping $x \in X \sim G(x) := (G_1(x), \ldots, G_n(x)) \in \mathbb{C}^n$, $P = q \circ G$.

Let $<p>$ be smallest integer number that is greater than or equal to $p$. In [8] is proved that the polynomials $F_k(\sum a_i \xi_i) = \sum a_i^k$ for $k =< p >$, $< p > +1, \ldots$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$. So there are no symmetric polynomials of degree less than $<p>$ in $\mathcal{P}_s(\ell_p)$ and if $<p_1> =< p_2>$ then $\mathcal{P}_s(\ell_{p_1}) = \mathcal{P}_s(\ell_{p_2})$. Thus, without loss of generality we can consider $\mathcal{P}_s(\ell_p)$ only for integer $p$. Throughout we will assume that $p$ is an integer number, $1 \leq p < \infty$.

It is well known ([9] XI §52) that for $n < \infty$ any polynomial in $\mathcal{P}_s(\mathbb{C}^n)$ is uniquely representable as a polynomial in the elementary symmetric polynomials $(R_i)_{i=1}^n$, $R_i(x) = \sum_{k_1<\cdots<k_i} x_{k_1} \cdots x_{k_i}$.

**Lemma 1.1** Let $\{G_1, \ldots, G_n\}$ be an algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $G_i(x) = \xi_i$, $i = 1, \ldots, n$. If for some $y = (y_1, \ldots, y_n)$, $G_i(y) = \xi_i$, $i = 1, \ldots, n$, then $x = y$ up to a permutation.

**Proof.** First we suppose that $G_i = R_i$. Then according to the Vieta formulae [9], the solutions of the equation

$$x^n - \xi_1 x^{n-1} + \cdots + (-1)^n \xi_n = 0$$

satisfy the conditions $R_i(x) = \xi_i$ and so $x = (x_1, \ldots, x_n)$ as required. Let now $G_i$ be an arbitrary algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. Then $R_i(x) = v_i(G_1(x), \ldots, G_n(x))$ for some polynomials $v_i$ on $\mathbb{C}^n$. Setting $v$ as the polynomial mapping $x \in \mathbb{C}^n \sim v(x) := (v_1(x), \ldots, v_n(x)) \in \mathbb{C}^n$, we have $R = v \circ G$.

As the elementary symmetric polynomials also form a basis, there is a polynomial mapping $w : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $G = w \circ R$, hence $R = (v \circ w) \circ R$, so $v \circ w = id$. Then $v$ and $w$ are inverse each other since $w \circ v$ coincides with the identity on the open set $\text{Im}(w)$. In particular, $v$ is one to one.

Now, the solutions $x_1, \ldots, x_n$ of the equation

$$x^n - v_1(\xi_1, \ldots, \xi_n) x^{n-1} + \cdots + (-1)^n v_n(\xi_1, \ldots, \xi_n) = 0$$

satisfy the conditions $R_i(x) = v_i(\xi)$, $i = 1, \ldots, n$. That is, $v(\xi) = R(x) = v(G(x))$, hence $\xi = G(x)$. $\square$
Corollary 1.2. Given \((\xi_1, \ldots, \xi_n) \in \mathbb{C}^n\) there is \(x \in \ell_p^{n+p-1}\) such that
\[
F_p(x) = \xi_1, \ldots, F_{p+n-1}(x) = \xi_n.
\]

This results shows that any \(P \in \mathcal{P}_a(\ell_p)\) has a “unique” representation in terms of \(\{F_k\}\), in the sense that if \(q \in \mathcal{P}(\mathbb{C}^n)\) for some \(n\) is such that \(P(x) = q(F_p(x), \ldots, F_{n+p}(x))\), and if \(q' \in \mathcal{P}(\mathbb{C}^m)\) for some \(m\) is such that \(P(x) = q'(F_p(x), \ldots, F_{m+p}(x))\), with, say \(n \leq m\), then \(q'(\xi_1, \ldots, \xi_m) = q(\xi_1, \ldots, \xi_n)\).

For \(x, y \in \ell_p\), we will write \(x \sim y\), whenever there is a permutation \(T\) of the basis in \(\ell_p\) such that \(x = T(y)\). For any point \(x \in \ell_p\), \(\delta_x\) will denote the linear multiplicative functional on \(\mathcal{P}_a(\ell_p)\) “evaluation” at \(x\). It is clear that if \(x \sim y\) then \(\delta_x = \delta_y\).

Theorem 1.3. Let \(x, y \in \ell_p\) and \(F_i(x) = F_i(y)\) for every \(i > p\). Then \(x \sim y\).

Proof. Call \(x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)\). Without loss of generality, we can assume that \(1 = |x_1| = \cdots = |x_k| > |x_{k+1}| \geq \cdots\) and \(1 \geq |y_1| \geq |y_2| \geq \cdots\)

If \(|y_1| < 1\) then for many big \(j\), \(|F_j(x)|\) will be close to \(k\) while for all big \(j\), \(|F_j(y)|\) will be close to \(0\). Thus \(|y_1| = 1\). Suppose that \(1 = |y_1| = \cdots = |y_m| > |y_{m+1}| \geq \cdots\). Claim: \(m = k\).

Suppose for a contradiction, that \(m < k\). Then, for many big \(j\), \(|F_j(x)|\) is close to \(k\), while for all big \(j\), \(|F_j(y)| < m + 1/2 < k\). This contradiction shows that \(m < k\) is false; similarly, \(k < m\) is false, and so \(m = k\).

Let \(\tilde{x} = (x_1, \ldots, x_k)\) and \(\tilde{y} = (y_1, \ldots, y_k)\). Also, for \(z = (z_i) \in \ell_p\), let \(z'\) denote the point \((z_1^1, z_1^2, \ldots)\). We claim that \(\tilde{x} \sim \tilde{y}\), where we associate \(\tilde{x} = (x_1, \ldots, x_k) \in \mathbb{C}^k\), for example, with \((x_1, \ldots, x_k, 0, 0, \ldots)\). Consider the function \(f : (S^1)^{2k} \rightarrow \mathbb{C}\) given by
\[
f(\tilde{u}, \tilde{v}) = f(u_1, \ldots, u_k, v_1, \ldots, v_k) = [u_1 + \cdots + u_k] - [v_1 + \cdots + v_k].
\]

Since \(F_j(x - \tilde{x})\) and \(F_j(y - \tilde{y}) \rightarrow 0\) as \(j \rightarrow \infty\) and since we are assuming that \(F_j(x) = F_j(y)\) for all \(j \geq p\), it follows that \(f(\tilde{x}', \tilde{y}') \rightarrow 0\) as \(j \rightarrow \infty\). Now, \(f\) is obviously a continuous function, and so it follows that for any point \((u, v) \in (S^1)^{2k}\) which is a limit point of \(\{(\tilde{x}', \tilde{y}') : j \geq p\}\), \(f(u, v) = 0\).

Next, the point \((1, \ldots, 1) \in (S^1)^{2k}\) is a limit point of \(\{(\tilde{x}', \tilde{y}') : j \geq p\}\). If the net \((\tilde{x}', \tilde{y}')_t \rightarrow (1, \ldots, 1)\), then \((\tilde{x}'^t, \tilde{y}'^t)_t \rightarrow (\tilde{x}, \tilde{y})\). Consequently, \(f(\tilde{x}, \tilde{y}) = 0\) or in other words \(F_j(\tilde{x}) = F_j(\tilde{y})\). Similarly, \(F_j(\tilde{x}) = F_j(\tilde{y})\) for all \(j\). From Lemma 1.1 it follows that \(\tilde{x} \sim \tilde{y}\). So \(F_j(x - \tilde{x}) = F_j(y - \tilde{y})\) for every \(j \geq p\) i.e.
\[
F_j(0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots) = F_j(0, \ldots, 0, y_{k+1}, y_{k+2}, \ldots)
\]
for every \(j \geq p\). If \(|x_{k+1}| = 0\) and \(|y_{k+1}| = 0\) then \(x_i = 0\) and \(y_i = 0\) for \(i > k\). Let \(|x_{k+1}| = a \neq 0\) then we can repeat the above argument for vectors \(x' = (x_{k+1}/a, x_{k+2}/a, \ldots)\) and \(y' = (y_{k+1}/a, y_{k+2}/a, \ldots)\) and by induction we will see that \(x \sim y\). \(\square\)

Corollary 1.4. Let \(x, y \in \ell_p\). If for some integer \(m \geq p\), \(F_i(x) = F_i(y)\) for each \(i \geq m\), then \(x \sim y\).

Proof. Since \(m \geq p\) then \(x, y \in \ell_m\) and from Theorem 1.3 it follows that \(x \sim y\) in \(\ell_m\). So \(x \sim y\) in \(\ell_p\). \(\square\)
Proposition 1.5. (Nullstellensatz) Let \( P_1, \ldots, P_m \in \mathcal{P}_s(\ell_p) \) be such that \( \ker P_1 \cap \cdots \cap \ker P_m = \emptyset \). Then there are \( Q_1, \ldots, Q_m \in \mathcal{P}_s(\ell_p) \) such that
\[
\sum_{i=1}^{m} P_i Q_i = 1.
\]

Proof. Let \( n = \max_i (\deg P_i) \). We may assume that \( P_i(x) = g_i(F_p(x),\ldots,F_n(x)) \) for some \( g_i \in \mathcal{P}([0,1]) \). Let us suppose that at some point \( \xi \in \mathbb{C}^{n-p+1} \), \( \xi = (\xi_1, \ldots, \xi_{n-p+1}) \), \( g_i(\xi) = 0 \). Then by Corollary 1.2 there is \( x_0 \in \ell_p \) such that \( F_i(x_0) = \xi_i \). So the common set of zeros of all \( g_i \) is empty. Thus by the Hilbert Nullstellensatz there are polynomials \( q_1, \ldots, q_m \) such that \( \sum_i g_i q_i = 1. \) Put \( Q_i(x) = q_i(F_p(x),\ldots,F_n(x)) \). \( \square \)

2. Finitely generated symmetric algebras

Let us denote by \( \mathcal{P}_s^n(\ell_p) \), \( n \geq p \) the subalgebra of \( \mathcal{P}_s(\ell_p) \) generated by \( \{F_p, \ldots, F_n\} \). By appealing to Corollary 1.2, one easily verifies that \( \mathcal{P}_s^n(\ell_p) \cap \mathcal{P}(k\ell_p) \) is a sup-norm closed subspace of \( \mathcal{P}(k\ell_p) \) for every \( k \in \mathbb{N} \).

Let \( A^n_{us}(B_{\ell_p}) \) and \( \mathcal{H}^n_{bs}(\ell_p) \) be the closed subalgebras of \( A_{us}(B_{\ell_p}) \) and \( \mathcal{H}_{bs}(\ell_p) \) generated by \( \{F_p, \ldots, F_n\} \), that is the closure of \( \mathcal{P}_s^n(\ell_p) \) in each of the corresponding algebras. Note that for any \( f \in \mathcal{H}^n_{bs}(\ell_p) \), with \( f \) having Taylor series \( f = \sum P_k \) about 0, we have \( P_k \in \mathcal{P}_s^n(\ell_p) \). Indeed, if \( f \in \mathcal{P}_s^n(\ell_p) \), it is immediate that \( P_k \in \mathcal{P}_s^n(\ell_p) \cap \mathcal{P}(k\ell_p) \) for all \( k \). Then the same holds for any \( f \in \mathcal{H}^n_{bs}(\ell_p) \) by recalling the continuity of the map which assigns to a holomorphic function its \( k \)th Taylor polynomial.

By [6] III. 1.4, we may identify the spectrum of \( A^n_{us}(B_{\ell_p}) \) with the joint spectrum of \( \{F_p, \ldots, F_n\} \), \( \sigma(F_p, \ldots, F_n) \). It is well known that \( \mathcal{M}(\mathcal{H}(\mathbb{C}^n)) = \mathbb{C}^n \) in the sense that all continuous homomorphisms are evaluations at some point in \( \mathbb{C}^n \).

Let us denote by \( \mathcal{F}_p^n \) the mapping from \( \ell_p \) to \( \mathbb{C}^{n-p+1} \) given by \( \mathcal{F}_p^n : x \mapsto (F_p(x),\ldots,F_n(x)) \). Then \( D^n_p := \mathcal{F}^n_p(\overline{B_{\ell_p}}) \) is a subset of the closed unit ball of \( \mathbb{C}^{n-p+1} \) with the max-norm.

Let \( K \) be a bounded set in \( \mathbb{C}^n \). Recall that a point \( x \) belongs to the polynomial convex hull of \( K, [K] \), if for every polynomial \( f \), \( |f(x)| \leq \sup_{z \in K} |f(z)| \). A set is polynomially convex if it coincides with its polynomial convex hull. Recall that the sup norm on \( K \) of a polynomial coincides with the sup norm on \( [K] \). It is well known (see e.g. [6]) that the spectrum of the uniform Banach algebra \( P(K) \) generated by polynomials on the compact set \( K \) coincides with the polynomially convex hull of this set. Thus, \( [D^n_p] \) denotes the polynomial convex hull of \( D^n_p \).

Theorem 2.1.

(i) The composition operator \( C_{\mathcal{F}_p^n} : \mathcal{H}(\mathbb{C}^{n+1-p}) \to \mathcal{H}^n_{bs}(\ell_p) \) given by \( C_{\mathcal{F}_p^n}(g) = g \circ \mathcal{F}_p^n \) is a topological isomorphism.

(ii) The composition operator \( C_{\mathcal{F}_p^n} : P([D^n_p]) \to A^n_{us}(B_{\ell_p}) \) given by \( C_{\mathcal{F}_p^n}(g) = g \circ \mathcal{F}_p^n \) is a topological isomorphism.

(iii) \( \mathcal{M}(\mathcal{H}^n_{bs}(\ell_p)) = \mathbb{C}^{n+1-p} \).

(iv) \( \mathcal{M}(A^n_{us}(B_{\ell_p})) = [D^n_p] \).

Proof. Clearly the composition operators are well defined and one to one, so it remains to prove that they are onto.
In (i), let \( f \in \mathcal{H}_s^p(\ell_p) \) and \( f = \sum P_k \) be the Taylor series expansion of \( f \) at 0. Since \( P_k \in \mathcal{P}_s^p(\ell_p) \), there is a homogeneous polynomial \( g_k \in \mathcal{P}(\mathbb{C}^{n+1}) \) such that \( P_k(x) = g_k(F_p(x), \ldots, F_n(x)) \). Put \( g(\xi_1, \ldots, \xi_{n-p+1}) = \sum_{k=1}^{\infty} g_k(\xi_1, \ldots, \xi_{n-p+1}) \); since \( g \) is a convergent power series in each variable, it is separately holomorphic, hence holomorphic. Note that \( f = g \circ F^n_p \).

In (ii), observe that for any \( g \in \mathcal{P}(D^n_p) \), \( ||C_{F^n_p}(g)|| = \sup_{x \in B_{F^n_p}} |g \circ F^n_p(x)| = ||g||D^n_p = ||g||D^n_p \). Thus \( C_{F^n_p} \) is an isometry, hence its range is a closed subspace, which moreover contains \( \mathcal{P}_s^n(\ell_p) \), therefore \( C_{F^n_p} \) is onto \( A_{us}(B_{F^n_p}) \).

To conclude, we record the following elementary result which will be needed in Section 3.

**Lemma 2.2.** If \((\xi_1^0, \ldots, \xi_n^0) \in [D^m_p] \) and \( n < m \) then \((\xi_1^0, \ldots, \xi_n^0) \in [D^m_p] \).

**Proof.** If \((\xi_1^0, \ldots, \xi_n^0) \notin [D^m_p] \), there is a polynomial of \( n \) variables such that
\[
|q(\xi_1^0, \ldots, \xi_n^0)| > \sup_{(\xi_1, \ldots, \xi_n) \in D^m_p} |q(\xi_1, \ldots, \xi_n)|.
\]
Consider the polynomial \( \tilde{q} \) in \( m \) variables given by \( \tilde{q}(\xi_1, \ldots, \xi_m) = q(\xi_1, \ldots, \xi_n) \). Then,
\[
\sup_{(\xi_1, \ldots, \xi_m) \in D^m_p} |\tilde{q}(\xi_1, \ldots, \xi_m)| = \sup_{x \in B_{F^n_p}} |\tilde{q}(F_p(x), \ldots, F_{p+m-1}(x))| = \\
\sup_{x \in B_{F^n_p}} |q(F_p(x), \ldots, F_{p+m-1}(x))| < |q(\xi_1^0, \ldots, \xi_n^0)| = |q(\xi_1, \ldots, \xi_m)|.
\]
But this means \((\xi_1^0, \ldots, \xi_m^0) \notin [D^m_p] \), a contradiction. \( \square \)

### 3. Spectrum of \( A_{us}(B_{\ell_p}) \)

In the study of the spectrum of \( A_{us}(B_{\ell_p}) \) the most decisive feature is that the polynomials \( \{F^n_p\}_{n=p}^{\infty} \) generate a dense subalgebra. Actually for every \( f \in A_{us}(B_{\ell_p}) \) its Taylor polynomials are easily seen to be symmetric, using the fact (see, e.g., [3]) each such polynomial can be calculated by integrating \( f \).

Note that there are symmetric holomorphic functions on \( B_{\ell_p} \), which are not in \( A_{us}(B_{\ell_p}) \). One such example is \( f = \sum_{k=p}^{\infty} F_k \). To see that \( f \) is holomorphic on the open ball \( B_{\ell_p} \), let \( x \in B_{\ell_p} \) be arbitrary and choose \( \rho < 1 \) such that \( ||x|| < \rho \). Then, \( \sum_{k=p}^{\infty} |F_k(x)| \) converges since the sequence \( (F_k(\xi)) = (f_k(\xi) / \rho^k) \) is null. On the other hand, \( f \notin A_{us}(B_{\ell_p}) \) since \( f(\ell_{v_1}) = \lim_{t \to 1} \frac{f(t \ell_{v_1})}{t^{p^{v_1}}} \to \infty \) as \( t \uparrow 1 \).

First we will show that the spectrum of the uniform algebra of symmetric holomorphic functions on \( B_{\ell_p} \) does not coincide with equivalence classes of point evaluation functionals.

**Example 3.1.** For every \( n \) put \( v_n = \frac{1}{n^{p^2}}(e_1 + \ldots + e_n) \in \mathbb{B}_{\ell_p} \). Then \( \delta_{v_n}(F_p) = 1 \) and \( \delta_{v_n}(F_j) \to 0 \) as \( n \to \infty \) for every \( j > p \). By compactness of \( \mathcal{M}(A_{us}(B_{\ell_p})) \) there is an accumulation point \( \phi \) of the sequence \( \{\delta_{v_n}\} \). Then \( \phi(F_p) = 1 \) and \( \phi(F_j) = 0 \) for all \( j > p \). From Corollary 1.4 it follows that there is no point \( z \in \ell_p \) such that \( \delta_z = \phi \). Another, more
geometric, way of looking at this example is to fix $k \in \mathbb{N}$ and consider $D_p^{p+k} \subset \mathbb{C}^{k+1}$. It is straightforward that $(1,0,\ldots,0) \notin D_p^{p+k}$, although this point is a limit of the sequence $(F_p^{p+k}(v_n)) = (1, \frac{1}{n+1}, \ldots, \frac{1}{n(k+1)p})$. Intuitively, the accumulation point $\phi$ corresponds to the point $(1,0,0,\ldots) \in B_{\ell_\infty}$.

Let us denote by $\Sigma_p := \{(a_i)_{i=p}^{\infty} \in \ell_\infty : (a_i)_{i=p}^{n} \in [D_p^n] \text{ for every } n\}$. As a consequence of Lemma 2.2, $\Sigma_p$ is the limit of the inverse sequence $\{[D_p^n], \pi_n^m, \mathbb{N}\}$ where $\pi_n^m : \mathbb{C}^{m} \to \mathbb{C}^{n}$ is the projection onto the first $n$ coordinates. When $\Sigma_p$ is endowed with the product topology, that is the topology of coordinatewise convergence, it is a non-empty compact Hausdorff space by ([4] 3.2.13). $\Sigma_p$ is a weak-star compact subset of the closed unit ball $\ell_\infty$ since the weak star topology and the pointwise convergence topology coincide on the closed unit ball of $\ell_\infty$.

Now we describe the spectrum of $A_{us}(B_{\ell_p})$. It is immediate that it is a connected set; it suffices to recall Shilov’s idempotent theorem ([6], III.6.5) and notice that there are no idempotent elements in $A_{us}(B_{\ell_p})$.

**Theorem 3.2.** $\Sigma_p$ is homeomorphic to the spectrum of $A_{us}(B_{\ell_p})$.

**Proof.** (cf ([10], 8.3)) First of all, observe that any $\Psi \in \mathcal{M}(A_{us}(B_{\ell_p}))$ is completely determined by the sequence of values $\{\Psi(F_n)\}$ since $\Psi$ is determined by its behaviour on $\mathcal{P}_s(\ell_p)$, the algebra generated by $\{F_n\}$, which in turn is dense in $A_{us}(B_{\ell_p})$.

We construct an embedding

$$j : (a_i)_{i=p}^{\infty} \in \Sigma_p \rightsquigarrow \Phi \in \mathcal{M}(A_{us}(B_{\ell_p})), \quad \text{and prove that it is a homeomorphism. Given } (a_i)_{i=p}^{\infty} \in \Sigma_p \text{ a homomorphism } j[(a_i)_{i=p}^{\infty}] := \Phi \text{ on } A_{us}(B_{\ell_p}) \text{ is defined in the following way: Every polynomial } P \in \mathcal{P}_s(\ell_p) \text{ may be written as } g \circ F_p^n \text{ for some } n \in \mathbb{N} \text{ and some polynomial } g \text{ in } n-p+1 \text{ variables. Thus we may define } \Phi(P) := g(a_p, \ldots, a_n).$$

Certainly $\Phi(P)$ is well defined since if $P = h \circ F_p^m$ for some other polynomial $h$, and, say, $m > n$, then by Corollary 1.2, $h = \hat{g}$, where $\hat{g}$ has the same meaning as in Lemma 2.2. Hence $g(a_p, \ldots, a_n) = \hat{g}(a_p, \ldots, a_n, \ldots, a_m) = h(a_p, \ldots, a_n, \ldots, a_m)$. It is easy now to see that $\Phi$ is linear and multiplicative on the subalgebra of symmetric polynomials. Also $|\Phi(P)| = |g(a_p, \ldots, a_n)| \leq ||g||_{D_p^n} = ||g||_{D_p^\infty} \leq ||P||$. Therefore $\Phi$ is uniformly continuous on $\mathcal{P}_s(\ell_p)$, and hence it has a continuous linear and multiplicative extension to the closure of $\mathcal{P}_s(\ell_p)$ that is, to $A_{us}(B_{\ell_p})$. We still denote this extension by $\Phi$.

Obviously, $j$ is one to one. Moreover $j$ is also an onto mapping: Indeed, for any $\Psi \in \mathcal{M}(A_{us}(B_{\ell_p}))$, the sequence $\{\Psi(F_n)\} \in \Sigma_p$ because $\{\Psi(F_n)_{n=p}^{m}\}$ is an element of the joint spectrum of $\mathcal{M}(A_{us}(B_{\ell_p}))$ (obtained just by taking the restriction of $\Psi$ to $A_{us}(B_{\ell_p})$ which we know to be $[D_p^m]$. Of course, $j[\{\Psi(F_n)\}] = \Psi$ since they coincide on each $F_n$.

Next, this embedding is continuous. To see this, observe first that the spectrum $\mathcal{M}(A_{us}(B_{\ell_p}))$ is an equicontinuous subset of the dual space $(A_{us}(B_{\ell_p}))^*$. Therefore, the weak-star topology coincides on it with the topology of pointwise convergence on the elements of the dense set of all symmetric polynomials, and hence on the generating system $\{F_n\}_{n=p}^{\infty}$.

Finally $j$ is a homeomorphism as the continuous bijection between two compact Hausdorff spaces. $\square$
We can view $\Sigma_p$ as “the joint spectrum” of the sequence $\{F_n\}_{n=p}^\infty$, since $\Phi(F_n) = a_n$.

We denote by $F_p$ the mapping $x \in \overline{B_{\ell_p}} \sim (F_p^n(x)) \in \mathbb{C}^n$. Note that $F_p(\overline{B_{\ell_p}}) \subset \Sigma_p$. So we may remark that the set $D_p = F_p(\overline{B_{\ell_p}}) \subset \Sigma_p$ corresponds to the set of point evaluation multiplicative functionals on $A_{us}(B_{\ell_p})$. Actually, we have that $D_p \subset B_{\ell_\infty} \cup \{(|e^{i\theta}e^{\ell_\infty}|, \ldots, |e^{i\theta}|, \ldots) | \theta \in [0, 2\pi]\}$. To see this, we first let $x \in \overline{B_{\ell_p}}$ be such that $|x_m| < 1$ for all $m \in \mathbb{N}$. Then, as we observed in the proof of Theorem 1.3, the sequence $(F_n(x))_{n=p}^\infty$ converges to 0. In case $x \in \overline{B_{\ell_p}}$ is such that $|x_m'| = 1$ for some $m' \in \mathbb{N}$, then $m'$ is unique, $x_m' = e^{i\theta}$ and further, $x_m = 0$ if $m \neq m'$. Thus $F_n(x) = e^{i\theta}$.

It is clear that $\overline{D_p}^\infty \subset [D_p^\infty]$ but we do not know whether this embedding is proper. This is related to a corona type theorem for $A_{us}(B_{\ell_p})$ since $D_p$ is dense in $\Sigma_p$ if $\overline{D_p}^\infty = [D_p^\infty]$ for all $n \in \mathbb{N}$.

Note that if $q > p$ then $D_p \subset D_q$ and the inclusion is strict. Indeed, let $x \in B_{\ell_q}$ so that $x \notin \ell_p$. If $F_q(y) = F_p(x)$ for some $y \in \ell_q$ then $x \sim y$ in $\ell_q$ and so $x \sim y$ in $\ell_p$, which is a contradiction.

**Proposition 3.3.** $\Sigma_p \subset \ell_\infty$ is polynomially convex and coincides with the polynomial convex hull of $D_p \subset (\ell_\infty, \tau_{pw})$.

**Proof.** Let $(a_i)_{i=p}^\infty \in \ell_\infty$ be such that $|P((a_i))| \leq ||P||_{\Sigma_p}$ for all polynomials $P \in \mathcal{P}(\ell_\infty)$. For any $n \geq p$ and any $g \in \mathcal{P}(\mathbb{C}^{n+1-p})$, the mapping $Q$ given by $(x_i)_{i=p}^\infty \in \ell_\infty \sim g(x_p, \ldots, x_n)$ is a polynomial on $\ell_\infty$. Hence

$$|g(a_p, \ldots, a_n)| = |Q((a_i))| \leq ||Q||_{\Sigma_p} \leq ||g||_{D_p^\infty}.$$ 

Therefore $(a_p, \ldots, a_n) \in [D_p^\infty]$, as we want and $\Sigma_p$ is polynomially convex. So to finish, it is enough to check that $\Sigma_p$ is contained in the polynomial convex hull of $D_p$. To do this, let $(a_i)_{i=p}^\infty \in \Sigma_p$ and $P \in \mathcal{P}(\ell_\infty, \tau_{pw})$. As $P$ is pointwise continuous, it depends on a finite number of variables, say $x_p, \ldots, x_n$. Thus the mapping $q$ given by $(x_p, \ldots, x_n) \sim P(x_p, \ldots, x_n, 0, \ldots, 0, \ldots)$ is a polynomial on $\mathbb{C}^{n+1-p}$. Since $(a_p, \ldots, a_n) \in [D_p^\infty]$,

$$|P((a_i))| = |P(a_p, \ldots, a_n, 0, \ldots, 0, \ldots)| = ||q(a_p, \ldots, a_n)|| \leq ||q||_{D_p^\infty} \leq ||P||_{D_p},$$

it follows that $(a_i)_{i=p}^\infty$ belongs to the polynomial convex hull of $D_p$. □

**Theorem 3.4.** There is an algebraic and topological isomorphism between $A_{us}(B_{\ell_p})$ and the uniform Banach algebra on $\Sigma_p$ generated by the $w^*(\ell_\infty, \ell_1)$ continuous coordinate functionals $\{\pi_k\}_{k=p}^\infty$.

**Proof.** For every $f \in A_{us}(B_{\ell_p})$ and $\Phi \in M(A_{us}(B_{\ell_p}))$ denote by $\hat{f}(\Phi) = \Phi(f)$ the standard Gelfand transform which is known to be an algebraic isometry into $C(\Sigma_p)$. Recall that the range of the Gelfand transform is a closed subalgebra which, as we are going to see, will coincide with $A_p$, the uniform Banach subalgebra of $C(\Sigma_p)$ generated by the coordinate functionals $\{\pi_k\}_{k=p}^\infty$. 
Since \( \hat{F}_k(\xi) = \xi_k \) for \( \xi = (\xi_i)_i \in \Sigma_p \), it follows that the Gelfand transform of \( F_k \) is the \( k^{th} \) coordinate functional on \( \ell_\infty \). As \( A_{us}(B_{\ell_p}) \) is the closure of the algebra generated by \( \{ F_k : k \geq p \} \), it follows that \( \hat{f} \in A_p \) for every \( f \in A_{us}(B_{\ell_p}) \). Therefore \( A_p \) is precisely the range of the Gelfand transform. \( \Box \)

**Proposition 3.5.** The mapping \( S : f \in A(D) \rightarrow F \in A_{us}(B_{\ell_p}) \) defined by \( F((x_i)) = \sum_{i=1}^\infty x_i^p f(x_i) \) is an isometry onto the closed subspace \( F \) of \( A_{us}(B_{\ell_p}) \) generated by \( \{ F_{k+p} \}_{k=0}^\infty \).

**Proof.** Let \( f(z) = \sum_{k=0}^\infty c_k z^k \) be the Taylor series expansion. For each \((x_i) \in B_{\ell_p}\), put

\[
F((x_i)) := \sum_{k=0}^\infty c_k F_{k+p}(x_i) = \sum_{k=0}^\infty \sum_{i=1}^\infty c_k x_i^{p+k}.
\]

Since \( |F_{k+p}((x_i))| \leq \| (x_i) \|^{p+k} \) and the series \( \sum_{k=0}^\infty c_k x_i^k \) is absolutely convergent in the open unit disc,

\[
\sum_{k=0}^\infty \sum_{i=1}^\infty |c_k x_i^{p+k}| = \sum_{k=0}^\infty |c_k| \sum_{i=1}^\infty |x_i^{p+k}| = \sum_{k=0}^\infty |c_k| F_{k+p}((|x_i|)) \leq \sum_{k=0}^\infty |c_k| (\| (x_i) \|^{p+k} = \| (x_i) \|^{p+k} \sum_{k=0}^\infty |c_k| (\| (x_i) \|^{k}) < \infty.
\]

So \( F((x_i)) \) is well defined and \( F((x_i)) = \sum_{i=1}^\infty \sum_{k=0}^\infty c_k x_i^{p+k} = \sum_{i=1}^\infty x_i^p f(x_i) \).

Also \( |F((x_i))| = |\sum_{i=1}^\infty x_i^p f(x_i)| \leq \sum_{i=1}^\infty |x_i|^p |f(x_i)| \leq \| f \|_D \| (x_i) \|^p \), and hence \( \| F \|_{B_{\ell_p}} \leq \| f \|_D \). On the other hand, if \( a \in D \) and \( x_0 = (a,0,\ldots,0,\ldots) \), we have \( x_0 \in B_{\ell_p} \) and \( |F(x_0)| = |a|^p |f(a)| \). By the maximum principle, it follows that \( \| F \|_{B_{\ell_p}} \geq \| f \|_D \).

Consequently, \( \| F \|_{B_{\ell_p}} = \| f \|_D \).

Now we check that \( F \in A_{us}(B_{\ell_p}) \) and then that actually, \( F \in F \). To do this, let \( s_m(t) = \sum_{k=0}^m c_k t^k \) be the partial sums of the Taylor series of \( f \) and let \( \psi_n = \frac{1}{n}(s_0 + s_1 + \cdots + s_n) \) be the Cesàro means. Put \( \Psi_n((x_i)) = \sum_{k=0}^m c_k F_{k+p}((x_i)) = \sum_{i=1}^\infty x_i^p s_m(x_i) \). Then

\[
\Psi_n((x_i)) = \frac{1}{n}(S_0((x_i)) + S_1((x_i)) + \cdots + S_n((x_i))) = \frac{1}{n} \sum_{i=1}^\infty x_i^p (s_0(x_i) + s_1(x_i) + \cdots + s_n(x_i)) = \sum_{i=1}^\infty x_i^p \psi_n(x_i)
\]

are the Cesàro means partial sums of \( \sum_{k=0}^\infty c_k F_{k+p} \).

Since

\[
|\Psi_n((x_i)) - F((x_i))| = \left| \sum_{i=1}^\infty x_i^p (\psi_n(x_i) - f(x_i)) \right| \leq \|\psi_n - f\| \cdot \| (x_i) \|,
\]

the uniform convergence of \( \psi_n \) to \( f \) on \( D \) implies the uniform convergence of \( \Psi_n \) to \( F \) on \( B_{\ell_p} \). So \( F \in A_{us}(B_{\ell_p}) \) and moreover \( F \in F \) since every \( \Psi_n \) is obviously in \( F \).

The mapping \( S \) being an isometry, its range is a closed subspace of \( A_{us}(B_{\ell_p}) \). Therefore, its range is onto \( F \) since \( F_{k+p} \) is the image of \( z^k \). \( \Box \)

**Proposition 3.6.** \( \Sigma_p \neq \bar{B}_{\ell_\infty} \) for every positive integer \( p \).

**Proof.** We show that no point of the form \( (e^{i\theta}, \pm 1, 0, \ldots, 0, \ldots) \) is in \( \Sigma_p \). This will follow from Proposition 3.5 applied to every linear fractional transformation \( f(z) = \frac{z-a}{1-az} \), \( |a| < 1 \),
whose Taylor series $f(z) = -a + \sum_{n=1}^{\infty} a_n (1 - |a|^2)z^n$ has radius of convergence bigger than 1. Its image $F$ by the mapping $S$ in 3.5 is $F = -aF_p + \sum_{n=1}^{\infty} a_n (1 - |a|^2)F_{n+p}$. Moreover the convergence of this series is uniform on $B_{\ell_p}$, and therefore the Gelfand transform of $F$ is $\hat{F} = -a\pi_p + \sum_{n=1}^{\infty} a_n (1 - |a|^2)\pi_{n+p}$. Pick $\theta$ such that $-ae^{i\theta} = |a|$ and assume that the point $(e^{i\theta}, 1, 0, \ldots, 0, \ldots)$ is in $\Sigma_p$. Then $|\hat{F}(e^{i\theta}, 1, 0, \ldots, 0, \ldots)| \leq ||F|| = ||f|| = 1$. However, $|\hat{F}(e^{i\theta}, 1, 0, \ldots, 0, \ldots)| = \left| (-a\pi_p + \sum_{n=1}^{\infty} a_n (1 - |a|^2)\pi_{n+p})(e^{i\theta}, 1, 0, \ldots, 0, \ldots) \right| = | -ae^{i\theta} + 1 - |a|^2 | = |a| + 1 - |a|^2 > 1$, which is a contradiction. □

We remark that arguments similar to those in Theorem 1.3 enable us to show that no point of the form $(1, -1, -1, z_4, z_5, \ldots) \in B_{\ell_\infty}$ can be in $\Sigma_p$.

Our final result describes the class of functionals on $\ell_\infty$ which belong to the range of of $A_{us}(B_{\ell_p})$ under the Gelfand transform, thereby completing a circle of connections between $A_{us}(B_{\ell_p}), A(D), C(\Sigma_p)$, and certain functionals on $\ell_\infty$. Recall that such Gelfand transforms are weak-star continuous on $\Sigma_p$.

**Proposition 3.7.** Let $\phi$ be a linear functional on $\ell_\infty$ weak-star continuous on $\Sigma_p$. Then $\phi$ is the Gelfand transform of some $F \in A_{us}(B_{\ell_p})$ and, furthermore, there is $f \in A(D)$ with $||\phi||_{\Sigma_p} = ||f||_D$ and such that $\phi(\mathcal{F}_p(x)) = \sum_{i=1}^{\infty} a_i^p f(a_i)$, $x = (a_i) \in B_{\ell_p}$.

**Proof.** Every $(a_i)_{i=p}^{\infty} \in \Sigma_p$ is the $w(\ell_\infty, \ell_1)$ convergent series $\sum_{i=p}^{\infty} a_i e_i$. Therefore, $\phi((a_i)) = \sum_{i=p}^{\infty} a_i \phi(e_i)$ and, setting $c_i = \phi(e_i)$, we have that the series $\sum_{i=p}^{\infty} c_i \pi_i$ is pointwise convergent in $\Sigma_p$ to $\phi$. Moreover, the partial sums of this series are uniformly bounded on $\Sigma_p$ since

$$\left| \sum_{j=p}^{l} c_j \pi_j ((a_i)) \right| = \left| \sum_{j=p}^{l} c_j a_j \right| = \left| \sum_{j=p}^{l} \phi(e_j) a_j \right|$$

Thus $\phi$ is the weak* limit in $C(\Sigma_p)$ of the series $\sum_{i=p}^{\infty} c_i \pi_i$. Since each of the terms in the series belongs to the range of the Gelfand transform, it follows that there is $F \in A_{us}(B_{\ell_p})$ such that $\hat{F} = \phi$ and also that the series $F = \sum_{i=p}^{\infty} c_i F_i$ converges weakly in $A_{us}(B_{\ell_p})$.

Note that $||\phi||_{\Sigma_p} = ||f||_{B_{\ell_p}}$ and also that $F$ belongs to the weakly closed subspace $\mathcal{F}$ generated by $\{F_{k+p}\}_{k=0}^{\infty}$. Thus by Proposition 3.5 there is $f \in A(D)$ such that $F(x) = \sum_{i=1}^{\infty} x_i e_i = \sum_{i=1}^{\infty} x_i^p f(x_i)$. Therefore, $\phi(\mathcal{F}_p(x)) = \hat{F}(\mathcal{F}_p(x)) = \hat{F}(x)$ as we wanted. □

**References**


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