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# ALGEBRAS OF VIRASORO TYPE, ENERGY–MOMENTUM TENSOR, AND DECOMPOSITION OPERATORS ON RIEMANN SURFACES

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## INTRODUCTION

The present paper is the direct continuation of the authors' preceding papers [1, 2], in which the realization of the program of successive operator quantization of "multiloop diagrams" in the theory of boson strings was started.

At the base of all our constructions lies the following function theoretic construction on a nonsingular Riemann surface $\Gamma$ of genus $g < \infty$ with a pair of distinguished points $P_\pm$. For any integer $\lambda$, and any integer $n + \frac{1}{2}g$ a tensor $f_n^\lambda(z)$, $z \in \Gamma$, of weight $\lambda$ which is holomorphic on $\Gamma$ away from the points $P_\pm$, where it may have poles, is defined uniquely up to a factor, and it has the following asymptotic behavior:

$$f_n^\lambda = z_\pm^{\pm n - S(\lambda)} O(1)(dz_\pm)^\lambda, \quad S(\lambda) = S(\lambda, g) = g/2 - \lambda(g-1), \tag{0.1}$$

$z_\pm$ are coordinates in neighborhoods of the points $P_\pm$.

Strictly speaking, this assertion is true for $\lambda = 0, 1$ for all $n$, except a finite number $|n| \le g/2$. For all $\lambda \ne 0, 1$ it is true for all $g > 1$, if the triple $\Gamma$, $P_\pm$ is typical; and if $g = 1$, then this assertion is true for all $n \ne 1/2$. The definition of the tensors $f_n^\lambda$ given generalizes to half-integral $\lambda$, where it depends in addition on the spinor structure.

The cases $\lambda = -1, 0, 1/2, 1, 2$ (of vector fields, functions, spinors, differentials, quadratic differentials) are the most important. For them we use the special notation $f_n^{-1} = e_n$, $f_n^0 = A_n$, $f_n^{1/2} = \Phi_n$, $f_n^1 = d\omega_{-n}$, $f_n^2 = d^2\Omega_{-n}$.

The tensors $f_n^\lambda$ have the following important multiplicative property of "almost gradedness":

$$f_n^\lambda f_m^\mu = \sum_{k=-g/2}^{g/2} Q_{n,m}^{\lambda,\mu,k} f_{n+m-k}^{\lambda+\mu}, \tag{0.2}$$

$$[e_n, f_m^\lambda] = \sum_{k=-g_0}^{g_0} R_{n,m}^{\lambda,k} f_{n+m-k}^\lambda, \quad g_0 = 3g/2. \tag{0.3}$$

In particular, for $\lambda = 0, 1$ we obtain a commutative almost-graded algebra $\mathcal{A}^\Gamma$ and an almost-graded Lie algebra $\mathcal{L}^\Gamma$

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$$[e_n, e_m] = \sum_{k=-g_0}^{g_0} c_{n,m}^{k} e_{n+m-k}. \tag{0.4}$$

Almost-graded central extensions of these algebras, the analogs of the Heisenberg and Virasoro algebras on Riemann surfaces, play a fundamental role in the operator theory of an interacting string.

The triple $(\Gamma, P_\pm)$ uniquely determines an Abelian differential of the third kind dk such that: a) away from the points $P_\pm$ it is holomorphic and at these points it has simple poles with residues $\pm 1$; b) the periods of dk with respect to any cycle on $\Gamma$ are purely imaginary. The function $\tau(z) = \operatorname{Re} k(z)$ is called "time," its level lines are denoted by $C_\tau$, the domain of $z$ such that $a \le \tau(z) \le b$ is denoted by $C_{[a,b]}$.

THEOREM. For any smooth tensor $F^\lambda(\sigma)$ of weight $\lambda$ on the contour $C_\tau$, $\sigma \in C_\tau$, or tensor which is holomorphic in the domain $C_{[a,b]}$, there is a decomposition analogous to the Fourier–Laurent decomposition

$$F^\lambda(\sigma) = \frac{1}{2\pi i} \sum_n f_n^\lambda(\sigma) \left( \oint_{C_\tau} F^\lambda f_{-n}^{1-\lambda} \right). \tag{0.5}$$

Tensors $f_n^\lambda(z)$ also have a close connection with the theory of solitons. We recall that the Baker–Akhiezer function $F_N^\lambda$ in general position of a discrete argument $N \in Z$ is defined by the following analytic properties:

a) as $Q \to P_\pm$ it has asymptotics:

$$F_N^\lambda = z_\pm^{\pm N} O(1)(dz_\pm)^\lambda, \quad z_\pm = z_\pm(Q); \tag{0.6}$$

b) away from the points $P_\pm$ it has divisor of poles $D_\lambda$ independent of N of degree $\deg D_\lambda = 2S(\lambda) + k$, $k \ge 0$;

c) it is uniquely normalized with the help of k linearly independent conditions (for example, as in [1, 3] or somewhat more general ones).

Basically in the soliton literature the case $\lambda = k = 0$ has been discussed. For all $\lambda$ and $k = 0$ the Baker–Akhiezer tensor reduces to a scalar:

$$\psi_N = F_N^\lambda / F_0^\lambda.$$

Hence in this case according to the results of [5, 6], the Baker–Akhiezer tensor $F_N^\lambda$ is a simultaneous eigenfunction of commuting differences of operators with respect to N whose coefficients can be expressed in terms of theta-functions and as a consequence are quasiperiodic in the variable N (the use of this function for $k \neq 0$ in the theory of solitons is discussed in [1, 3]).

The tensors $f_n^\lambda$ are special limit cases of general Baker–Akhiezer tensors when $k = 0$ and the divisor of poles tends to a linear combination of the points $P_\pm$. For even g it is necessary to set $f_n^\lambda = F_n^\lambda$, $D_\lambda \to S(\lambda)(P_+ + P_-)$, and for odd g one should set $f_n^\lambda = F_{n-1/2}^\lambda$, $D_\lambda \to (S(\lambda) + 1/2)P_+ + (S(\lambda) - 1/2)P_-$. Obviously the asymptotics (0.6) go into the asymptotics (0.1). In this construction the tensor weight $\lambda$ plays an important role. In recalculation for scalar constructions we get a countable set of divisors of degree g which, for $\lambda \neq 0$, lie away from the points $P_\pm$.

For real hyperelliptic curves, defined by an equation

$$y^2 = \prod_{i=1}^{2g+2} (x - x_i), \tag{0.7}$$

and points $P_+ = (\infty, +)$, $P_- = (\infty, -)$ one of the contours $C_\tau$ coincides, as A. A. Gonchar showed the authors, with the collection of $a$-cycles corresponding to segments of the real axis $[x_{2i-1}, x_{2i}]$, $i = 1, \ldots, g$. In this special case it is possible to compare our constructions with some constructions of classical analysis. For example, the general functions $F_N(x)$ occur in classical analysis as the analogs of orthogonal polynomials on this system of intervals, $x \in R^1$. No classical analogs of the multiplicative properties of the tensors $f_n^\lambda$ which arise under special choice of the poles of $F_N$ have been discussed in classical analysis.

## 1. Basic Concepts. Regularity of Vacuum

According to the representations of [1, 2] the algebrogeometric model of a boson string is a nonsingular Riemann surface $\Gamma$ of genus g with two distinguished points $P_\pm$, which correspond to the conformal compactifications of the world surface of the string as $t \to \mp\infty$ in Minkowsky space. The intermediate positions of the string are defined as the images of the contours $C_\tau$ on $\Gamma$ under the imbedding of $\Gamma$ in Minkowsky space.

Let $X^\mu(\sigma)$ and $P^\mu(\sigma)$, $\mu = 1, \ldots, D$ be the coordinate and momentum operators of a boson string with the standard commutation relations. As was shown in [2], the coefficients of the decomposition of the differential

$$\partial_\sigma X^\mu(\sigma)\,d\sigma + \pi P^\mu(\sigma) = \sum_n \alpha_n^\mu d\omega_n = J^\mu(\sigma) \tag{1.1}$$

with respect to the basis differentials $d\omega_n(z)$ together with one operator are generators of the analog of the Heisenberg algebra

$$[\alpha_n^\mu, \alpha_m^\nu] = \eta^{\mu\nu}\gamma_{nm}, \quad \gamma_{nm} = \frac{1}{2\pi i}\oint_{C_\tau} A_m dA_n. \tag{1.2}$$

Here $\eta^{\mu\nu}$ is the Minkowsky metric with signature $(-1, 1, 1, \ldots, 1)$.

The Fock spaces of "in-" and "out-" states $H_\Gamma^{in}$ and $H_\Gamma^{out}$ are defined as the spaces generated by the operators $\alpha_n^\mu$ from the vacuum vectors $|0\rangle$ and $\langle 0|$ respectively, which are defined by the relations

$$\begin{aligned} &\alpha_n^\mu\,|\,0\rangle = 0, \quad n > g/2, \quad n = -g/2, \\ &\langle 0\,|\,\alpha_n^\mu = 0, \quad n \leqslant -g/2. \end{aligned} \tag{1.3}$$

Remark. Here and later in the paper we consider only the "analytic" part of the Fock space. The analytic part is generated by operators $\overline{\alpha}_n^\mu$ which are the coefficients of the decomposition of the differential $X_\sigma^\mu d\sigma - \pi P^\mu$ with respect to the antiholomorphic differentials $\overline{d\omega_n}$. As was proved in [2], the operators $\alpha_n^\mu$ and $\overline{\alpha}_m^\nu$ commute and hence the analytic and antianalytic parts of the complete Fock space can be considered independently and absolutely in parallel.

We recall that in the notation for the functions $A_n(z)$ with indices $n > g/2$, $n = -g/2$ used in [2], they form a basis for the subspace of functions holomorphic away from $P_-$ and those with indices $n \le -g/2$ form a basis for the subspace of functions holomorphic away from $P_+$, $A_{-g/2} \equiv 1$. This lets us establish isomorphisms of $H_\Gamma^{in}$ and $H_\Gamma^{out}$ with standard Fock spaces $H_0^{in}$ and $H_0^{out}$ constructed on small contours about the points $P_\pm$ with the help of the "free" operators $a_N^{\mu,\pm}$, N being an integer, satisfying the commutation relations

$$[a_N^{\mu,\pm}, a_M^{\nu,\pm}] = \eta^{\mu\nu} N \delta_{N+M,0}. \tag{1.4}$$

The right and left vacuum vectors are defined by the relations

$$a_N^{\mu,+}\,|\,0\rangle = 0, \quad N \geqslant 0; \quad \langle 0\,|\,a_N^{\mu,-} = 0, \quad N \leqslant 0. \tag{1.5}$$

We denote by $z_\pm$ the "free" local coordinates in small neighborhoods of the points $P_\pm$, i.e., such that the differential dk in them has "free" form $dk = \pm z_\pm^{-1} dz_\pm$.

By definition the functions $A_n$ in neighborhoods of the points have the form

$$A_n = z_\pm^{\pm n - g/2}\Big(\sum_{s=0}^{\infty} \xi_s^\pm(n)\, z_\pm^s\Big) \tag{1.6}$$

for $|n| > g/2$ and are slightly changed for $|n| \le g/2$ (cf. [1, 2] for more details). One can verify directly that then the operators

$$\alpha_n^\mu = \sum_{s=0}^{\infty} \xi_s^+(n)\, a_{n+s-g/2}^{\mu,+} = \sum_{s=0}^{\infty} \xi_s^-(n)\, a_{n+g/2-s}^{\mu,-} \tag{1.7}$$

for $|n| > g/2$ and analogously for $|n| \le g/2$ satisfy the commutation relations (1.2). Using (1.7), one can formally construct the Bogolyubov transformation between the "free" Fock spaces of in and out states. However the expressions one obtains here for the coefficients of the decomposition of $a_N^{\mu,+}$ in terms of $a_M^{\mu,-}$ are in the form of infinite series which apparently converge nowhere.

Both the free operators and the energy-momentum tensors $T_0^{\pm}(z)$ defined in the standard way with their help make sense only in neighborhoods of the points $P_{\pm}$ and diverge at points of bifurcation of the contours $C_\tau$ (i.e., at the zeros of the differential dk). In [2] there was defined an energy-momentum tensor

$$T(z) = \frac{1}{2} : J^2(z) := \frac{1}{2} \sum : \alpha_n \alpha_m : d\omega_n(z) d\omega_m(z), \tag{1.8}$$

which is holomorphic on $\Gamma$ away from the points $P_{\pm}$ for any method of introduction of normal ordering considered there: $\alpha_n \alpha_m$: of the operators $\alpha_n^{\mu}$.

If $d^2\Omega_k(z)$ is a basic collection of quadratic differentials, then one can define operators $L_k$ which decompose the energy-momentum tensor with respect to this collection

$$T(z) = \sum_k L_k d^2\Omega_k(z), \quad L_k = \frac{1}{2} \sum_{n,m} l^k_{n,m} : \alpha_n \alpha_m :. \tag{1.9}$$

These operators satisfy the commutation relations

$$[L_n, L_m] = \sum_{k=-g_0}^{g_0} c^k_{mn} L_{n+m-k} + D\chi^{\Sigma}_{nm}, \tag{1.10}$$

where

$$\chi^{\Sigma}_{nm} = \frac{1}{48\pi i} \oint ((e_n''' e_m - e_n e_m''') - 2(e_n' e_m - e_n e_m') R^{\Sigma})\, dz \tag{1.11}$$

and $R^{\Sigma} = R^{\Sigma}(z)$ is a projective connection on $\Gamma$ depending on $\Sigma$, the method of normal ordering. Thus, the operators $L_k$ define a representation of a central extension of the algebra (0.4), which generalizes the Virasoro algebra to the case of Riemann surfaces. The central charge D is equal to the dimension of the Minkowsky space.

The admissible methods of normal ordering are those for which from (1.3) follow the "regularity relations of the vacuum"

$$\text{a) } L_n \mid 0\rangle = 0, \ n \geqslant g_0 - 1; \quad \text{b) } \langle 0 \mid L_n = 0, \ n \leqslant -g_0 + 1. \tag{1.12}$$

In these cases the corresponding projective connections $R^{\Sigma}$ are holomorphic everywhere on $\Gamma$ including the points $P_{\pm}$. Among such admissible methods are all the methods described by (2.26) and (2.27) of [2] in which $\sigma_{\pm g} = \pm g/2$, $\sigma_{\pm(g-1)} = \pm(g/2 - 1)$.

One should stress particularly that the conditions of (1.12) are necessary for the construction of an arbitrary conformal field theory on Riemann surfaces. They follow from the requirement of "regularity" of the vacuum, which means that for any primary field $\Phi(z)$ which one has in the theory, the states $\Phi(z)|0\rangle$ and $\langle 0|\Phi(z)$ should be holomorphic in neighborhoods of the points $P_+$ and $P_-$ respectively. We note that (1.3) is a consequence of this principle if $\Phi(z) = J(z)$.

In order to follow the whole process of interaction of the string it is necessary not only to define the global objects (of the type of the energy-momentum tensor) on the surface $\Gamma$ with distinguished points $P_{\pm}$, but also to introduce a bilinear scalar product between the spaces $H^{in}$ and $H^{out}$. In [2] a scalar product was introduced between certain right and left (in and out) Verma modules. Briefly we give the corresponding construction in a form which is equivalent to the original one of [2], but is better adapted for what follows.

We denote by $H_{\lambda}^{+}(p)$ the space generated by the basis consisting of semiinfinite forms, "exterior products," of the form

$$f^{\lambda}_{i_0} \wedge f^{\lambda}_{i_1} \wedge \cdots \wedge f^{\lambda}_{i_{m-1}} \wedge f^{\lambda}_{S-p+m} \wedge f^{\lambda}_{S-p+m+1} \wedge \cdots, \ S = S(\lambda), \tag{1.13}$$

where the indices of the basic fields $f_n^{\lambda}$ starting from some number run through all values in succession to $+\infty$.

Remark. The more general case of tensor fields holomorphic away from the line joining the points $P_{\pm}$ on which they satisfy specific boundary conditions was considered in [1, 2]. This generalization lets one define the modules $H_{\lambda}^{+}(p)$ for any, not just intergral, p. We shall dwell on this question in more detail in the next section for the case $\lambda = 1/2$.

As was shown in [1], the action of the vector fields $e_k$ on the basic fields $f_n^{\lambda}$ induces a representation of the algebra (1.10) with central charge

$$D = -2\,(6\lambda^2 - 6\lambda + 1). \qquad (1.14)$$

The "generating" vector of the right Verma module

$$|\Psi^+_{\lambda, p}\rangle = f^\lambda_{S-p} \wedge f^\lambda_{S-p+1} \wedge \ldots, \quad S = S(\lambda), \qquad (1.15)$$

satisfies the relations

$$L_n |\Psi^+_{\lambda, p}\rangle = 0, \quad n > g_0 = 3g/2; \quad L_{g_0} |\Psi^+_{\lambda, p}\rangle = h^{\lambda, p}_+ |\Psi^+_{\lambda, p}\rangle. \qquad (1.16)$$

One defines the space $H^-_\lambda(p)$ of "left" semiinfinite forms, generated by forms of the form

$$\ldots \wedge f^\lambda_{-S+p-m-1} \wedge f^\lambda_{-S+p-m} \wedge f^\lambda_{j-m+1} \wedge \ldots \wedge f^\lambda_{j_0}, \qquad (1.17)$$

analogously, where the indices of $f^\lambda_n$ go from $-\infty$ to some integer in succession.

The "generating" vector of the left Verma module

$$\langle \Psi^-_{\lambda, p}| = \ldots \wedge f^\lambda_{-S+p-2} \wedge f^\lambda_{-S+p-1} \wedge f^\lambda_{-S+p}, \quad S = S(\lambda) \qquad (1.18)$$

satsifies the relations

$$\langle \Psi^-_{\lambda, p}| L_n = 0, \quad n < -g_0, \quad \langle \Psi^-_{\lambda, p}| L_{-g_0} = \langle \Psi^-_{\lambda, p}| h^{\lambda, p}_-. \qquad (1.19)$$

We define a scalar product between the spaces $H^+_\lambda$ and $H^-_\lambda$ of all right and left "semiinfinite" forms

$$H^\pm_\lambda = \sum_{p=-\infty}^{\infty} H^\pm_\lambda(p), \quad p \in Z, \qquad (1.20)$$

as follows. For any two basic semiinfinite forms $\Psi^\pm_\lambda \in H^\pm_\lambda$ we define a bilaterally infinite form by "placing" $\Psi^-_\lambda$ and $\Psi^+_\lambda$ next to one another. If after permutation of the factors $f^\lambda_n$ in it we obtain the standard form

$$\ldots \wedge f_{s-2} \wedge f_{s-1} \wedge f_s \wedge f_{s+1} \wedge \ldots, \qquad (1.21)$$

in which all the indices go in succession from $-\infty$ to $\infty$ then we set $\langle \Psi^-_\lambda | \Psi^+_\lambda \rangle = \pm 1$ where the sign is equal to the sign of the corresponding permutation. In all other cases we set $\langle \Psi^-_\lambda | \Psi^+_\lambda \rangle = 0$ and we extend the scalar product of basis elements on $H^-_\lambda$ and $H^+_\lambda$ so defined by linearity. We note that the product of the "generating" vectors (1.15) and (1.17) is equal to

$$\langle \Psi^-_{\lambda, p_-} | \Psi^+_{\lambda, p_+} \rangle = \delta_{p_+ + p_-,\, 2S(\lambda)-1}. \qquad (1.22)$$

As established in [2], the operators $L_n$ are self-adjoint with respect to the scalar product so defined, i.e., for any two elements $\Psi^-_\lambda$, $\Psi^+_\lambda$ we have

$$\langle \Psi^-_\lambda | L_n \Psi^+_\lambda \rangle = \langle \Psi^-_\lambda L_n | \Psi^+_\lambda \rangle = \langle \Psi^-_\lambda | L_n | \Psi^+_\lambda \rangle. \qquad (1.23)$$

Now we discuss the requirement of "regularity" of the vacuum (1.12). As is clear from (1.13) and (1.19), the generating vectors $|\Psi^+_{\lambda, p_+}\rangle$ and $\langle \Psi^-_{\lambda, p_-}|$ satisfy (1.12) if:

a) $$h_\pm = 0,$$

b) $$L_{g_0-1} |\Psi^+_{\lambda, p_+}\rangle = 0, \quad \langle \Psi^-_{\lambda, p_-}| L_{-g_0+1} = 0. \qquad (1.24)$$

<u>LEMMA 1.1</u>. The vector $|\Psi^+_{\lambda, p_+}\rangle$ or $\langle \Psi^-_{\lambda, p_-}|$ satisfies (1.12a) or (1.12b) respectively (the vacuum regularity conditions) if the corresponding semiinfinite form coincides with the exterior product of all negative powers of the local parameter $z_+$ or $z_-$. This holds if

$$p_+ = p_- = 0 \qquad (1.25)$$

(it is assumed here that $g \neq 1$).

<u>Proof</u>. The basis tensors $f^\lambda_n$ in neighborhoods of the points $P_\pm$ have the form

$$f^\lambda_n = \varphi^\pm_{n, \lambda} z_\pm^{\pm n - S(\lambda)} (1 + O(z_\pm)) (dz_\pm)^\lambda, \qquad (1.26)$$

where $\varphi^\pm_{n,\lambda}$ are constants depending on $\Gamma$, $P_\pm$. The action of $e_i$ on $f^\lambda_j$ has the form (0.3). It follows from this that the equations

$$R^{\pm g_0}_{\pm(g_0-1),\, \pm S \mp p} = 0, \quad S = S(\lambda) \qquad (1.27)$$

are necessary conditions for (1.24) to hold. From [2, (3.8)], we get (1.25). The lemma is proved.

The tensors $f_n^\lambda$ holomorphic on $\Gamma$ away from the points $P_\pm$, are determined by their form (1.25) in neighborhoods of the points $P_\pm$ uniquely up to proportionality. It follows from (1.25) that the "generating" vector $|\Psi^+_{\lambda,p_+}\rangle$ for $p_+ = 0$ coincides with the usual "free" vacuum vector of the "in-string," i.e., as $t \to -\infty$

$$|\Psi^+_{\lambda,0}\rangle = 1 \wedge z_+ \wedge z_+^2 \wedge \ldots = |0_\lambda\rangle \tag{1.28}$$

for the choice of "in-normalization" $\varphi^+_{n,\lambda} \equiv 1$.

Correspondingly the left "generating" vector $\langle\Psi^-_{\lambda,p_-}|$ for $p_- = 0$ coincides with the left "free" vacuum

$$\langle 0_\lambda| = \ldots \wedge z_-^3 \wedge z_-^2 \wedge z_-^1 \wedge 1 \tag{1.29}$$

(we recall that in the case of the ordinary complex plane $z_- = z_+^{-1}$) for the choice of "out-normalization" $\varphi^-_{n,\lambda} = 1$. In order not to complicate the notation, we shall in what follows use only the "in-normalization." Under this normalization the left generating vector does not coincide with the left vacuum, but is only proportional to it:

$$\langle\Psi^-_{\lambda,0}| = \Big(\prod_{n=-\infty}^{-S(\lambda)} \varphi^-_{n,\lambda}\Big)\langle 0_\lambda|. \tag{1.30}$$

Remark. As will be clear in what follows from the structure of the explicit formulas for $\varphi^-_{n,\lambda}$ the product in (1.30) is divergent and needs suitable regularization. We shall return to this question below, interpreting this product formally for now.

COROLLARY. a) For all integral $\lambda$

$$\langle 0_\lambda | 0_\lambda\rangle = 0. \tag{1.31}$$

b) The unique value of $\lambda$ for which for the numbers $p_+ = 0$, $p_- = 0$ the condition $2S(\lambda) = p_+ + p_- + 1$ (necessary for the nontriviality of the scalar product defined above between the generating vectors) holds is $\lambda = 1/2$. One has

$$\langle 0_{1/2}|0_{1/2}\rangle = \Big(\prod_{n=-\infty}^{-1/2} \varphi^-_{n,1/2}\Big)^{-1}. \tag{1.32}$$

Remark. As shown to the authors by R. Iengo, the formal product on the right side of (1.32) arises in calculating, with the help of the basis $f_n^{1/2}$, the continuous integral with respect to spinor fields, equal to the determinant of the Dirac operator, and hence the problem of regularization of (1.32) is the ordinary problem of regularization of such a determinant.

## 2. Spinor Structures and Operator Realization of a Boson String

We fix an arbitrary representation $\rho\colon \pi_1(\Gamma) \to C^*$ of the fundamental group of the surface $\Gamma$ in the nonzero complex numbers and a contour $\sigma$ on $\Gamma$ joining the distinguished points $P_\pm$. The collection $(\Gamma, P_\pm, \rho, \sigma)$ will be called a normalized diagram. We denote by $\mathcal{F}^{1/2}(\rho, p)$ the space of half differentials $\Phi$ on $\Gamma$ which are holomorphic away from the contour $\sigma$, on which the limiting values of these differentials must satisfy the condition

$$\Phi^+ = e^{2\pi i p}\Phi^-. \tag{2.1}$$

Moreover, upon circuit about any cycle $\gamma$ these differentials are multiplied by the number $\rho(\gamma)$. To representations $\rho_0$ such that $\rho_0(\gamma) = \pm 1$ correspond ordinary spinor structures on $\Gamma$.

LEMMA 2.1. For representations $\rho$ in general position for any half-integral $\nu - p$ there exists a unique half differential $\Phi_\nu(z;\rho) \in \mathcal{F}^{1/2}(\rho, p)$ such that in neighborhoods of the points $P_\pm$ it has the form

$$\Phi_\nu = \varphi^\pm_{\nu,1/2} z_\pm^{\pm\nu-1/2}(1 + O(z_\pm))(dz_\pm)^{1/2}, \quad \varphi^+_{\nu,1/2} \equiv 1, \tag{2.2}$$

where the constants $\varphi^-_{\nu,1/2}$ depend on $\Gamma$, $P_\pm$, $\rho$, $\sigma$.

Remark. For $p = 0$ the spinor structures $\rho_0$ are representations for which the assertion of the theorem is valid if there do not exist holomorphic sections of the corresponding bundles. Hence these spinor structures will necessarily be even.

We note further that for integral p, or what is the same, half-integral $\nu$, the half differentials $\Phi_\nu$ do not depend on $\sigma$.

All assertions of the preceding section automatically carry over after introduction of a basis of half differentials $\Phi_\nu(z;\ \rho)$ to the case $\lambda = 1/2$. In particular, the spaces of right and left semiinfinite forms $H^{\pm}_{1/2,\rho}(p)$ constructed from $\Phi_\nu(z,\ \rho)$, $\nu - 1/2 - p \in Z$, are modules over the algebra (1.10) with central charge $D = 1$. The "generating" vectora (1.15) and (1.18) for $\lambda = 1/2$, $p_{\pm} = 0$ satisfy the vacuum "regularity" conditions (1.12), and formulas (1.30) and (1.32) in which now all quantities depend in addition on $\rho$ hold for them. In what follows the vacuum vectors $H^{\pm}_{1/2,\rho}(0)$ will be denoted for brevity by $|0_\rho\rangle$ and $\langle 0_\rho|$.

We consider the standard Fock spaces $\mathcal{H}^\pm$ of Dirac fermions, generated by the operators $\psi_\nu$ and $\psi^+_\nu$ with half-integral indices $\nu$, satisfying the anticommutation relations

$$[\psi_\nu, \psi_\mu]_+ = 0, \quad [\psi^+_\nu, \psi^+_\mu]_+ = 0, \quad [\psi_\nu, \psi^+_\mu] = \delta_{\mu+\nu,0}, \tag{2.2}$$

from the "vacuum vectors" $|0_F\rangle$ and $\langle 0_f|$ such that

$$\psi_\nu|0_F\rangle = \psi^+_\nu|0_F\rangle = 0, \quad \nu > 0; \quad \langle 0_F|\psi_\nu = \langle 0_F|\psi^+_\nu = 0, \quad \nu < 0. \tag{2.3}$$

If to the operators $\psi_\nu$ and $\psi^+_\nu$ we ascribe "charges" equal to 1 and $-1$ respectively, then the spaces $\mathcal{H}^\pm$ decompose into the direct sum of subspaces with fixed charge p (integral)

$$\mathcal{H}^\pm = \sum_{p=-\infty}^{\infty} \mathcal{H}^\pm_p. \tag{2.4}$$

The correspondence under which the operator $\psi_\nu$ corresponds to the exterior product of the semiinfinite forms on $\Phi_\nu$ and the operator $\psi^+_\nu$ corresponds to the differentiation $\partial/\partial\Phi_{-\nu}$ establishes an isomorphism between the spaces $\mathcal{H}^\pm_p \approx H^\pm_{1/2\rho}(p)$. For fermion vacuums one has

$$|0_\rho\rangle = |0_F\rangle, \quad \langle 0_F| = \Big(\prod_{\nu=-\infty}^{-1/2} \varphi^-_{\nu,1/2}(\rho)\Big)\langle 0_\rho|. \tag{2.5}$$

Besides the basis $\Phi_\nu(z;\ \rho)$ we introduce the basis of "dual" half differentials by setting, by definition,

$$\Phi^+_\nu = \Phi^+_\nu(z;\rho) = \Phi_\nu(z;\rho^{-1}). \tag{2.6}$$

For indices $\nu$, $\mu$ such that $\nu + \mu$ is integral, the product $\Phi_\nu\Phi^+_\mu$ is a single-valued holomorphic differential on $\Gamma$ which is holomorphic away from the points $P_\pm$. It follows from (2.1) that

$$\frac{1}{2\pi i}\oint \Phi_\nu\Phi^+_\mu = \delta_{\mu+\nu,0}. \tag{2.7}$$

We define "fermion" fields $\psi(z;\ \rho)$ and $\psi^+(z;\ \rho)$ on the surface $\Gamma$ by formulas in which the summation is over all half-integral $\nu$:

$$\psi(z;\rho) = \sum_\nu \psi_\nu\Phi_{-\nu}(z;\rho), \quad \psi^+(z;\rho) = \sum_\nu \psi^+_\nu\Phi^+_{-\nu}(z;\rho). \tag{2.8}$$

The "time" function $\tau(z)$ on $\Gamma$ introduced with the help of the differential dk lets one define the "chronologically" ordered product of operators.

THEOREM 2.1. The chronologically ordered product of operators $\psi(z;\ \rho)\psi^+(w;\ \rho)$, where $\tau(z) > \tau(w)$, is well-defined. As $z \to w$ one has the operator decomposition

$$\psi(z;\rho)\psi^+(w;\rho) = \frac{\sqrt{dz\,dw}}{z-w} + J(z,\rho) + O(z-w). \tag{2.9}$$

The coefficients of the decomposition of the operator-valued 1-form $J(z,\ \rho)$ defined from (2.9),

$$J(z,\rho) = \sum_n \alpha_n(\rho)\, d\omega_n(z),$$

satisfy the commutation relations of the analog of the Heisenberg algebra

$$[\alpha_n(\rho), \alpha_m(\rho)] = \gamma_{nm}. \tag{2.10}$$

Proof. That the operator $\psi(z;\rho)\psi^+(w;\rho)$ is well-defined means that its action on any element of $\mathcal{H}^\pm$ is a well-defined element of $\mathcal{H}^\pm \otimes \mathcal{F}_z^{1/2} \otimes \mathcal{F}_w^{1/2}$. Here the coefficient of each basis element of $\mathcal{H}^\pm$ contains only a finite number of products $\Phi_\nu(z;\rho)\Phi_\mu^+(w;\rho)$. We represent the operator $\psi(z;\rho)\psi^+(w;\rho)$ in the form

$$\psi(z;\rho)\,\psi^+(w;\rho)=\sum_{\nu,\mu}:\psi_\nu\psi_\mu^+:\Phi_{-\nu}(z,\rho)\,\Phi_{-\mu}^+(w,\rho)+\sum_{\nu<0}\Phi_\nu(z;\rho)\,\Phi_{-\nu}^+(w,\rho), \tag{2.11}$$

where : : is the ordinary Dirac normal product of fermion operators in which the death operators of the right vacuum stand to the right of all birth operators. By virtue of this definition the first summand in (2.11) correctly defines an operator which is holomorphic away from the points. We consider the second summand.

We denote by $S_p(z, w;\rho)$ the analogs of the Szegö kernels which are uniquely determined by the following analytic properties. With respect to the variables z and w the kernel $S_p$ is a half differential which is transformed upon circuit about any cycle according to the representations $\rho$ and $\rho^{-1}$ respectively. For fixed w the kernel $S_p$ is holomorphic in the variable z away from the points $P_\pm$, $z = w$, and the contour $\sigma$ on which for the boundary values of $S_p$ (2.1) holds ($S_p^+ = \exp(2\pi i p)S_p^-$). In neighborhoods of the points $P_\pm$ one has the decompositions

$$S_p = z_\pm^{\pm p}\, O(1)\,(dz_\pm)^{1/2},\quad w = \text{const}. \tag{2.12}$$

In the variable w the kernal $S_p$ satisfies the same conditions as in the variable z, but with p replaced by $-p$. Finally, the last condition which determines $S_p$ uniquely: in a neighborhood of $z = w$ the kernel $S_p$ has the form

$$S_p(z,w;\rho)=\frac{\sqrt{dz\,dw}}{z-w}+ds_p(z;\rho)+O(z-w). \tag{2.13}$$

The differential $ds_p(z,\rho)$ defined from (2.13) is holomorphic away from the points $P_\pm$ where it has simple poles with residues $\pm p$ (i.e., the differential $ds_0$ is holomorphic everywhere on $\Gamma$).

LEMMA 2.2. For $\tau(z) > \tau(w)$ the series (2.14) converges and is equal to

$$S_p(z,w;\rho)=\sum_{\nu=-\infty}^{p-1/2}\Phi_\nu(z;\rho)\,\Phi_{-\nu}^+(w;\rho). \tag{2.14}$$

For $\tau(z) < \tau(w)$ the series (2.15) converges

$$S_p(z,w;\rho)=-\sum_{p+1/2}^{\infty}\Phi_\nu(z;\rho)\,\Phi_{-\nu}^+(w;\rho) \tag{2.15}$$

(the summation in (2.14) and (2.15) is over $\nu$ such that $\nu - 1/2 - p \in \mathbb{Z}$).

The proof of the lemma is completely analogous to the proof of Lemma 2.2 of [2]. Its assertion for $p = 0$ proves that the chronological product of operators is well defined and proves (2.9), where

$$J(z,\rho)=\sum_{\nu,\mu}:\psi_\nu\psi_\mu^+:\Phi_{-\nu}(z;\rho)\,\Phi_{-\mu}^+(z,\rho)+ds_0(z,\rho). \tag{2.16}$$

Decomposing (2.16) with respect to the basis differentials $d\omega_n(z)$ we get

$$\alpha_n(\rho)=\sum_{\nu,\mu}a_{\nu,\mu}^n:\psi_\nu\psi_\mu^+:+a_n, \tag{2.17}$$

where the constants $a_{\nu,\mu}^n$ and $a_n$ which depend on $\Gamma$, $P_\pm$, $\rho$ are equal to

$$a_{\nu,\mu}^n=\frac{1}{2\pi i}\oint(A_n\Phi_{-\nu}\Phi_{-\mu}^+);\quad a_n=\frac{1}{2\pi i}\oint A_n\,ds_0. \tag{2.18}$$

For $|n| > g/2$ we have by virtue of (1.6) and (2.2) that

$$a_{\nu,\mu}^n=0,\quad |n-\nu-\mu|>g/2. \tag{2.19}$$

For $|n| \le g/2$ the width of the strip in the $\nu$, $\mu$ plane for which $a_{\nu,\mu}^n\neq 0$ is somewhat larger, but remains finite as before. Since the differential $ds_0$ is holomorphic, one has

$$a_n=a_n(\rho)=0,\quad |n|>g/2,\quad n=-g/2. \tag{2.20}$$

We omit the proof of (2.10) since it goes analogously to the proof of (1.11) in Theorem 2.1 of [2]. The theorem is proved.

In what follows, for any operator H, by $\langle H\rangle_\rho$ we shall for brevity denote its vacuum mean, i.e.,

$$\langle H\rangle_\rho = \frac{\langle o_\rho | H | o_\rho\rangle}{\langle o_\rho | o_\rho\rangle}.$$

COROLLARY 1. The vacuum mean of the operators $\psi(z;\ \rho)\psi^+(w;\ \rho)$ is equal to

$$\langle\psi(z;\rho)\psi^+(w;\rho)\rangle_\rho = S_0(z, w; \rho). \tag{2.21}$$

This equation follows from (2.11) and (2.14) and also the fact that by definition of normal products of fermion operators we have $\langle:\psi_\nu\psi^+_\mu:\rangle_\rho = 0$.

Remark. (2.21) for the case of spinor structures $\rho_0$ coincides with the well-known expression for the "fermion propagator," However in the absence of an operator realization for $g > 0$ it appears in the physical literature rather as the definition of the left side of the equation.

(2.17) generalizes the formulas of "ferminization" of the birth and death operators of a free boson string to the case $g > 0$. The appearance of the anomalous term $a_n$ in these formulas is a consequence of a definition of $\alpha_n$ which would preserve the operator decomposition (2.9) on arbitrary Riemann surfaces.

COROLLARY 2. One has

$$\langle\alpha_n(\rho)\rangle_\rho = a_n = a_n(\rho). \tag{2.22}$$

For representations $\rho_0$ corresponding to fixations of a spinor structure on $\Gamma$ it follows from (2.14) and (2.15) that

$$S_0(z, w; \rho_0) = -S_0(w, z; \rho_0). \tag{2.23}$$

It follows from (2.23) that $ds_0(z, \rho_0) = 0$. From this we get

COROLLARY 3. For spinor structures $\rho_0$ we have for all n

$$\langle\alpha_n(\rho_0)\rangle_{\rho_0} = 0. \tag{2.24}$$

(We recall that $\rho_0$ is an even spinor structure).

It is clear from the proof of Theorem 2.1 that the chronological product of any number of "fermion" operators is well-defined, and for this product, just as in the case $g = 0$, one has Vick's theorem:

$$\Psi(z_1;\rho)\ldots\Psi(z_N;\rho) = \sum_I \pm \prod_{i,\,j\in I} \langle\Psi(z_i;\rho)\Psi(z_j;\rho)\rangle_\rho : \prod_{k\notin I}\Psi(z_k;\rho): \tag{2.25}$$

Here $\Psi(z, \rho)$ is either $\psi(z;\ \rho)$ or $\psi^+(z, \rho)$. The summation in (2.25) is over all even subsets of the collection of indices (1, ..., N) and over all methods of partition of these collections into pairs (i, j). The sign in front of the product is the sign of the corresponding permutation. In order to be able to use (2.25) for calculating vacuum means, it is necessary to mention some other equations

$$\langle\psi(z;\rho)\psi(w;\rho)\rangle_\rho = 0;\quad \langle\psi^+(z;\rho)\psi^+(w;\rho)\rangle_\rho = 0. \tag{2.26}$$

Now we consider not only vacuum but also other generating vectors defined by (1.15) and (1.18) for $\lambda = 1/2$ (and which depend on $\rho$ and for nonintegral p on $\sigma$ also). As an abbreviation we write them as

$$|\Psi^+_{1/2,\rho,p}\rangle = |p_\rho\rangle, \tag{2.27}$$

$$\langle\Psi^-_{1/2,\rho,p}| = \Big(\prod_{\nu=-\infty}^{p-1/2}\varphi^-_{\nu,1/2}(\rho)\Big)\langle p_\rho|. \tag{2.28}$$

The appearance of the normalizing factor in (2.28) is connected with the use of the "in"-normalization $(\varphi^+_\nu \equiv 1)$. The asymptotic vectors $|p_\rho\rangle$ and $\langle p_\rho|$ coincide in neighborhoods of points $P_\pm$ with the corresponding vectors of the free string.

Let p be an integer. Then the operators $\alpha_n(\rho)$ defined by (2.17) act on the vectors $|p_\rho\rangle$ and $\langle p_\rho|$ as follows:

$$\alpha_n \mid p_\rho\rangle = 0, \quad n > g/2; \qquad \alpha_{-g/2} \mid p_\rho = p \mid p_\rho\rangle, \tag{2.29}$$

$$\langle p_\rho \mid \alpha_n = 0, \quad n < -g/2; \quad \langle p_\rho \mid \alpha_{-g/2} = p\langle p_\rho \mid. \tag{2.30}$$

These formulas follow from the assertion of the following lemma.

LEMMA 2.3. The operator $J(z;\rho)$ defined from the decomposition (2.9) and having the form (2.16) can be represented in the form

$$J(z;\rho) = \sum_{\nu,\mu} :\psi_\nu\psi_\mu^+: \Phi_{-\nu}(z;\rho)\Phi_{-\mu}^+(z;\rho) + ds_p(z;\rho), \tag{2.31}$$

where $: \ :_p$ is the normal ordering of fermion operators in relation to the vector $|p_\rho\rangle$ for which its death operators stand to the right of its birth operators.

The proof of the lemma follows from (2.14).

For any p (not just integers) we define the action of the operators $\alpha_n$ in the space $H^+_{1/2,\rho}(\pm p)$ with the help of the formula

$$\alpha_{n,p}(\rho) = \sum a^n_{\nu,\mu} :\psi_\nu\psi_\mu^+:_p + a_{n,p};\ \nu - p - 1/2 \in Z,\ \mu + \nu \in Z, \tag{2.32}$$

where the constants $a^n_{\nu,\mu}$ are given by (2.18) and the constant $a_{n,p}$ is also given by (2.18) but after replacing $ds_0$ by $ds_p$ in the latter.

Before formulating the assertion of the following lemma we note that the operators $\alpha_{n,p}$ defined by (2.32) for any p act in the spaces $H^\pm_{1/2,\rho}$ $(\pm p')$, $p - p' = k$ being an integer.

LEMMA 2.4. The operators $\alpha_{n,p}$ defined by (2.32) for p and p' such that $p - p' = k$ is an integer coincide. (2.29) and (2.30) are valid for these operators.

The vectors $|p_\rho\rangle$ and $\langle p_\rho|$ as well as any generating vectors are annihilated by the operators $L_n$ with $n > g_0$ and $n < -g_0$ respectively. Moreover,

$$L_{g_0}|p_\rho\rangle = \frac{p^2}{2}|p_\rho\rangle; \quad \langle p_\rho|L_{-g_0} = \varphi^-_{-g_0,-1}\cdot\frac{p^2}{2}\langle p_\rho| \tag{2.33}$$

(we recall that the operator $L_{-g_0}$ corresponds to the vector field $e_{-g_0}$ which, in a neighborhood of the point $P_-$, has the form $\varphi^-_{-g_0,-1}\cdot z_-(1 + O(z_-)\partial/\partial z_-)$.

(2.33) means that the vectors $|p_\rho\rangle$ and $\langle p_\rho|$ have, as one says, conformal dimension $p^2/2$. This agrees with the tensor weight of

$$\mathcal{A}(p; P_+, P_-, \Gamma, \rho) = \frac{\langle -p_\rho | p_\rho\rangle}{\langle 0_\rho | 0_\rho\rangle}, \tag{2.34}$$

which we consider first for the case of integral p. Both infinite products (1.30) and (2.28) are divergent and in need of regularization. At the same time, for the case of integral p their ratio is well-defined. We have

$$\mathcal{A}(\pm p; P_+, P_-, \Gamma, \rho) = \prod_{\mp 1/2}^{\mp p \pm 1/2} (\varphi^-_{\nu, 1/2}(\rho))^{\pm 1}, \quad p > 0. \tag{2.35}$$

The normalizing constants $\varphi^-_{\nu,1/2}$ depend even for their definition not only on $\Gamma$, $P_\pm$, $\rho$ but also on the choice of local coordinates in neighborhoods of the points $P_\pm$. Thus, they are tensors of weight $\nu$ in the variables $P_\pm$. In addition $\mathcal{A}(p, P_+, P_-)$ is a tensor of weight $\lambda_p = p^2/2$.

We only give expressions for $\varphi^-_{\nu,1/2}(\rho)$ for the cases of representations $\rho_0$ corresponding to even spinor structures. (The general case differs from this by insignificant changes). For this we fix on $\Gamma$ a basis of cycles $a_i$, $b_j$ with canonical intersection matrix. Fixing this basis lets us introduce: a basis of normalized holomorphic differentials $\omega_i$ on $\Gamma$, a matrix of b-periods, the Jacobian $J(\Gamma)$ of the curve $\Gamma$, the Abel map $A\colon \Gamma \to J(\Gamma)$ and the Riemann theta function $\theta(v)$, $v = (v_1, \ldots, v_g)$. Analogously to the way formulas for the solutions of the equations of a Toda lattice are constructed in [5, 7] one can obtain the following expression:

$$\varphi^{-}_{\nu,1/2}=E(P_+,P_-)^{-2\nu}\frac{\theta[\rho_0]((\nu-1/2)(A(P_+)-A(P_-)))}{\theta[\rho_0]((\nu+1/2)(A(P_+)-A(P_-)))}, \tag{2.36}$$

where $\theta[\rho_0](v)$ is a theta-function with even characteristic $\rho_0$, and $E(P_+, P_-)$ is a so-called Prym-form (cf. [8, 9]):

$$E^2(P_+,P_-)=\frac{\theta^2[m](A(P_+)-A(P_-))}{\left(\sum\omega_i(P_+)\theta_i[m](0)\right)\left(\sum\omega_i(P_-)\theta_i[m](0)\right)}. \tag{2.37}$$

By choice of the local coordinates $z_\pm$ one can make the value of the Prym-form equal to 1 at the points $P_\pm$. For such a choice of $z_\pm$ it is natural to set the infinite product in (1.32) equal to

$$\langle o_\rho \mid o_\rho\rangle = \theta[\rho_0](0). \tag{2.38}$$

In these same local coordinates we have

$$\mathcal{A}(p,P_+,P_-,\Gamma,\rho)=\frac{\theta[\rho_0](p(A(P_+)-A(P_-))}{\theta[\rho_0](0)}. \tag{2.39}$$

One can also arrive at the expression (2.39) for nonintegral p if first one regularizes the product in (2.28) similarly to (2.38) and afterwards takes the ratio of the corresponding expression and (2.38). It is true that for nonintegral p the quantity obtained depends on the contour $\sigma$ through the difference vector $A(P_+) - A(P_-)$ with coordinates

$$A_k(P_+)-A_k(P_-)=\int_\sigma\omega_k. \tag{2.40}$$

Averaging $\mathcal{A}(p, P_+, P_-, \Gamma, \rho_0)$ over all homology classes of the cycle $\sigma$, we get for rational $p = r/n$ the expression

$$\mathcal{A}=\frac{(\varepsilon_m(P_+,P_-))^{P^2}}{\theta[\rho_0](0)}\cdot\prod_{N,M}\left[\frac{\theta[\rho_0](p(A_+-A_-))+N+BM))}{\theta^{p^2}[m]((A_+-A_-)+N+BM)}\right]^{n-2g}, \tag{2.41}$$

$$A_\pm=A(P_\pm);\quad \varepsilon_m^2(P_+,P_-)=\left(\sum\omega_i(P_+)\theta_i[m](0)\right)\left(\sum\omega_i(P_-)\theta_i[m](0)\right),$$

where the product is taken over all integral vectors $N = (N_1, \ldots, N_g)$, $M = (M_1, \ldots, M_g)$, $|N_i|$, $|M_i| \le n$, and B is the matrix of b-periods of holomorphic differentials on $\Gamma_1$.*

For irrational p the result of averaging has the form

$$\mathcal{A}(p,P_+,P_-,\Gamma,\rho_0)=\left[\frac{\theta[\rho_0](A(P_+)-A(P_-))}{E(P_+,P_-)\theta[\rho_0](0)}\right]^{p^2}. \tag{2.42}$$

The last expression admits analytic continuation to complex values of p which is necessary for defining the two-point amplitude of scattering in the case of the Minkowsky metric. In fact, as is clear from the results of the present section, the in and out Fock spaces of a boson string in D-dimensional Minkowsky space are isomorphic to the D-fold tensor power of the spaces $H^{\pm}_{1/2,\rho}(0)$ which are isomorphic to the subspaces with zero charge of the Fock space of Dirac fermions

$$H^{\text{in}}_{\text{out}\,\Gamma}\approx(H^{\pm}_{1/2,\rho}(0))^{\otimes D}\approx(\mathcal{H}_0^{\pm})^{\otimes D}. \tag{2.43}$$

Under this isomosphism the operators $\alpha_n^\mu$ go into operators which act on the $\mu$-th factor as the operator $\alpha_n(\rho)$ if $\mu > 1$ and as the operator $i\alpha_n(\rho)$ for $\mu = 1$. On the remaining factors the $\alpha_n^\mu$ act as identity operators.

Under this definition the state with momentum $|\vec{p}\rangle$, $\vec{p} = (p^\mu)$ coincides with the tensor product of the corresponding generating vectors

$$|\vec{p}\rangle=\bigotimes_{\mu=1}^{D}|\sqrt{\eta^\mu}\,p_\rho^\mu\rangle,\quad \eta^{\mu\nu}=\eta^\mu\delta_\mu^\nu. \tag{2.44}$$

Consequently,

$$\tilde{\mathcal{A}}(\vec{p},P_+,P_-,\Gamma,\rho)=\prod_{\mu=1}^{D}\mathcal{A}(\sqrt{\eta^\mu}\,p^\mu,P_+,P_-,\Gamma,\rho). \tag{2.45}$$

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*Another version is averaging only over contours $\sigma = \sigma(\tau)$ from $P_+$ to $P_-$ along which the "time" $\tau(\sigma)$ increases monotonically. At the present time we do not know which version is right.

Another version of the construction in the case of the Minkowsky metric is the following. Changing the definition of "vertex" operator carrying $|0\rangle$ into $|p\rangle$ by changing the in-normalizations of basis fields to out-normalizations, we arrive at the expression (p being an integer)

$$\tilde{\mathcal{A}}(\tilde{p}, P_+, P_-, \Gamma, \rho) = \prod_{\mu=1}^{D} [\mathcal{A}(p^\mu, P_+, P_-, \Gamma, \rho)]^{\eta^\mu}. \tag{2.46}$$

The authors intend to return to the analysis of this possibility in later papers. Now we do not know which version is right.

## 3. Energy-Momentum "Pseudotensor"

The absence of a natural normal ordering of products of boson operators $\alpha_n^\mu$ when $g > 0$ leads, as already said above, to the nonuniqueness of the definition (1.8) of the energy-momentum tensor. The basic goal of the present section is the introduction in an invariant way of an energy-momentum "pseudotensor" $\tilde{T}(z)$ depending only on the "diagram" $\Gamma$, $P_+$, $P_-$.

THEOREM 3.1. The chronological product of operators $J(z)J(w)$, $J = (J^\mu)$ is well-defined. As $z \to w$ one has the operator decomposition

$$J(z)\,J(w) = D\frac{dz\,dw}{(z-w)^2} + 2\tilde{T}(z) + O(z-w). \tag{3.1}$$

For any projecive connection $R_\Sigma$ holomorphic away from the points $P_\pm$ the coefficients $L_k$ of decomposition of the operator-valued quadratic differential

$$T = \tilde{T} - \frac{D}{2}R_\Sigma = \sum_k L_k d^2\Omega_k \tag{3.2}$$

satisfy the commutation relations (1.11), (1.12) of the analog of the Virasoro algebra.

Proof. We fix the following method of normal ordering:

$$\begin{aligned} &:\alpha_n\alpha_m: = \alpha_m\alpha_n, && \text{if} \quad n > g/2,\ m < -g/2, \\ &:\alpha_n\alpha_m: = \alpha_n\alpha_m, && \text{otherwise}. \end{aligned} \tag{3.3}$$

One can represent the operator $J(z)J(w)$ in the form

$$J(z)\,J(w) = \sum :\alpha_n\alpha_m: d\omega_n(z)\,d\omega_m(w) + D\sum_{n>g/2}^{m<-g/2} \gamma_{nm}\,d\omega_n(z)\,d\omega_m(w). \tag{3.4}$$

(If one changes the method of normal ordering, then (3.4) remains valid if one changes the limits of summation in the second summand of the right side of (3.4) correspondingly).

The first summand in (3.4) well defines an operator which is holomorphic away from the points $P_\pm$. We consider the second summand. Let $\Omega(z, w)$ be a bidifferential on $\Gamma$ which is uniquely determined by its analytic properties: with respect to each of the variables z and w it is a holomorphic differential away from the diagonal $z = w$, with respect to the variable z it has a zero of order g at the point $P_-$ and with respect to the variable w, a zero of order g at the point $P_+$. In a neighborhood of $z = w$ it has the form

$$\Omega(z, w) = \frac{dz\,dw}{(z-w)^2} + R_0(z) + O(z-w). \tag{3.5}$$

The value of the regular part of $\Omega(z, w)$ on the diagonal, $R_0(z)$, is a holomorphic projective connection.

LEMMA 3.1. For $\tau(z) > \tau(w)$ the series (3.6) converges and is equal to

$$\Omega(z, w) = \sum_{n>g/2}^{m<-g/2} \gamma_{nm}\,d\omega_n(z)\,d\omega_m(w). \tag{3.6}$$

Proof. By definition of $\gamma_{nm}$ we have that the right side of (3.6) is equal to

$$\frac{1}{2\pi i}\oint_{C_\tau} \sum dA_n(z_1)\,A_m(z_1)\,d\omega_n(z)\,d\omega_m(w) \tag{3.7}$$

(here and in (3.8) the summation is as in (3.5) over $n > g/2$, $m < -g/2$). The integral is taken over the contour $\tau(z_1) = \tau_1$, where $\tau(z) > \tau_1 > \tau(w)$. By virtue of Lemma 2.2 of [2] the series under the integral sign in (3.7) converge. Using the expressions obtained we find

$$\sum \gamma_{nm}\, d\omega_n(z)\, d\omega_m(w) = \frac{1}{2\pi i} \oint dz_1 \left(\frac{\partial}{\partial z_1} K_{g/2+1}(z_1, z)\, dz\right) K_{-g/2-1}(z_1, w)\, dw, \tag{3.8}$$

where $K_N(z, w)dw$ is the kernel of Cauchy type defined in [2]. The assertion of the lemma follows from (3.8) and the analytic properties of these kernels.

Lemma 3.1 and (3.4) prove the first assertion of the theorem and (3.1), from which it follows that

$$\tilde{T}(z) = \frac{1}{2} \sum :\alpha_n \alpha_m: d\omega_n(z)\, d\omega_m(z) + \frac{D}{2} R_0(z). \tag{3.9}$$

It suffices to prove the last assertion of the theorem just for any one connection, for example, for the connection $R_0(z)$. In this case we have

$$T(z) = \tilde{T} - \frac{D}{2} R_0 = \frac{1}{2} \sum :\alpha_n \alpha_m: d\omega_n\, d\omega_m, \tag{3.10}$$

where the normal ordering is given by (3.3). For the cocycle $\chi_{nm}^{\Sigma}$, corresponding to this method of normal ordering, one can obtain an expression analogous to [2, (2.54)], which was derived in [2] for other methods of normal ordering. Comparison of the expression obtained with (3.8) proves the assertion of the theorem.

The proof of the following operator decomposition of the product of energy-momentum tensors (more precisely, pseudotensors) which is fundamental in conformal field theory is analogous to the proof of Theorems 2.1 and 3.1 and is hence omitted.

<u>THEOREM 3.2</u>. The chronological product of operators $\tilde{T}(z)\tilde{T}(w)$ is well defined. As $z \to w$ one has the decomposition

$$\tilde{T}(z)\tilde{T}(w) = \frac{D}{2(z-w)^4} + \frac{2\tilde{T}(z)}{(z-w)^2} + \frac{\tilde{T}_z(z)}{z-w} + O(1). \tag{3.11}$$

The operator decompositions obtained above, combined with Vick's theorem, let us easily find the mean of the products of operators $J(z_i)\tilde{T}(z_j)$. We consider as an example the mean of the operator $\tilde{T}(z)$ corresponding to the conformal anomaly. It follows from Theorem 2.1 that

$$\langle J(z) J(w)\rangle_\rho = \lim_{z_1 \to z} \lim_{w_1 \to w} \left\langle \left(\psi(z)\psi^+(z_1) - \frac{1}{z - z_1}\right) \left(\psi(\omega)\psi^+(\omega_1) - \frac{1}{w - w_1}\right) \right\rangle_\rho. \tag{3.12}$$

Using (2.24) for calculating the mean of the product of four fermion operators, we get

$$\langle J(z) J(w)\rangle_\rho = -S_0(z, w; \rho)\, S_0(w, z; \rho). \tag{3.13}$$

The decomposition of the bidifferential

$$-S_0(z, w; \rho)\, S_0(w, z, \rho) = \frac{dz\, dw}{(z-w)^2} + R_\rho(z) + O(z - w) \tag{3.14}$$

on the diagonal $z = w$ defines a projective connection $R_\rho(z)$ depending on the representation $\rho$ but not on the points $P_\pm$. Hence although our whole construction depends on fixing the pair of points $P_\pm$ the vacuum mean $\langle\tilde{T}(z)\rangle_\rho$ does not depend on this choice. It follows from (3.1) and (3.13) that

$$\langle \tilde{T}(z)\rangle_\rho = \frac{D}{2} R_\rho(z). \tag{3.15}$$

<u>Example</u>. $g = 1$. For the three spinor structures $\rho_\alpha$ corresponding to fixing three even half-periods $\omega_\alpha$, $\alpha = 1, 2, 3$, the elliptic curve $\Gamma$ corresponding to the Szego kernel has the form

$$S_0(z, \omega, \rho_\alpha) = \frac{\sigma(z - w + \omega_\alpha)}{\sigma(z - w)\, \sigma(w_\alpha)} e^{-\eta_\alpha(z-w)}. \tag{3.16}$$

From this and (3.14) we get

$$\langle \tilde{T}(z)\rangle_{\rho_\alpha} = \frac{D}{2} \wp(\omega_\alpha), \tag{3.17}$$

where $\sigma$, $\wp$ are the Weierstrass functions. This result coincides with the results of the calculations for the mean of the energy-momentum tensor on an elliptic curve obtained in [10] with the help of a different approach based on using the Ward identities.

## 4. Supplement and Remarks

In the preceding sections as in [1, 2] only the "boson sector" in the theory of a closed string was considered. It is well known that the construction of a conformally invariant theory requires the introduction of supplementary "ghost" fields b(z) and c(z) having tensor weight 2 and −1 respectively. Decomposing these fields with respect to the basis tensors

$$b(z)=\sum_n b_n d^2\Omega_n(z), \tag{4.1}$$

$$c(z)=\sum_n c_n e_n(z), \tag{4.2}$$

we define operators $b_n$ and $c_n$ with the usual anticommutation relations

$$[b_n, b_m]_+ = 0, \quad [c_n, c_m]_+ = 0, \quad [b_n, c_m]_+ = \delta_{n,m}. \tag{4.3}$$

As was shown in [11], the use of our constructions (in particular, the bases $e_n$ and $d^2\Omega_n$) lets one naturally generalize the definition of the energy-momentum tensor $T^{b,c}(z)$ of the ghost fields and the definition of the operator BRST of charge Q to the case of a surface $\Gamma$ of genus $g > 0$. Without dwelling on the details in these definitions we only make some comments about vacuum means in the "ghostly" sector. In correspondence with the principle of "regularity" the vacuum vectors $|o_{gh}\rangle$ and $\langle o_{gh}|$ of the ghost sector should satisfy the relations

$$\begin{aligned} b_n\,|\,o_{gh}\rangle &= 0, \quad n \geqslant g_0 - 1; \quad \langle o_{gh}\,|\,b_n = 0, \quad n \leqslant -g_0 + 1; \\ c_n\,|\,o_{gh}\rangle &= 0, \quad n < g_0 - 1; \quad \langle o_{gh}\,|\,c_n = 0, \quad n > -g_0 + 1. \end{aligned} \tag{4.4}$$

One can identify these vectors with the "vacuum" vectors $|o_2\rangle$ and $\langle o_2|$ defined in Sec. 1 (formulas (1.28), (1.30) for $\lambda = 2$), if one gives a representation of the operators $b_n$ and $c_n$ in $H_2^{\pm}$ for which the operator $c_n$ corresponds to exterior multiplication of semiinfinite forms by $f_n^2 = d^2\Omega_{-n}$, and the operator $b_n$ to "inner" differentiation $\partial/\partial f_n^2$. As already noted above, the scalar product of such vectors is equal to zero:

$$\langle o_2\,|\,o_2\rangle = 0. \tag{4.5}$$

The simplest nonzero quantity which one can form with the help of scalar products has the form

$$\begin{aligned} \langle o_2|c_{-1}c_0c_1|o_2\rangle &= 1, \quad g=0, \\ \langle o_2|b_{-1/2}c_{1/2}|o_2\rangle &\neq 0, \quad g=1, \\ \langle o_2|b_{-g_0+2}\dots b_{g_0-2}|o_2\rangle &= \Big(\prod_{n=-\infty}^{-g_0+1} \varphi^-_{n,2}\Big)^{-1}, \quad g>1. \end{aligned} \tag{4.6}$$

Since the operators $b_n$ with $|n| \leq g_0 - 2$ correspond for $g > 1$ to holomorphic quadratic differentials forming a basis in the cotangent bundle to the manifold of moduli of curves of genus g, the square of the modulus of (4.6) defines a measure on the manifold of moduli of curves of genus g with a pair of distinguished points. We shall return to the question of the expression in terms of these quantities and the norms of vacuum vectors of the boson sector of the Polyakov measure on the manifold of moduli.

We note in addition that the question of the definition of the mean $\langle T^{b,c}\rangle_{gh}$ of the energy-momentum tensor of ghosts still remains open. The fact is that analogously to (4.5) we have

$$\langle o_2\,|\,T^{b,c}(z)|\,o_2\rangle = 0. \tag{4.7}$$

Expressions of the type

$$\langle o_2\,|\,T^{b,c}(z)b_{-g_0+2}\dots b_{g_0-2}\,|\,o_2\rangle \neq 0 \tag{4.8}$$

are different from zero. Possibly the arbitrariness which one has here, connected with the fact that one can put the operator $T^{b,c}$ to the right of the operators $b_n$ or between them, is compensated for by the arbitrariness in the boson sector connected with fixing a spinor structure on Γ.

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# PRYMIANS OF REAL CURVES AND THEIR APPLICATIONS TO THE EFFECTIVIZATION OF SCHRÖDINGER OPERATORS


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In accord with the papers of A. P. Veselov and S. P. Novikov [1, 2], purely potential real two-dimensional Schrödinger operators that are two-dimensional finite-gap in relation to one level of the energy [3] are given by the following data: a Riemann surface P; commuting antiholomorphic involutions $\tau_1$, $\tau_2$: $P \to P$ such that the involution $\tau = \tau_1\tau_2$ has exactly two fixed points $p_1, p_2 \in P$; a local parameter $w_1$: $W_1 \to C$, $p_1 \in W_1$ such that $w_1\tau = -w_1$; a divisor $D \in P$ possessing some property of symmetry of a type [4] with respect to the involution $\tau_i$. An operator constructed according to such initial data has the form

$$L = \partial\bar\partial + 2\partial\bar\partial \ln \theta\,(zU_1 + \bar z U_2 - e \mid V) - \varepsilon_0.$$

An algorithm is presented in the paper, allowing one to find the elements of the Prym matrix V of the curve (P, τ), the vectors $U_1$, $U_2$ of the Prymian $P_r = P_r(P, \tau)$ and $\varepsilon_0 \in R$ in the form of convergent series of parameters defining the curve $(P, \tau_1, \tau_2)$ and the map $w_1$. The selection of nonsingular operators from the real algebraic ones leads to questions of independent interest of the theory of real algebraic curves.

In Sec. 1 properties are analyzed of real abelian differentials on real algebraic curves $(P^*, \tau^*)$, that is, abelian differentials $\omega^*$ on a compact Riemann surface $P^*$, invariant with respect to the antiholomorphic involution $\tau^*$: $P^* \to P^*$. In a map symmetric with respect to $\tau^*$ the differential $\omega^*$ assumes real values on the ovals. All the relations between the signs of these values are found.

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