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ALGEBRAS WHOSE CONGRUENCE LATTICES ARE DISTRIBUTIVE

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1. Introduction.

This note is concerned with equational classes $\boldsymbol{K}$ of algebras, subject to the condition

$\Delta(\boldsymbol{K})$ *For all* $A \in \boldsymbol{K}$, $\Theta(A)$ *is distributive.*

Here $\Theta(A)$ is the lattice of all congruence relations over A. In Section 2 necessary and sufficient conditions are obtained in order for $\Delta(\boldsymbol{K})$ to hold. This result is inspired by a theorem in Malcev [7] which gives necessary and sufficient conditions in order for all the algebras in $\boldsymbol{K}$ to have permutable congruence relations.

The remainder of the paper, which is independent of Section 2, is based on Birkhoff's theorem ([2]) about equational classes of algebras. For any class $\boldsymbol{K}$ of (similar) algebras, let $\boldsymbol{K}^e$ be the smallest equational class that contains $\boldsymbol{K}$. Also, let $I(\boldsymbol{K})$, $H(\boldsymbol{K})$ and $S(\boldsymbol{K})$ be the classes consisting, respectively, of all isomorphic images, homomorphic images, and subalgebras of members of $\boldsymbol{K}$, and let $P(\boldsymbol{K})$, $P_s(\boldsymbol{K})$ and $P_u(\boldsymbol{K})$ be the classes consisting, respectively, of all direct products, subdirect products, and ultraproducts of members of $\boldsymbol{K}$. (For general information about ultraproducts (prime products) see e.g. Frayne, Morel and Scott [4].) As a corollary to Birkhoff's theorem, it is shown in Tarski [10] that

$$\boldsymbol{K}^e = HSP(\boldsymbol{K}) .$$

The principal result in Section 3 states that

$$\Delta(\boldsymbol{K}^e) \Rightarrow \boldsymbol{K}^e = IP_sHSP_u(\boldsymbol{K}) .$$

The proof is quite simple, but some of the consequences are rather unexpected. For instance, as a special case of Corollary 3.5 we find that for any two finite, non-isomorphic, subdirectly irreducible lattices there exists an identity that holds in one but not in the other. Once such

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identities are known to exist, it is of course theoretically possible to find them in each specific instance, simply by arranging all the lattice identities into a sequence and testing them one by one. However, this method is clearly too inefficient to be actually carried out except in the most trivial cases, and no practical procedure is known for finding such identities. In particular, we do not know any identity that distinguishes between the lattices of all subspaces of two n-dimensional vector spaces over non-isomorphic fields with the same characteristic.

Section 4 contains some simple results about the lattice of all equational classes of algebras, and Sections 5 and 6 are concerned with equational classes of lattices.

2. A condition equivalent to $\Delta(\boldsymbol{K})$.

Since we regard binary relations as sets of ordered pairs, the lattice product of two congruence relations φ and ψ over an algebra A is simply their set-theoretic intersection, and the lattice sum of φ and ψ is the set-theoretic union of the relative products $\varphi;\psi$, $\varphi;\psi;\varphi$, $\varphi;\psi;\varphi;\psi, \ldots$. For $a,b \in A$ let $\theta_{a,b}$ be the smallest member φ of $\Theta(A)$ such that $a\varphi b$. If t is a term (in the formalized language corresponding to the similarity type of A), then we let t^A be the corresponding operation over A. A binary relation α over A is said to preserve the operation t^A if, for all $a,a', b,b',\ldots$ the conditions $a\alpha a'$, $b\alpha b'$, $\ldots$ jointly imply that $t^A(a,b,\ldots)\alpha t^A(a',b',\ldots)$.

THEOREM 2.1. *If $\boldsymbol{K}$ is an equational class of algebras, then $\Delta(\boldsymbol{K})$ if and only if for some integer $n \geqq 2$ the following condition holds:*

$\Delta_n(\boldsymbol{K})$. There exist terms $t_0,t_1,\ldots,t_n$ in three variables such that for $i=0,1,\ldots,n-1$ the identities

$$t_0(x,y,z) = x, \quad t_n(x,y,z) = z, \quad t_i(x,y,x) = x,$$
$$t_i(x,x,z) = t_{i+1}(x,x,z) \quad (i \text{ even}), \qquad t_i(x,z,z) = t_{i+1}(x,z,z) \quad (i \text{ odd})$$

hold in every member of $\boldsymbol{K}$.[1]

PROOF. Assume $\Delta(\boldsymbol{K})$. Choose an algebra A that is $\boldsymbol{K}$-freely generated by a three-element set $\{a,b,c\}$. Then

$$\langle a,c\rangle \in \theta_{a,c} \cap (\theta_{a,b}+\theta_{b,c}) = (\theta_{a,c}\cap\theta_{a,b}) + (\theta_{a,c}\cap\theta_{b,c}).$$

Hence for some integer $n \geqq 2$ there exist $d_0,d_1,\ldots,d_n \in A$ such that

$$a = d_0(\theta_{a,c}\cap\theta_{a,b})\, d_1(\theta_{a,c}\cap\theta_{b,c})\, d_2(\theta_{a,c}\cap\theta_{a,b})\ldots d_n = c.$$

[1] The fact that $\Delta_2(\boldsymbol{K})$ implies $\Delta(\boldsymbol{K})$ was shown in Prixley [8].

Since A is generated by $\{a,b,c\}$, there exist terms $t_0,t_1,\ldots,t_n$ in three variables such that

(1) $$d_i = t_i^A(a,b,c) \quad \text{for} \quad i=0,1,\ldots,n\,,$$

and since A is $\boldsymbol{K}$-freely generated by $\{a,b,c\}$ it suffices to show that

(2) $$t_0^A(a,b,c) = a, \quad t_n^A(a,b,c) = c, \quad t_i^A(a,b,a) = a\,,$$

(3) $$t_i^A(a,a,c) = t_{i+1}^A(a,a,c)\ (i \text{ even}), \qquad t_i^A(a,c,c) = t_{i+1}^A(a,c,c)\ (i \text{ odd})\,.$$

The first two equations in (2) follow from (1) and the fact that $d_0=a$ and $d_n=c$. Observe that the elements d_i are all in the same $\theta_{a,c}$-class. Therefore $a\,\theta_{a,c}t_i^A(a,b,c)$, and since $a\,\theta_{a,c}c$ this implies that $a\,\theta_{a,c}t_i^A(a,b,a)$. But $\theta_{a,c}$ is trivial on the subalgebra generated by $\{a,b\}$, and we therefore infer that the last equation in (2) holds.

For i even we have

$$t_i^A(a,b,c)\,\theta_{a,b}t_{i+1}^A(a,b,c) \quad \text{and} \quad a\,\theta_{a,b}b\,,$$

whence it follows that

$$t_i^A(a,a,c)\,\theta_{a,b}t_{i+1}^A(a,a,c)\,.$$

Since $\theta_{a,b}$ is trivial on the subalgebra generated by $\{a,c\}$, this yields the first part of (3). The proof of the second part is similar.

Assume $\Delta_n(\boldsymbol{K})$ and consider any algebra $A \in \boldsymbol{K}$. We shall show that if $\varphi \in \Theta(A)$ and if α and β are any binary relations over A that include the identity relation and preserve all the operations t^A, then

(4) $$\varphi\cap(\alpha;\beta) \subseteq (\varphi\cap\alpha^{-1});(\varphi\cap\alpha);(\varphi\cap\beta);(\varphi\cap\beta^{-1});(\varphi\cap\alpha^{-1});\ldots$$

with $2n$ factors on the right. In fact, assuming that $a\varphi c$ and $a\alpha b\beta c$, define d_i, $i=0,1,\ldots,n$, by (1). Then $d_0=a$, $d_n=c$ and, for $i=0,1,\ldots,n-1$,

$$d_i = t_i^A(a,b,c)\ \varphi t_i^A(a,b,a) = t_{i+1}^A(a,b,a)\ \varphi t_{i+1}^A(a,b,c) = d_{i+1}\,,$$
$$t_i^A(a,a,c)\ \varphi t_i^A(a,a,a) = a = t_i^A(a,b,a)\ \varphi t_i^A(a,b,c) = d_i\,.$$

Consequently all the elements d_i and the elements $t_i^A(a,a,c)$ belong to the same φ-class. For i even,

$$d_i = t_i^A(a,b,c)\ \alpha^{-1}t_i^A(a,a,c) = t_{i+1}^A(a,a,c)\ \alpha t_{i+1}^A(a,b,c) = d_{i+1}\,.$$

Consequently

$$d_i(\varphi\cap\alpha^{-1})t_i^A(a,a,c)(\varphi\cap\alpha)d_{i+1}\,.$$

A similar argument shows that if i is odd, then

$$d_i(\varphi\cap\beta)t_i^A(a,c,c)(\varphi\cap\beta^{-1})d_{i+1}\,.$$

Hence (4) follows.

To prove that $\Theta(A)$ is distributive it suffices to show that

$$\varphi \cap (\psi + \psi') \subseteq (\varphi \cap \psi) + (\varphi \cap \psi')$$

for all $\varphi, \psi, \psi' \in \Theta(A)$. Letting

$$\alpha_k = \psi;\psi';\psi;\ldots \quad (k \text{ factors})$$

we have

$$\varphi \cap (\psi + \psi') = (\varphi \cap \alpha_1) \cup (\varphi \cap \alpha_2) \cup (\varphi \cap \alpha_3) \cup \ldots ,$$

and the problem therefore reduces to showing that

$$(5) \qquad \varphi \cap \alpha_k \subseteq (\varphi \cap \psi) + (\varphi \cap \psi')$$

for $k = 1, 2, 3, \ldots$. We have $\alpha_{k+1} = \alpha_k ; \beta$ where β is either ψ or ψ'. Hence, by (4),

$$\varphi \cap \alpha_{k+1} \subseteq (\varphi \cap \alpha_k)^{-1};(\varphi \cap \alpha_k);(\varphi \cap \beta);(\varphi \cap \alpha_k)^{-1};\ldots .$$

From this it follows by an easy induction that (5) holds for every positive integer k. This completes the proof.

Example 1. *The class* $\boldsymbol{K}$ *of all lattices satisfies* $\Delta_2(\boldsymbol{K})$. In fact, we may take $t_0(x,y,z) = x$, $t_2(x,y,z) = z$ and

$$t_1(x,y,z) = xy + yz + zx .$$

Example 2. $\Delta_3(\boldsymbol{K})$ *does not imply* $\Delta_2(\boldsymbol{K})$. Consider algebras with two ternary operations, and let t_1 and t_2 be the corresponding operation symbols. Let $\boldsymbol{K}$ be the class of all models of the identities

$$t_1(x,y,z) = x, \quad t_2(x,y,z) = x ,$$

$$t_1(x,x,z) = x, \quad t_1(x,z,z) = t_2(x,z,z), \quad t_2(x,x,z) = z .$$

Then $\Delta_3(\boldsymbol{K})$ holds. To say that $\Delta_2(\boldsymbol{K})$ holds means that there exists a term $s(x,y,z)$ such that the identities

$$(1) \qquad s(x,x,z) = x, \quad s(x,y,x) = x, \quad s(x,z,z) = z$$

hold in $\boldsymbol{K}$. Our argument showing that no such term exists rests upon the following assertion: *If A is an algebra that is $\boldsymbol{K}$-freely generated by a set U, and if $u \in U$, then for all $a,b,c \in A$*

$$t_1{}^A(a,b,c) = u \Rightarrow a = u, \qquad t_2{}^A(a,b,c) = u \Rightarrow c = u .$$

This can be proved by constructing A as the union of an increasing sequence of partial algebras. A detailed proof is somewhat long but not difficult, and will be omitted. Now if (1) holds in $\boldsymbol{K}$, then $s(x,y,z)$ cannot be one of the terms x,y,z, and must therefore be of the form $t_i(s_1(x,y,z),$

$s_2(x,y,z)$, $s_3(x,y,z)$). By considering an algebra that is $\boldsymbol{K}$-freely generated by a three-element set, and by using the above property of $\boldsymbol{K}$-free algebras, we readily infer that the identities obtained from (1) by replacing s by s_1 if $i=1$ and by s_3 if $i=2$ hold in $\boldsymbol{K}$. This leads to a contradiction, for we may assume $s(x,y,z)$ to be the simplest term (in some easily defined sense) such that the identities (1) hold in $\boldsymbol{K}$.

3. The equational closure of a class.

If φ is a congruence relation over an algebra B, and if A is a subalgebra of B, then we let $\varphi|A$ be the restriction of φ to A. Given (similar) algebras C_i, $i \in I$, if D is a filter in the set I, then we let $D^\wedge$ be the congruence relation over the direct product

$$B = \prod_{i\in I} C_i$$

that is associated with D. Thus for $x,y \in B$, $xD^\wedge y$ if and only if the set

$$\{i : i \in I \text{ and } x_i = y_i\}$$

belongs to D. If D is the principal filter generated by a subset J of I, then we let $J^\wedge = D^\wedge$.

Lemma 3.1. *If A is a subalgebra of the direct product of algebras C_i, $i \in I$, $\Theta(A)$ is distributive, $\varphi \in \Theta(A)$, and A/φ is subdirectly irreducible, then there exists an ultrafilter U over I such that $U^\wedge|A \subseteq \varphi$.*

Proof. Let D be the family of all subsets J of I such that $J^\wedge \subseteq \varphi$, and let U be a maximal filter contained in D. Then $U^\wedge$ is the lattice sum, and in fact the set-theoretic union, of the relations $J^\wedge$, $J \in U$, and $U^\wedge|A$ is therefore contained in φ. We therefore need only show that U is an ultrafilter over I.

For all $J,K \subseteq I$,

$$(J\cup K)^\wedge|A = (J^\wedge|A))\cap(K^\wedge|A).$$

Hence, if $J\cup K \in D$, then

$$\varphi = \varphi+((J\cup K)^\wedge|A) = (\varphi+(J^\wedge|A))\cap(\varphi+(K^\wedge|A)). \tag{1}$$

Since A/φ is subdirectly irreducible, φ is multiplicatively irreducible in $\Theta(A)$, and (1) therefore implies that either

$$\varphi = \varphi+(J^\wedge|A) \quad \text{or} \quad \varphi = \varphi+(K^\wedge|A).$$

Thus

$$J\cup K \in D \Rightarrow J\in D \vee K\in D. \tag{2}$$

Obviously we also have

$$(3) \qquad I \in D, \qquad I \supseteq J \supseteq K \in D \Rightarrow J \in D .$$

If U is not an ultrafilter, then there exists a subset J of I such that neither J nor $I-J$ belongs to U. Together with (3) and the maximality of U, this implies that there exist sets K' and K'' in U such that neither $J \cap K'$ nor $(I-J) \cap K''$ belongs to D. Then the set $K = K' \cap K''$ belongs to U and hence to D. However, since K is the union of the two sets $J \cap K$ and $(I-J) \cap K$, neither of whom belongs to D, this contradicts (2).

COROLLARY 3.2. *If $\boldsymbol{K}$ is a class of algebras and $\Delta(\boldsymbol{K}^e)$, then every subdirectly irreducible member of $\boldsymbol{K}^e$ belongs to $HSP_u(\boldsymbol{K})$.*

PROOF. Every algebra in $\boldsymbol{K}^e$ is isomorphic to a quotient algebra A/φ where A is a subalgebra of the direct product B of algebras $C_i \in \boldsymbol{K}$, $i \in I$. If A/φ is subdirectly irreducible, then $U^\wedge | A \subseteq \varphi$ for some ultrafilter U over I. Thus A/φ is a homomorphic image of the algebra $A/(U^\wedge | A)$, which is isomorphic to a subalgebra of the ultraproduct $B/U^\wedge$.

THEOREM 3.3. *If $\boldsymbol{K}$ is a class of algebras and $\Delta(\boldsymbol{K}^e)$, then*

$$\boldsymbol{K}^e = IP_s HSP_u(\boldsymbol{K}) .$$

PROOF. Since every member of $\boldsymbol{K}^e$ is isomorphic to a subdirect product of subdirectly irreducible members of $\boldsymbol{K}^e$, this follows from 3.2.

COROLLARY 3.4. *If $\boldsymbol{K}$ is a finite set of finite algebras and $\Delta(\boldsymbol{K}^e)$, then every subdirectly irreducible member of $\boldsymbol{K}^e$ belongs to $HS(\boldsymbol{K})$, and hence*

$$\boldsymbol{K}^e = IP_s HS(\boldsymbol{K}) .$$

PROOF. In this case $P_u(\boldsymbol{K}) \subseteq I(\boldsymbol{K})$ (cf. Frayne, Morel and Scott [4, Corollary 2.3]), and the conclusion therefore follows from 3.2 and 3.3.

COROLLARY 3.5. *Suppose $\boldsymbol{K}$ is an equational class of algebras and $\Delta(\boldsymbol{K})$. If A and B are non-isomorphic, finite, subdirectly irreducible members of $\boldsymbol{K}$, and if the order of A does not exceed the order of B, then there exists an identity that holds in A but does not hold in B.*

PROOF. Since B cannot be a homomorphic image of a subalgebra of A, this follows from 3.4 with $\boldsymbol{K}$ replaced by $\{A\}$.

4. Lattices of equational classes.

We may regard the equational classes of algebras (of a given similarity type) as members of a lattice $\mathscr{K}$, the lattice product of two classes

$\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ being their intersection, and their lattice sum being the equational closure of their union,

$$\boldsymbol{K}_0+\boldsymbol{K}_1 = (\boldsymbol{K}_0 \cup \boldsymbol{K}_1)^e .$$

There is a valid objection, that equational classes are not sets, and that one therefore cannot speak of the class of all equational classes, but our use of lattice-theoretic terminology is merely intended to be suggestive, and all the statements that we make could easily be reformulated in such a way that no set-theoretic principles would be violated.

The lattice $\mathscr{K}$ is dually isomorphic to the lattice of all logically closed sets of identities. We simply associate with each such set Σ the class $M(\Sigma)$ consisting of all its models, and verify that

$$M(\Sigma_0 \cap \Sigma_1) = M(\Sigma_0) + M(\Sigma_1), \qquad M(\Sigma_0+\Sigma_1) = M(\Sigma_0) \cap M(\Sigma_1) ,$$

where $\Sigma_0+\Sigma_1$ is the set of all identities that are consequences of $\Sigma_0 \cup \Sigma_1$.

Let F be an algebra that is absolutely freely generated by an infinite set X, that is, let F be A-freely generated by X, where A is the class of all algebras of the given similarity type. For each equational class $\boldsymbol{K}$ let $\theta(\boldsymbol{K})$ be the smallest congruence relation φ over F such that $F(\varphi) \in \boldsymbol{K}$. Then θ maps $\boldsymbol{K}$ dually isomorphically onto a sublattice of $\Theta(F)$. In fact, if F' is the algebra obtained by adjoining to F all the endomorphisms of F as new operations, then θ is a dual isomorphism of $\boldsymbol{K}$ onto $\Theta(F')$. (This observation was made jointly by E. Engeler and the author.)

Either one of these observations could be used to reformulate our results in such a way that no set-theoretic objections would apply, but we find it more suggestive to work directly with the equational classes.

LEMMA 4.1. *If $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ are equational classes of algebras such that $\Delta(\boldsymbol{K}_0+\boldsymbol{K}_1)$, then every member of $\boldsymbol{K}_0+\boldsymbol{K}_1$ is isomorphic to a subdirect product of a member of $\boldsymbol{K}_0$ and a member of $\boldsymbol{K}_1$ and, in particular, every subdirectly irreducible member of $\boldsymbol{K}_0+\boldsymbol{K}_1$ belongs to either $\boldsymbol{K}_0$ or $\boldsymbol{K}_1$.*

PROOF. The first part of the conclusion obviously follows from the second, and this in turn is a consequence of 3.2 and the fact that, for any classes $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$,

$$HSP_u(\boldsymbol{K}_0 \cup \boldsymbol{K}_1) = HSP_u(\boldsymbol{K}_0) \cup HSP_u(\boldsymbol{K}_1) .$$

COROLLARY 4.2. *If $\boldsymbol{K}$ is an equational class of algebras and $\Delta(\boldsymbol{K})$, then the lattice of all equational subclasses of $\boldsymbol{K}$ is distributive.*

PROOF. If $\boldsymbol{K}_0$, $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are equational subclasses of $\boldsymbol{K}$, then it follows from 4.1 that every subdirectly irreducible member of $\boldsymbol{K}_0 \cap (\boldsymbol{K}_1+\boldsymbol{K}_2)$ belongs to either $\boldsymbol{K}_0 \cap \boldsymbol{K}_1$ or $\boldsymbol{K}_0 \cap \boldsymbol{K}_2$, and from this the conclusion follows.

It is easy to check that under the hypothesis of 4.2 the infinite distributive law

$$\mathbf{K}' + \bigcap_{i \in I} \mathbf{K}_i = \bigcap_{i \in I} (\mathbf{K}' + \mathbf{K}_i)$$

also holds for equational subclasses $\mathbf{K}'$, $\mathbf{K}_i$ of $\mathbf{K}$, but the dual law is obviously false.

COROLLARY 4.3. *If* $\mathbf{K}$ *is a finite set of finite algebras and* $\mathbf{K}'$ *is an equational class of algebras, and if* $\Delta(\mathbf{K}^e + \mathbf{K}')$, *then there are only finitely many equational classes* $\mathbf{K}''$ *such that* $\mathbf{K}' \subseteq \mathbf{K}'' \subseteq \mathbf{K}^e + \mathbf{K}'$.

PROOF. If $\mathbf{K}' \subseteq \mathbf{K}'' \subseteq \mathbf{K}^e + \mathbf{K}'$, then by 4.1 every subdirectly irreducible member A of $\mathbf{K}''$ belongs to either $\mathbf{K}'$ or $\mathbf{K}^e$. In the latter case $A \in HS(\mathbf{K})$ by 3.4. Since every equational class is completely determined by its subdirectly irreducible members, the proof is completed by observing that $HS(\mathbf{K})$ has only finitely many non-isomorphic members.

5. Equational classes of lattices.

Let $\mathbf{L}$ be the class of all lattices.

In the proof of Theorem 3.5 of Jónsson [6] it was shown that $\mathbf{L}$ has the amalgamation property: *Given* $A,B,B' \in \mathbf{L}$ *and monomorphisms* $f: A \to B$, $f': A \to B'$, *there exist* $C \in \mathbf{L}$ *and monomorphisms* $g: B \to C$, $g': B' \to C$ *such that* $gf = g'f'$. (Actually the amalgamation property considered in [6] is stronger than the one formulated here.) According to Theorem 3.2 of Jónsson [5] it follows that $\mathbf{L}$ also has the following property: *If, for each* $i \in I$, A_i *is a sublattice of the lattice* B_i *and* f_i *is a monomorphism of* A_i *into a lattice* C, *then there exists an extension* $D \in \mathbf{L}$ *of* C *such that for each* $i \in I$ *there is a monomorphism of* B_i *into* D *that agrees with* f_i *on* A_i.

LEMMA 5.1. *Every lattice is a sublattice of a subdirectly irreducible lattice.*

PROOF. Obviously every finite chain is a sublattice of a simple lattice. Now consider a lattice C and let C' be the lattice obtained by adjoining to C two new elements u and 1 with $u < 1$ and $x < u$ for all $x \in C$. For all $x,y \in C'$ the chain

$$A_{x,y} = \{xy, x+y, u, 1\}$$

can be embedded in a simple lattice $B_{x,y}$, and by the corollary to the amalgamation property there exist monomorphisms of the lattices $B_{x,y}$ into an extension $D \in \mathbf{L}$ of C' taking the elements of the lattices $A_{x,y}$

into themselves. We may therefore assume that all the lattices $B_{x\,y}$ are sublattices of D.

Now consider a maximal congruence relation φ over D with the property that, for all $x,y \in C'$, $x\varphi y$ implies $x=y$. Observe that if $\varphi \subset \psi \in \Theta(D)$, then $u\psi 1$, for if x and y are distinct elements of C' with $x\psi y$, then $xy\psi(x+y)$ and hence $u\psi 1$ by the simplicity of $B_{x,y}$. This implies that D/φ is subdirectly irreducible, for any non-trivial congruence relation over D/φ identifies u/φ and $1/\varphi$. Since C is isomorphic to a sublattice of D/φ, this completes the proof.

COROLLARY 5.2. *If $\boldsymbol{F}$ is the class of all finite lattices, then*

$$\boldsymbol{L} = HSP_u(\boldsymbol{F}) .$$

PROOF. It is well known that every finite substructure of a lattice can be embedded in a finite lattice. Hence $\boldsymbol{L}=\boldsymbol{F}^e$, and it follows by 5.1 and 3.2 that

$$\boldsymbol{L} = SHSP_u(\boldsymbol{F}) = HSP_u(\boldsymbol{F}) .$$

COROLLARY 5.3. *If $\boldsymbol{K} \neq \boldsymbol{L}$ is an equational class of lattices, then there exists an equational class $\boldsymbol{K}'$ of lattices that covers $\boldsymbol{K}$.*

PROOF. By 5.2 there exists a finite lattice A that does not belong to $\boldsymbol{K}$. By 4.3 there are only finitely many equational classes $\boldsymbol{K}'$ such that $\boldsymbol{K} \subset \boldsymbol{K}' \subseteq \{A\}^e + \boldsymbol{K}$, and at least one of these classes must therefore cover $\boldsymbol{K}$.

THEOREM 5.4. *If $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ are equational classes of lattices such that $\boldsymbol{K}_0 + \boldsymbol{K}_1 = \boldsymbol{L}$, then $\boldsymbol{K}_0 = \boldsymbol{L}$ or $\boldsymbol{K}_1 = \boldsymbol{L}$.*

PROOF. If $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ are both proper subclasses of $\boldsymbol{L}$, then there exist lattices A_0 and A_1 such that $A_0 \notin \boldsymbol{K}_0$ and $A_1 \notin \boldsymbol{K}_1$. By 5.1 there exists a subdirectly irreducible lattice B such that A_0 and A_1 are isomorphic to sublattices of B. Consequently B belongs to neither $\boldsymbol{K}_0$ or $\boldsymbol{K}_1$, whence it follows by 4.1 that $B \notin \boldsymbol{K}_0 + \boldsymbol{K}_1$. Thus $\boldsymbol{K}_0 + \boldsymbol{K}_1 \neq \boldsymbol{L}$.

COROLLARY 5.5. *There exists no equational class $\boldsymbol{K}$ of lattices such that $\boldsymbol{L}$ covers $\boldsymbol{K}$.*

PROOF. If the equational class $\boldsymbol{K}$ is properly contained in $\boldsymbol{L}$, then there exists a finite lattice A such that $A \notin \boldsymbol{K}$. Since $\{A\}^e \neq \boldsymbol{L}$, it follows by 5.4 that $\boldsymbol{K} \subset \boldsymbol{K} + \{A\}^e \subset \boldsymbol{L}$. Thus $\boldsymbol{L}$ does not cover $\boldsymbol{K}$.

6. Open problems.

We conclude by summarizing some of the known facts about the lattice of all equational classes of lattices, and by mentioning some open problems. The zero element 0 of this lattice is of course the class of all one-element lattices. Since every lattice with more than one element has a two-element sublattice D_2, the class $\boldsymbol{D} = \{D_2\}^e$, which is of course known to be the class of all distributive lattices, covers 0 and is contained in all the other equational classes. According to our results, $\boldsymbol{D}$ must in turn be covered by one or more classes. In fact, it is known that every lattice that is not distributive must contain either a five-element lattice M_5 that is modular but not distributive or else a five-element non-modular lattice N_5, and that these lattices are unique up to isomorphism. Consequently $\boldsymbol{M}_5 = \{M_5\}^e$ and $\boldsymbol{N}_5 = \{N_5\}^e$ cover $\boldsymbol{D}$ and every equational class other than 0 and $\boldsymbol{D}$ contains either $\boldsymbol{M}_5$ of $\boldsymbol{N}_5$. Because of the distributivity of the lattice of all equational classes of lattices, $\boldsymbol{M}_5 + \boldsymbol{N}_5$ covers both $\boldsymbol{M}_5$ and $\boldsymbol{N}_5$. For similar reasons, if $\boldsymbol{M}$ is the class of all modular lattices then $\boldsymbol{M} + \boldsymbol{N}_5$ covers $\boldsymbol{M}$ and is contained in every class that properly contains $\boldsymbol{M}$. If M_6 and $M_{5,5}$ are the lattices indicated in Fig. 1, then $\boldsymbol{M}_6 = \{M_6\}^e$ and $\boldsymbol{M}_{5,5} = \{M_{5,5}\}^e$ cover $\boldsymbol{M}_5$.

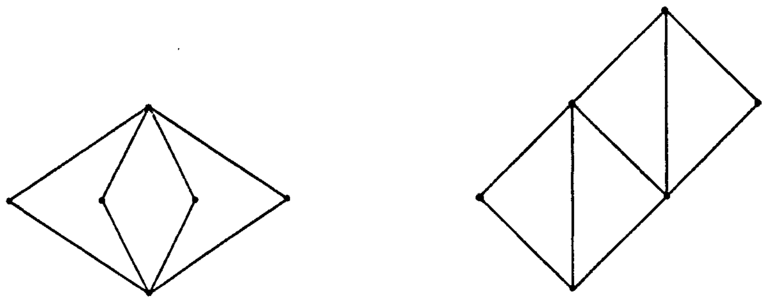

Fig. 1.

According to an unpublished result by G. Grätzer, if an equational class $\boldsymbol{K}$ of modular lattices contains a finite lattice that does not belong to $\boldsymbol{M}_5$, then that class contains either $\boldsymbol{M}_6$ or $\boldsymbol{M}_{5,5}$. However, Grätzer's results do not appear to give answers to the following two related questions:

PROBLEM 1. *Are* $\boldsymbol{M}_6$ and $\boldsymbol{M}_{5,5}$ *the only equational classes of modular lattices that cover* $\boldsymbol{M}_5$*?*

PROBLEM 2. *Is it true that every equational class of modular lattices that properly contains* $\boldsymbol{M}_5$ *contains either* $\boldsymbol{M}_6$ *or* $\boldsymbol{M}_{5,5}$*?*

More generally we may ask:

PROBLEM 3. *Is it true that each proper equational subclass of* $\boldsymbol{L}$ *is covered by only finitely many equational subclasses of* $\boldsymbol{L}$*?*

PROBLEM 4. *Is it true that if $\boldsymbol{K}$ and $\boldsymbol{K}'$ are equational classes of lattices with $\boldsymbol{K} \subset \boldsymbol{K}'$, then some equational subclass of $\boldsymbol{K}'$ covers $\boldsymbol{K}$?*

PROBLEM 5. *Is it true that, for every equational class $\boldsymbol{K}$ of lattices, $\boldsymbol{K} = (\boldsymbol{K} \cap \boldsymbol{F})^e$, where $\boldsymbol{F}$ is the class of all finite lattices?*

It seems rather too optimistic to hope for an affirmative answer to this last question, for the following statement obviously holds: *If $\boldsymbol{K}$ is a finitely axiomatizable class of lattices and $\boldsymbol{K} = (\boldsymbol{K} \cap \boldsymbol{F})^e$, then the word problem for $\boldsymbol{K}$-free lattices is decidable.* In particular, therefore, if it could be shown that $\boldsymbol{M} = (\boldsymbol{M} \cap \boldsymbol{F})^e$, then this would yield a solution to one of the most important problems in lattice theory, the word problem for free modular lattices.

Our last two questions suggest a somewhat different area of investigation:

PROBLEM 6. *Find systems of axioms (in the form of identities) for the classes $\boldsymbol{M}_5$, $\boldsymbol{N}_5$, $\boldsymbol{M}_6$ and $\boldsymbol{M}_{5,5}$.*

PROBLEM 7. *Give a workable method for finding, for any finite lattice A, a system of axioms for the class $\{A\}^e$. Are such classes always finitely axiomatizable?*

In Birkhoff [1], p. 70, a system of axioms for $\boldsymbol{M}_5$ is proposed, and in Schützenberger [9] axioms are proposed for both $\boldsymbol{M}_5$ and $\boldsymbol{N}_5$, but in neither case have the proofs been published. If the answers to the questions in Problem 1 and 2 are affirmative, then it is clearly a simple matter to test any finite system of proposed axioms (identities) for $\boldsymbol{M}_5$. The questions raised in Problems 3 and 4 are of course partly motivated by their possible significance in connection with the last problem.

7. Added April 1967.

A preliminary account of the work presented here was prepared and privately circulated during the winter 1963–64, and the results were reported at the 1964 Congress of Scandinavian Mathematicians. This manuscript was submitted in September 1964, and was accepted for publication in the Proceedings of the Congress. Because of a delay in the publication of these Proceedings, the manuscript was transferred to Mathematica Scandinavica in February 1967, and accepted for publication in April 1967.

During the intervening period G. Grätzer's results, referred to in Section 6, have appeared in print (*Equational classes of lattices*, Duke

Math. J. 33 (1966), 613–622). Also, Corollary 3.5, for the special case of lattices, has been discovered independently by Alfred L. Foster and Alden F. Pixley (*Algebraic and equational semi-maximality; equational spectra* II, Math. Zeitschr. 93 (1966), 122–133). Two other papers have been written, which refer to and make use of the results presented here: M. I. Gould and G. Grätzer, *Boolean extensions and normal subdirect powers of finite universal algebras*, Math. Zeitschr. 99 (1967), 16–25, and G. Grätzer, *On the spectra of classes of algebras*, to appear in Math. Scand.

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