

Antanas Žilinskas

Algorithm. 44. MIMUN. Optimization of one-dimensional multimodal functions in the presence of noise

*Aplikace matematiky*, Vol. 25 (1980), No. 3, 234–240

Persistent URL: <http://dml.cz/dmlcz/103855>

## Terms of use:

© Institute of Mathematics AS CR, 1980

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

## ALGORITMUS

## 44. MIMUN

OPTIMIZATION OF ONE-DIMENSIONAL MULTIMODAL FUNCTIONS  
IN THE PRESENCE OF NOISE

ANTANAS ŽILINSKAS

Institute of mathematics and cybernetics, Academy of sciences of the Lithuanian SSR,  
Lenino pr. 3, 232600, Vilnius, USSR

The problems of minimization in the presence of noise occur in various fields of science and engineering. But, as far as it is known to the author, among the currently available issues there is not a single publication of a computer program for solving such problems. A rather efficient and quick-operating algorithm for one-dimensional multimodal minimization in the presence of noise is proposed in [1]. This algorithm is based on the usage of a Wiener process for a statistical model of an objective function [2]. The results of investigation of a former version of this algorithm are given in [3], [4]. The algorithm for one-dimensional multimodal minimization without noise based on similar assumptions [5] is more efficient than other algorithms of analogous destination as shown in [6]. Only a brief description of this algorithm is given here to explain the meaning of formal parameters while its full description is given in [1].

Let a function  $f(x)$ ,  $a \leq x \leq b$ , be minimized where only the values  $z(x_i) = f(x_i) + \xi_i$  may be observed where  $\xi_i$  are independent Gaussian random numbers (noise), whose mean is equal to zero and the dispersion is  $\sigma_i^2$ ,  $i$  being the number of observation. Before the minimization the variance analysis of the results  $z_{ij}$  is carried out, where  $z_{ij} = z(x(i))$ ,  $x(i) = a + (b - a)(i - 1)/(m_2 - 1)$ ,  $i = 1, \dots, m_2$ ,  $j = 1, \dots, m_3$ , i.e. at each point  $x(i)$  the objective function  $z(\cdot)$  is observed  $m_3$  times. If the hypothesis of equality of  $f(x(i))$  is accepted (the significance level being equal to 0.05) then the algorithm terminates indicating that the noise level is too high. If the variance of  $f(\cdot)$  is significant then the dispersion of noise and the parameter of a Wiener process, chosen as a statistical model of an objective function, are estimated [1]. After that the minimization begins. To simplify the algorithm the lattice  $x^i = a + (b - a)(i - 1)/(m_4 - 1)$ ,  $i = 1, \dots, m_4$ , is substituted for the interval of

minimization  $[a, b]$ . The additional error caused by discretization of  $[a, b]$  may obviously be reduced to a desirable value choosing sufficiently large  $m_4$ ; the value  $m_4 = 101$  is large enough for many practical problems. The coordinate of the current observation is defined by the condition of maximum of the expected improvement [1, 3]. The algorithm terminates if the number of objective function evaluations reaches the maximally allowable amount or if the P-probability of evaluating the global minimum with given accuracy  $\varepsilon_1$  exceeds 0.9; this probability is calculated according to the chosen statistical model of objective function [1]. Note that in the case when  $\sigma_t/\varepsilon_1 \geq 10$  and  $\sigma_t$  is of the same order as variance of  $f(\cdot)$ , more than a thousand observations of an objective function are necessary for P to reach 0.9. On the other hand, if the noise level is not so high, a practically acceptable solution is usually obtained after 200–500 observations [1].

Using this algorithm the following remarks must be taken into account:

1. The variable *kmax* is the machine dependent constant which is initialised as 19. If the maximal real number of the user's computer is  $10^k$  where  $k < 19$  then the value of the variable *kmax* must be set equal to  $k$ .

2. The formal parameter *ifault* is the failure indicator. The normal termination of the algorithm is indicated by *ifault* = 0. If *ifault* = 1 then the cause of termination of the algorithm is a too high level of noise. *ifault* = 2 means that the number of observations reaches the maximally allowable value. The scale of values of an objective function must be chosen so that  $|f(x)|$  does not exceed  $10^K$  where  $K = kmax/2$ ; the violation of this condition is indicated as *ifault* = 3. The termination with *ifault* = 4 means that the variance of the objective function is insignificant as shown by the results before the minimization investigation; the scale of the values of an objective function must be changed or the different number of points  $m_2$  must be taken (for example,  $2m_2 + 1$ ).

3. The algorithm calls the auxiliary **real procedure** *ndtr*, which is the ALGOL version of *SUBROUTINE NDTR* [7] and which calculates the value of the Gaussian distribution function.

Algorithm *mimun*:

**procedure** *mimun*(*be, en, sn1, am, m, f, e1, e, nf, xm, ym, ifault,*  
*anm, b, c, y*);

**comment:** *be* ... input ... start of interval of optimization,

*en* ... input ... end of interval of optimization,

*sn1* ... input ... if (*sn1* > 0) then *sn1* is variance of noise,  
if (*sn1* ≤ 0) then variance of noise to be  
evaluated by algorithm,

*am* ... input ... if *am* = 1 then minimization,

if *am* = -1 then maximization,

*m* ... input ... *m*[1] ... maximal allowed number of observations of objective function *f*,  
*m*[2] ... number of observation points for parameters estimation, it is recommended = 6,  
*m*[3] ... number of observations at each point for parameters estimation, it is recommended = 5,  
*m*[4] ... number of points of lattice. it is recommended = 101,  
*f* ... input ... objective function,  
*e1* ... input ... if(*e1* ≥ 0) then *e1* is required accuracy of *ym*, if(*e1* < 0) then required accuracy is equal to  $\sqrt{\text{variance of noise}/\text{abs}(e1)}$ ,  
*e* ... output ... estimation of mean-root-square error of *ym*,  
*nf* ... output ... number of observations of *f*,  
*xm* ... output ... estimation of optimum point,  
*ym* ... output ... estimation of optimum,  
*ifault* ... output ... failure indicator  
*anm*, *b*, *c*, *y* ... workspace, dimension of these arrays ≥ *m* [4];  
**value** *be*, *en*, *sn1*, *am*, *e1*; **integer** *nf*, *ifault*;  
**real** *be*, *en*, *sn1*, *am*, *e1*, *e*, *xm*, *ym*; **integer array** *m*;  
**array** *anm*, *b*, *c*, *y*; **real procedure** *f*;  
**begin integer** *n*, *kmax*, *k*, *k1*, *n1*, *n2*, *j*, *k3*, *k4*, *km*, *km2*;  
**real** *dt*, *cv*, *eps2*, *an*, *pp*, *amax*, *ym1*, *p1*, *p2*, *p3*, *p4*, *p5*, *p6*, *cv2*, *ym2*, *sn2*,  
*aw*, *av1*, *av2*, *am1*, *v1*, *v2*, *c1*, *c2*, *pr*, *pr1*, *ppab*, *va*, *d*, *d1*,  
*a11*, *a12*, *a21*, *a22*, *eb*, *sf*, *eps1*, *eps3*;  
**real procedure** *av*(*k*, *co*);  
**comment:** auxiliary procedure for *mimun*: calculates conditional mean and variance of Wiener process;  
**integer** *k*; **real** *co*;  
**begin integer** *i*; **real** *a*, *p*;  
*a* := *y*[*k*]; *p* := *b*[*k*]; *co* := 1.0; *i* := *k*;  
**for** *i* := *i* - 1 **while** *co* × *eps2* < *p* ∧ *i* > 0 **do**  
**if** *anm*[*i*] > 0.0 **then begin** *a* := *a* + *p* × *y*[*i*]; *co* := *co* + *p*; *p* := *p* × *b*[*i*]  
**end;**  
*p* := *c*[*k*]; *i* := *k*;  
**for** *i* := *i* + 1 **while** *co* × *eps2* < *p* ∧ *i* ≤ *n* **do**  
**if** *anm*[*i*] > 0.0 **then begin** *a* := *a* + *p* × *y*[*i*]; *co* := *co* + *p*; *p* := *p* × *c*[*i*]  
**end;**  
*av* := *a*/*co*;  
**end** *av*;  
**real procedure** *fi*(*x*, *nr*);

```

value  $x$ ,  $nr$ ; integer  $nr$ ; real  $x$ ;
comment: auxiliary procedure for mimun;
begin integer  $k$ ; real  $a$ ;
 $a := 0.0$ ; for  $k := 1$  step 1 until  $nr$  do  $a := a + f(x)$ ;
 $fi := a/(nr \times cv)$ 
end  $fi$ ;
procedure  $updata$ ;
comment: auxiliary procedure for mimun:
updates array of parameters  $c$ ,  $b$ ;
begin integer  $k$ ,  $k1$ ,  $kp$ ,  $kp1$ ; real  $b1$ ,  $bs$ ,  $cs$ ;
 $kp := 1$ ;  $kp1 := n$ ;  $bs := cs := b[1] := c[n] := 1.0$ ;
for  $k := 2$  step 1 until  $n$  do
  begin if  $anm[k] > 0.0$  then
    begin  $b1 := dt/((dt/anm[kp] + (k - kp) \times an \times bs) \times anm[k])$ ;
       $bs := bs \times b1 + 1.0$ ;  $b[k] := b1$ ;  $kp := k$ 
    end;
     $k1 := n + 1 - k$ ; if  $anm[k1] > 0.0$  then
      begin  $b1 := dt/((dt/anm[kp1] + (kp1 - k1) \times an \times cs) \times anm[k1])$ ;
         $cs := cs \times b1 + 1.0$ ;  $c[k1] := b1$ ;  $kp1 := k1$ 
      end
    end
  end
end  $updata$ ;
real procedure  $ndtr(x)$ ; value  $x$ ; real  $x$ ;
comment: Gaussian distribution function, algol version of subroutine  $ndtr$ :
system/360 scientific subroutine package;
begin real  $t$ ,  $d$ ,  $p$ ,  $ax$ ;  $ax := abs(x)$ ;  $t := 1.0/(1.0 + 0.2316419 \times ax)$ ;
 $d := 0.3989423 \times exp(-x \times x/2.0)$ ;  $p := 1.0 - d \times t \times (((1.330274 \times t - 1.821256) \times t + 1.781478) \times t - 0.3565638) \times t + 0.3193815$ ;
if  $x > 0$  then  $ndtr := p$  else  $ndtr := 1.0 - p$ 
end Any other procedure of analogous destination may be used instead of  $ndtr$ ;
 $kmax := 19$ ;  $pp := 2.0$ ;  $eps2 := 0.001$ ;  $ifault := 0$ ;
 $n := m[4]$ ;  $n1 := m[2]$ ;  $n2 := m[3]$ ;  $cv := 1.0$ ;  $amax := 10 \uparrow (kmax \div 2 - 1)$ ;
 $eb := (en - be)/(n - 1)$ ;  $p2 := p5 := p6 := 0.0$ ;  $ym1 := amax$ ;  $an := 1/(n - 1)$ ;
for  $k := 1$  step 1 until  $n$  do begin  $y[k] := 0.0$ ;  $anm[k] := -1.0/amax$  end;
for  $k := 1$  step 1 until  $n1$  do
  begin  $p3 := 0.0$ ; for  $k1 := 1$  step 1 until  $n2$  do
    begin  $p4 := fi(be + eb \times (((n - 1) \times (k - 1)) \div (n1 - 1)), 1)$ ;
      if  $abs(p4) < amax$  then begin  $p3 := p3 + p4$ ;  $p2 := p2 + p4 \times p4$  end
      else begin  $ifault := 3$ ; go to  $fin$  end
    end;
     $y[k] := p3/n2$ ; if  $ym1 > p3$  then  $ym1 := p3$ ;
     $p5 := p5 + p3$ ;  $p6 := p6 + p3 \times p3$ 
  end

```

```

end;
nf :=  $n1 \times n2$ ; p5 :=  $p5 / nf$ ; p6 :=  $p6 / n2$ ; sf :=  $abs(p6 - p5) / (n1 - 1)$ ;
if sn1 > 0.0 then sn2 := sn1 else sn2 :=  $abs(p2 - p6) / (n1 \times (n2 - 1))$ ;
sf :=  $abs(p6 - p5) / (n1 - 1)$ ;
if sf <  $sn2 \times 2.5$  then begin ifault := 1; go to fin end;
comment: estimation of parameters;
p1 :=  $y[1]$ ; cv2 := 0.0;
for k := 2 step 1 until n1 do
    begin p2 :=  $y[k]$ ; cv2 :=  $cv2 + (p2 - p1) \uparrow 2$ ; p1 := p2
    end;
cv :=  $\sqrt{cv2}$ ; if cv <  $1.0 / amax$  then begin ifault := 4; go to fin end;
dt :=  $sn2 / cv2$ ;
for k := 1 step 1 until n1 do
    begin k1 :=  $((n - 1) \times (k - 1)) \div (n1 - 1) + 1$ ;
     $y[k1]$  :=  $y[k] / cv$ ; anm[k1] := n2
    end;
if e1 > 0.0 then eps3 :=  $e1 / cv$  else eps3 :=  $\sqrt{dt / abs(e1)}$ ;
eps1 :=  $eps3 / pp$ ;
comment: begin of optimization;
ym1 :=  $ym1 / (cv \times n2)$ ; v1 := 0.0; updata;
lopt: ppab := 1.0; if v1  $\geq eps1$  then ppab := 0.0; ym2 := av2 :=  $av(1, c2)$ ;
km := km2 := 1; pr := 0.0;
comment main loop, computing of point of current observations;
for k := 1 step 1 until n do
    begin if  $k < n \wedge anm[k] > 0.0$  then
        begin c1 := c2; av1 := av2; k3 := k;
        for j :=  $k + 1$  step 1 until n do if  $anm[j] > 0.0$  then
            begin av2 :=  $av(j, c2)$ ; if av2 < ym2 then
                begin ym2 := av2; km2 := j
                end;
            k4 := j; a11 :=  $1.0 / c1$ ; a12 :=  $a11 \times b[k3]$ ; a21 :=  $a11 \times c[k3]$ ;
            a22 :=  $1.0 / c2$ ; go to l1
            end
        end;
        l1: d :=  $(k - k3) / (k4 - k3)$ ; d1 :=  $(1 - d)$ ; aw :=  $av1 \times d1 + av2 \times d$ ;
        va :=  $\sqrt{d \times d1 \times (k4 - k3) / (n - 1) + (d1 \times (d1 \times a11 + d \times a21) / anm[k3] + d \times (d1 \times a12 + d \times a22) / anm[k4]) \times dt}$ ;
        if  $k = km2$  then v2 := va;
        am1 :=  $ym1 - aw$ ; p1 :=  $-0.2 \times am1$ ;
        comment: computing of probability of finding
        global optimum with required accuracy ppab;
        if  $v1 < eps1 \wedge va \geq 1.5 \times v1 \wedge ppab \geq 0.9$  then

```

```

ppab := ppab × (1 - ndtr((am1 - eps3)/va));
comment computing of mean improvement;
va := va × 7.0; if va > p1 then
  begin pr1 := am1 × 0.65 × exp(-0.443 × (0.75 - am1/va) ↑ 2) +
    va × 0.3989 × exp(-(am1 × am1)/(2.0 × va × va));
    if pr1 > pr ∧ va > eps1 then begin km := k; pr := pr1 end
  end
end main loop;
if ppab ≥ 0.9 then go to l2; d := anm[km];
j := 0.1 × d + 1.0; d1 := d + j; p4 := fi(be + eb × (km - 1), j);
if abs(p4) > amax then begin ifault := 3; go to fin end;
y[km] := (y[km] × d + am × j × p4)/d1; anm[km] := d1;
nf := nf + j; ym1 := ym2; v1 := v2; updata;
if nf < m[1] then go to lopt; ifault := 2;
l2: ym := ym2 × cv/am; xm := be + (km2 - 1) × eb; e := v2 × pp × cv;
fin:
end;

```

Example: The test function:

```

real procedure f(x); value x; real x;
comment: test function for mimun, integer
parameter kun must be declared in driver program
and initialised there as kun = 127;
begin real a, b; integer i;
comment: generation of pseudo-random number a;
kun := kun × 3125; kun := kun - entier(kun/67108864) × 67108864;
a := kun/33554432 - 1.0; b := 0.0;
for i := 1 step 1 until 5 do
  b := b - i × sin((i + 1) × x + i);
f := a + b
end

```

was minimized with the input parameters:  $be = -10.0$ ,  $en = 10.0$ ,  $am = 1.0$ ,  $sn1 = -1.0$ ,  $e1 = -5.0$ ,  $m[1] = 5000$ ,  $m[2] = 6$ ,  $m[3] = 5$ ,  $m[4] = 101$ . The following results were obtained (computer BESM-6):

$xm = 5.800000000$ ,  $ym = -12.07391983$ ,  $nf = 86$ ,  $e = 0.2136057320$ ,  $ifault = 0$ .

The FORTRAN codes of this algorithm are available from the author.

#### References

- [1] A. Žilinskas: Two algorithms for one-dimensional multimodal minimization. Math. Operat. Stat., ser. Optimization (in print).
- [2] A. Žilinskas: On statistical models for multimodal optimization. Math. Operat. Stat., ser. Statistics, 9 (1978), No. 2, 255—266.

- [3] *A. Жилинскас*: Одношаговый байесовский алгоритм минимизации одномерных функций в присутствии помех. В сб. Теория оптимальных решений, вып 1, Вильнюс, 1975, 9—22.
- [4] *J. Mockus*: On Bayesian methods of seeking the extremum and their applications. In Information Processing 77 (ed. by B. Gilchrist), North Holland, 1977, 195—200.
- [5] *A. Žilinskas*: Optimization of one-dimensional multimodal functions, statistical algorithm AS133. Applied Statistics, 27 (1978), No. 3, 367—375.
- [6] *A. Žilinskas*: On one-dimensional multimodal minimization. In Trans. of Eight Prague Conf. on Inform. Theory, Stat. Dec. Funct., Random Processes, vol. B, 1978, 393—402.
- [7] System/360 Scientific Subroutine package (360-CM-03X), Version III, New York, 1960—1970.