

## ALGORITHMIC AND HIGH-FREQUENCY TRADING

The design of trading algorithms requires sophisticated mathematical models, a solid analysis of financial data, and a deep understanding of how markets and exchanges function. In this textbook the authors develop models for algorithmic trading in contexts such as: executing large orders, market making, targeting VWAP and other schedules, trading pairs or collection of assets, and executing in dark pools. These models are grounded on how the exchanges work, whether the algorithm is trading with better informed traders (adverse selection), and the type of information available to market participants at both ultra-high and low frequency.

*Algorithmic and High-Frequency Trading* is the first book that combines sophisticated mathematical modelling, empirical facts and financial economics, taking the reader from basic ideas to the cutting edge of research and practice.

If you need to understand how modern electronic markets operate, what information provides a trading edge, and how other market participants may affect the profitability of the algorithms, then this is the book for you.

ÁLVARO CARTEA is a Reader in Financial Mathematics at University College London. Before joining UCL he was Associate Professor of Finance at Universidad Carlos III, Madrid-Spain (2009–2012) and from 2002 until 2009 he was a Lecturer (with tenure) in the School of Economics, Mathematics and Statistics at Birkbeck – University of London. He was previously JP Morgan Lecturer in Financial Mathematics at Exeter College, University of Oxford.

SEBASTIAN JAIMUNGAL is an Associate Professor and Chair, Graduate Studies in the Department of Statistical Sciences at the University of Toronto where he teaches in the PhD and Masters of Mathematical Finance programs. He consults for major banks and hedge funds focusing on implementing advance derivative valuation engines and algorithmic trading strategies. He is also an associate editor for the SIAM Journal on Financial Mathematics, the International Journal of Theoretical and Applied Finance, the journal Risks and the Argo newsletter. Jaimungal is the Vice Chair for the Financial Engineering & Mathematics activity group of SIAM and his research is widely published in academic and practitioner journals. His recent interests include High-Frequency and Algorithmic trading, applied stochastic control, mean-field games, real options, and commodity models and derivative pricing.

JOSÉ PENALVA is an Associate Professor at the Universidad Carlos III in Madrid where he teaches in the PhD and Master in Finance programmes, as well as at the undergraduate level. He is currently working on information models and market microstructure and his research has been published in Econometrica and other top academic journals.

Cambridge University Press  
978-1-107-09114-6 - Algorithmic and High-Frequency Trading  
Álvaro Cartea, Sebastian Jaimungal And José Penalva  
Frontmatter  
[More information](#)

---

Cambridge University Press  
978-1-107-09114-6 - Algorithmic and High-Frequency Trading  
Álvaro Cartea, Sebastian Jaimungal And José Penalva  
Frontmatter  
[More information](#)

ALGORITHMIC AND  
HIGH-FREQUENCY TRADING

ÁLVARO CARTEA  
*University College London*

SEBASTIAN JAIMUNGAL  
*University of Toronto*

JOSÉ PENALVA  
*Universidad Carlos III de Madrid*



Cambridge University Press  
978-1-107-09114-6 - Algorithmic and High-Frequency Trading  
Álvaro Cartea, Sebastian Jaimungal And José Penalva  
Frontmatter  
[More information](#)

CAMBRIDGE  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University’s mission by disseminating knowledge in the pursuit of  
education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781107091146](http://www.cambridge.org/9781107091146)

© Álvaro Cartea, Sebastian Jaimungal and José Penalva 2015

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without the written  
permission of Cambridge University Press.

First published 2015

Printed in the United Kingdom by Bell and Bain Ltd

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing in Publication data*  
Cartea, Álvaro.

Algorithmic and high-frequency trading / Álvaro Cartea, Sebastian Jaimungal, José Penalva.  
pages cm

Includes bibliographical references and index.

ISBN 978-1-107-09114-6 (Hardback : alk. paper)

1. Electronic trading of securities—Mathematical models. 2. Finance—Mathematical models.  
3. Speculation—Mathematical models. I. Title.

HG4515.95.C387 2015

332.64—dc23 2015018946

ISBN 978-1-107-09114-6 Hardback

Additional resources for this publication at [www.cambridge.org/9781107091146](http://www.cambridge.org/9781107091146)

Cambridge University Press has no responsibility for the persistence or accuracy of  
URLs for external or third-party internet websites referred to in this publication,  
and does not guarantee that any content on such websites is, or will remain,  
accurate or appropriate.

Cambridge University Press  
978-1-107-09114-6 - Algorithmic and High-Frequency Trading  
Álvaro Cartea, Sebastian Jaimungal And José Penalva  
Frontmatter  
[More information](#)

To my girls, in order of appearance, Victoria, Amaya, Carlota, and Penélope.  
— Á.C.

To my parents, Korisha and Paul, and my siblings Shelly, Cristina and  
especially my brother Curt for his constant injection of excitement and  
encouragement along the way.  
— S.J.

To Nuria, Daniel, Jose María and Adelina.  
For their patience and encouragement every step of the way, and for never  
losing faith.  
— J.P.

Cambridge University Press  
978-1-107-09114-6 - Algorithmic and High-Frequency Trading  
Álvaro Cartea, Sebastian Jaimungal And José Penalva  
Frontmatter  
[More information](#)

---

Contents

<i>Preface</i>	<i>page</i> xiii
<i>How to Read this Book</i>	xvi
<b>Part I Microstructure and Empirical Facts</b>	<b>1</b>
<i>Introduction to Part I</i>	3
<b>1 Electronic Markets and the Limit Order Book</b>	<b>4</b>
1.1 Electronic markets and how they function	4
1.2 Classifying Market Participants	6
1.3 Trading in Electronic Markets	9
1.3.1 Orders and the Exchange	9
1.3.2 Alternate Exchange Structures	10
1.3.3 Colocation	11
1.3.4 Extended Order Types	12
1.3.5 Exchange Fees	13
1.4 The Limit Order Book	14
1.5 Bibliography and Selected Readings	18
<b>2 A Primer on the Microstructure of Financial Markets</b>	<b>19</b>
2.1 Market Making	20
2.1.1 Grossman–Miller Market Making Model	21
2.1.2 Trading Costs	24
2.1.3 Measuring Liquidity	26
2.1.4 Market Making using Limit Orders	28
2.2 Trading on an Informational Advantage	30
2.3 Market Making with an Informational Disadvantage	34
2.3.1 Price Dynamics	36
2.3.2 Price Sensitive Liquidity Traders	37
2.4 Bibliography and Selected Readings	37
<b>3 Empirical and Statistical Evidence: Prices and Returns</b>	<b>39</b>
3.1 Introduction	39
3.1.1 The Data	39
3.1.2 Daily Asset Prices and Returns	41

viii	<b>Contents</b>	
	3.1.3 Daily Trading Activity	42
	3.1.4 Daily Price Predictability	42
	3.2 Asset Prices and Returns Intraday	46
	3.3 Interarrival Times	48
	3.4 Latency and Tick Size	49
	3.5 Non-Markovian Nature of Price Changes	52
	3.6 Market Fragmentation	54
	3.7 Empirics of Pairs Trading	57
	3.8 Bibliography and Selected Readings	60
<b>4</b>	<b>Empirical and Statistical Evidence: Activity and Market Quality</b>	<b>61</b>
	4.1 Daily Volume and Volatility	61
	4.2 Intraday Activity	63
	4.2.1 Intraday Volume Patterns	65
	4.2.2 Intraday Volume Patterns	67
	4.2.3 Price Patterns	68
	4.3 Trading and Market Quality	69
	4.3.1 Spreads	71
	4.3.2 Volatility	76
	4.3.3 Market Depth and Trade Size	79
	4.3.4 Price Impact	81
	4.3.5 Walking the LOB and Permanent Price Impact	87
	4.4 Messages and Cancellation Activity	90
	4.5 Hidden Orders	95
	4.6 Bibliography and Selected Readings	96
<b>Part II</b>	<b>Mathematical Tools</b>	<b>97</b>
	<i>Introduction to Part II</i>	99
<b>5</b>	<b>Stochastic Optimal Control and Stopping</b>	<b>100</b>
	5.1 Introduction	100
	5.2 Examples of Control Problems in Finance	101
	5.2.1 The Merton Problem	101
	5.2.2 The Optimal Liquidation Problem	102
	5.2.3 Optimal Limit Order Placement	103
	5.3 Control for Diffusion Processes	103
	5.3.1 The Dynamic Programming Principle	105
	5.3.2 Dynamic Programming Equation / Hamilton–Jacobi–Bellman Equation	107
	5.3.3 Verification	112
	5.4 Control for Counting Processes	113
	5.4.1 The Dynamic Programming Principle	114
	5.4.2 Dynamic Programming Equation / Hamilton–Jacobi–Bellman Equation	115



	Contents	ix
	5.4.3 Combined Diffusion and Jumps	120
5.5	Optimal Stopping	122
	5.5.1 The Dynamic Programming Principle	124
	5.5.2 Dynamic Programming Equation	124
5.6	Combined Stopping and Control	128
5.7	Bibliography and Selected Readings	130
<b>Part III</b>	<b>Algorithmic and High-Frequency Trading</b>	<b>131</b>
	<i>Introduction to Part III</i>	133
<b>6</b>	<b>Optimal Execution with Continuous Trading I</b>	<b>134</b>
	6.1 Introduction	134
	6.2 The Model	135
	6.3 Liquidation without Penalties only Temporary Impact	139
	6.4 Optimal Acquisition with Terminal Penalty and Temporary Impact	141
	6.5 Liquidation with Permanent Price Impact	144
	6.6 Execution with Exponential Utility Maximiser	150
	6.7 Non-Linear Temporary Price Impact	152
	6.8 Bibliography and Selected Readings	154
	6.9 Exercises	155
<b>7</b>	<b>Optimal Execution with Continuous Trading II</b>	<b>158</b>
	7.1 Introduction	158
	7.2 Optimal Acquisition with a Price Limiter	159
	7.3 Incorporating Order Flow	167
	7.3.1 Probabilistic Interpretation	174
	7.4 Optimal Liquidation in Lit and Dark Markets	175
	7.4.1 Explicit Solution when Dark Pool Executes in Full	178
	7.5 Bibliography and Selected Readings	182
	7.6 Exercises	182
<b>8</b>	<b>Optimal Execution with Limit and Market Orders</b>	<b>184</b>
	8.1 Introduction	184
	8.2 Liquidation with Only Limit Orders	185
	8.3 Liquidation with Exponential Utility Maximiser	193
	8.4 Liquidation with Limit and Market Orders	196
	8.5 Liquidation with Limit and Market Orders Targeting Schedules	206
	8.6 Bibliography and Selected Readings	209
	8.7 Exercises	209
<b>9</b>	<b>Targeting Volume</b>	<b>212</b>
	9.1 Introduction	212
	9.2 Targeting Percentage of Market's Speed of Trading	215
	9.2.1 Solving the DPE when Targeting Rate of Trading	216

x	<b>Contents</b>	
	9.2.2 Stochastic Mean-Reverting Trading Rate	220
	9.2.3 Probabilistic Representation	222
	9.2.4 Simulations	225
9.3	Percentage of Cumulative Volume	227
	9.3.1 Compound Poisson Model of Volume	231
	9.3.2 Stochastic Mean-Reverting Volume Rate	232
	9.3.3 Probabilistic Representation	233
9.4	Including Impact of Other Traders	235
	9.4.1 Probabilistic Representation	237
	9.4.2 Example: Stochastic Mean-Reverting Volume	238
9.5	Utility Maximiser	239
	9.5.1 Solving the DPE with Deterministic Volume	240
9.6	Bibliography and Selected Readings	243
9.7	Exercises	243
<b>10</b>	<b>Market Making</b>	<b>246</b>
	10.1 Introduction	246
	10.2 Market Making	247
	10.2.1 Market Making with no Inventory Restrictions	253
	10.2.2 Market Making At-The-Touch	254
	10.2.3 Market Making Optimising Volume	257
10.3	Utility Maximising Market Maker	259
10.4	Market Making with Adverse Selection	261
	10.4.1 Impact of Market Orders on Midprice	262
	10.4.2 Short-Term-Alpha and Adverse Selection	266
10.5	Bibliography and Selected Readings	271
10.6	Exercises	272
<b>11</b>	<b>Pairs Trading and Statistical Arbitrage Strategies</b>	<b>273</b>
	11.1 Introduction	273
	11.2 Ad Hoc Bands	274
	11.3 Optimal Band Selection	277
	11.3.1 The Optimal Exit Problem	278
	11.3.2 The Optimal Entry Problem	279
	11.3.3 Double-Sided Optimal Entry-Exit	281
11.4	Co-integrated Log Prices with Short-Term-Alpha	283
	11.4.1 Model Setup	284
	11.4.2 The Agent's Optimisation Problem	286
	11.4.3 Solving the DPE	288
	11.4.4 Numerical Experiments	292
11.5	Bibliography and Selected Readings	294
<b>12</b>	<b>Order Imbalance</b>	<b>295</b>
	12.1 Introduction	295

	Contents	xi
12.2	Intraday Features	295
12.2.1	A Markov Chain Model	297
12.2.2	Jointly Modelling Market Orders	300
12.2.3	Modelling Price Jumps	303
12.3	Daily Features	305
12.4	Optimal Liquidation	306
12.4.1	Optimisation Problem	308
12.5	Bibliography and Selected Readings	313
12.6	Exercises	313
<b>Appendix A</b>	<b>Stochastic Calculus for Finance</b>	<b>315</b>
A.1	Diffusion Processes	315
A.1.1	Brownian Motion	316
A.1.2	Stochastic Integrals	316
A.2	Jump Processes	319
A.3	Doubly Stochastic Poisson Processes	322
A.4	Feynman–Kac and PDEs	325
A.5	Bibliography and Selected Readings	326
	<i>Bibliography</i>	327
	<i>Glossary</i>	337
	<i>Subject index</i>	342

Cambridge University Press  
978-1-107-09114-6 - Algorithmic and High-Frequency Trading  
Álvaro Cartea, Sebastian Jaimungal And José Penalva  
Frontmatter  
[More information](#)

---

## Preface

We have written this book because we feel that existing ones do not provide a sufficiently broad view to address the rich variety of issues that arise when trying to understand and design a successful trading algorithm. This book puts together the diverse perspectives, and backgrounds, of the three authors in a manner that ties together the basic economics, the empirical foundations of high-frequency data, and the mathematical tools and models to create a balanced perspective of algorithmic and high-frequency trading.

This book has grown out of the authors' interest in the field of algorithmic and high-frequency finance and from graduate courses taught at University College London, University of Toronto, Universidad Carlos III de Madrid, IMPA, and University of Oxford. Readers are expected to have basic knowledge of continuous-time finance, but it assumes that they have no knowledge of stochastic optimal control and stopping. To keep the book self-contained, we include an appendix with the main stochastic calculus tools and results that are needed. The treatment of the material should appeal to a wide audience and it is ideal for a graduate course on Algorithmic Trading at a Master's or PhD level. It is also ideal for those already working in the finance sector who wish to combine their industry knowledge and expertise with robust mathematical models for algorithmic trading. We welcome comments! Please send them to [algo.trading.book@gmail.com](mailto:algo.trading.book@gmail.com).

### Brief guide to the contents

This book is organised into three parts that take the reader from the workings of electronic exchanges to the economics behind them, then to the relevant mathematics, and finally to models and problems of algorithmic trading.

Part I starts with a description of the basic elements of electronic markets and the main ways in which people participate in the market: as active traders exploiting an informational advantage to profit from possibly fleeting profit opportunities, or as market makers, simultaneously offering to buy and sell at advantageous prices.

A textbook on algorithmic trading would be incomplete if the development of strategies was not motivated by the information that market participants see in electronic markets. Thus it is necessary to devote space to a discussion of

data and empirical implications. The data allow us to present the context which determines the ultimate fate of an algorithm. By looking at prices, volumes, and the details of the limit order book, the reader will get a basic overview of some of the key issues that any algorithm needs to account for, such as the information in trades, properties of price movements, regularities in the intraday dynamics of volume, volatility, spreads, etc.

Part II develops the mathematical tools for the analysis of trading algorithms. The chapter on stochastic optimal control and stopping provides a pragmatic approach to material which is less standard in financial mathematics textbooks. It is also written so that readers without previous exposure to these techniques equip themselves with the necessary tools to understand the mathematical models behind some algorithmic trading strategies.

Part III of the book delves into the modelling of algorithmic trading strategies. The first two chapters are concerned with optimal execution strategies where the agent must liquidate or acquire a large position over a pre-specified window and trades continuously using only market orders. Chapter 6 covers the classical execution problem when the investor's trades impact the price of the asset and also adjusts the level of urgency with which she desires to execute the programme. In Chapter 7 we develop three execution models where the investor: i) carries out the execution programme as long as the price of the asset does not breach a critical boundary, ii) incorporates order flow in her strategy to take advantage of trends in the midprice which are caused by one-sided pressure in the buy or sell side of the market, and iii) trades in both a lit venue and a dark pool.

In Chapter 8 we assume that the investor's objective is to execute a large position over a trading window, but employs only limit orders, or uses both limit and market orders. Moreover, we show execution strategies where the investor also tracks a particular schedule as part of the liquidation programme.

Chapter 9 is concerned with execution algorithms that target volume-based schedules. We develop strategies for investors who wish to track the overall volume traded in the market by targeting: Percentage of Volume, Percentage of Cumulative Volume, and Volume Weighted Average Price, also known as VWAP.

The final three chapters cover various topics in algorithmic trading. Chapter 10 shows how market makers choose where to post limit orders in the book. The models that are developed look at how the strategies depend on different factors including the market maker's aversion to inventory risk, adverse selection, and short-term lived trends in the dynamics of the midprice.

Finally, Chapter 11 is devoted to statistical arbitrage and pairs trading, and Chapter 12 shows how information on the volume supplied in the limit order book is employed to improve execution algorithms.

### Style of the book

In choosing the content and presentation of the book we have tried to provide a rigorous yet accessible overview of the main foundational issues in market

microstructure, and of some of the empirical themes of electronic trading, using the US equities market as the one most familiar to readers. These provide the basis for a thorough mathematical analysis of models of trade execution, volume-based algorithms, market making, statistical arbitrage, pairs trading, and strategies based on order flow information. Most chapters in Part III end with exercises of varying levels of difficulty. Some exercises closely follow the material covered in the chapter and require the reader to: solve some of the problems by looking at them from a different perspective; fill in the gaps of some of the derivations; see it as an invitation to experiment further. We have set up a website, <http://www.algorithmic-trading.org>, from which readers can download datasets and MATLAB code to assist in such experimentation.

This book does not cover any of the information technology aspects of algorithmic trading. Nor does it cover in detail certain aspects of market quality or discuss regulation issues.

### Acknowledgements

We are thankful to those who took the time to read parts of the manuscript and gave us very useful feedback: Ali Al-Aradi, Gene Amromin, Robert Almgren, Ryan Francis Donnelly, Luhui Gan, John Goodacre, Hui Gong, Tianyi Jia, Hoi Kong, Tim Leung, Siyuan Li, Eddie Ng, Zhen Qin, Jason Ricci, Anton Rubisov, Mark Stevenson, Mikel Tapia and Jamie Walton. We also thank the students who have taken our courses at University College London, University of Toronto, University of Oxford, IMPA and Universidad Carlos III de Madrid.

Álvaro is grateful for the hospitality and generosity of the Finance Group at Saïd Business School, University of Oxford, with special thanks to Tim Jenkinson and Colin Mayer, and the Department of Statistical Sciences, University of Toronto, where a great deal of this book was written.

Sebastian is grateful for the hospitality of the Mathematical Institute, University of Oxford and the Department of Mathematics, University College London, where parts of this book were written.

José is grateful for the hospitality of the Department of Mathematics, University College London and the Department of Finance at Cass Business School where parts of this book were written, as well as his home institution, the Business Department of the Universidad Carlos III for allowing him to make these visits. He also wishes to thank Artem Novikov of TradingPhysics for his availability and help in accessing the data and clarifying specific issues faced by traders and technicians in high-frequency trading environments.

May 2015, Oxford, London, Toronto, Madrid, Mallorca

## How to Read this Book

This book is aimed at those who want to learn how to develop the mathematical aspects of Algorithmic Trading. It is ideal for a graduate course on Algorithmic Trading at a Master's or PhD level, and is also ideal for those already working in the finance sector who wish to combine their industry knowledge and expertise with robust mathematical models for algorithmic trading.

Much of this book can be covered in an intensive one semester/term course as part of a Graduate course in Financial Mathematics/Engineering, Computational Finance, and Applied Mathematics. A typical student at this stage will be learning stochastic calculus as part of other courses, but will not be taught stochastic optimal control, or be proficient in the way modern electronic markets operate. Thus, they are strongly encouraged to read Part I of the book to: gain a good understanding of how electronic markets operate; understand basic concepts of microstructure theory that underpin how the market reaches equilibrium prices in the presence of different types of risks; and, study stylised statistical issues of the dynamics of the prices of stocks in modern electronic markets. And to read Part II to learn the stochastic optimal control tools which are essential to Part III where we develop sophisticated mathematical models for Algorithmic and High-Frequency trading.

Those with a solid understanding of stochastic calculus and optimal control, may skip Part II of the book and cover in detail Part III. However, we still encourage them to read Part I to gain an understanding of the stylised statistical features of the market, and to develop a better intuition of why algorithmic models are designed in particular ways or with specific objectives in mind.

For a shorter and more compact course on algorithmic trading, students should focus on learning about the limit order book, Chapter 1, then optimal control in Part II, and then concentrate on selected Chapters in Part III, for instance Chapters 6, 8 and 10.

Readers in the financial industry who have some knowledge of how electronic markets are organised may want to skip Chapter 1 but are encouraged to read the other chapters which cover microstructure theory and the empirical and statistical evidence of stock prices before delving into the details of the mathematical models in Part III.