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Algorithmic Complexity of a Problem of Idempotent Convex Geometry

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Abstract—Properties of the idempotently convex hull of a two-point set in a free semimodule over the idempotent semiring $R_{\max\min}$ and in a free semimodule over a linearly ordered idempotent semifield are studied. Construction algorithms for this hull are proposed.

Key words: *idempotent geometry, convex hull, semimodule, idempotent semiring, idempotent semifield, algorithmic complexity.*

Some asymptotic physical problems (such as quasiclassical approximation in quantum mechanics [1]), as well as many problems of optimization theory, mathematical economics, etc., admit a natural and simple formulation in terms of algebraic structures involving the operations of minimization or maximization [2, 3]. Such algebraic structures are the object of the actively developing field of idempotent mathematics [3–5]. Interesting, important and useful constructions and results of traditional mathematics over number fields and similar structures have counterparts over idempotent semifields and semirings formulated in the spirit of Bohr's correspondence principle in quantum theory [6, 7]. This correspondence can be far from obvious, though. In this paper, we consider the simplest problem of idempotent convex geometry (developed, in particular, in [8] and [7]), the construction of the convex hull of a two-point set in an idempotent semimodule, and prove that the algorithmic complexity of its solution increases with the growth of the semimodule's dimension.

We consider the number line with the operations $\oplus = \max$ and $\odot = +$ and the additional element $-\infty$, which plays the part of $\mathbf{0}$, i.e., is assumed to have the properties $-\infty \oplus a = a$, $-\infty \odot a = -\infty$. The $\oplus$ operation is commutative, associative, and *idempotent* ($a \oplus a = a$), the $\odot$ operation is commutative and distributive with respect to $\oplus$. In idempotent analysis, the above-mentioned properties are considered to be the axioms of an *idempotent semiring.* The structure defined above has also the property of invertibility of the $\odot$ operation and, for that reason, is called the *idempotent semifield* $R_{\max +}$.

The idempotent semiring $R_{\max\min}$ is an important example of a semiring which is not a semifield. It includes the entire number line with the additional elements $-\infty$ and $+\infty$ and has two operations $\oplus = \max$ and $\odot = \min$. The elements $-\infty$ and $+\infty$ are considered to be the elements $\mathbf{0}$ and $\mathbf{1}$ of the semiring, i.e., are assumed to have the properties $-\infty \oplus a = a$, $a \odot (+\infty) = a$.

In any idempotent semiring, the $\oplus$ operation induces a partial order: $a \preceq b$ if and only if $a \oplus b = b$; $a \prec b$ if and only if $a \preceq b$ and $a \neq b$. In both semirings considered above, this order is linear, because for any elements a and b, we have $a \preceq b$ or $b \preceq a$; therefore, for any elements a and b of these semirings the operation $a \wedge b$ of taking the lower bound is also defined.

In the paper, we consider semimodules S^n of column vectors of the form $(a^1, \dots, a^n)$, $a^i \in S$, with coordinatewise operations of generalized addition and multiplication by a scalar from the

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semiring S. They are idempotent analogs of vector spaces. The $\oplus$ operation induces a partial order in these semimodules.

It is easy to carry the notion of convexity over to semimodules of this kind. A set $C \subset S^n$ is said to be *idempotently convex* if for any $x, y \in C$ and $\lambda, \mu \in S$ such that $\lambda \oplus \mu = \mathbf{1}$, the combination $\lambda x \oplus \mu y$ also belongs to C. A point y is called an *idempotently convex combination* of points $x_1, \dots, x_m$ if $y = \bigoplus_{i=1}^m \lambda_i x_i$, with $\bigoplus_{i=1}^m \lambda_i = \mathbf{1}$. From now on we omit the symbol $\odot$ in the notation of the generalized multiplication of a "vector" from S^n by a scalar from S. The *idempotently convex hull* of points $x_1, \dots x_m \in S^n$ is the set of their idempotently convex combinations.

In what follows, wherever it does not lead to misunderstanding, we shall call idempotently convex sets, combinations, and hulls simply convex. All the following results are formulated for semimodules S^n, where S is the linearly ordered idempotent semifield or the semiring $R_{\max\min}$.

In semimodules over these semirings, convex combinations of points x_1 and x_2 can be linearly ordered as follows: $y \preceq_{x_1,x_2} z$ if z belongs to the convex hull of $\{y, x_2\}$. It is readily seen that this relation satisfies all the order relation axioms. Now it is natural to give the following definition: a sequence of convex combinations $y_1, \dots, y_m$ of points x_1 and x_2 is called *monotone* if $y_i \preceq_{x_1,x_2} y_{i+1}$ for $i = 1, \dots, m-1$. The proof of the following lemma is obvious.

Lemma 1. *Let $y_1, \dots, y_m \in S^n$ be a monotone sequence of convex combinations of points y_0 and y_{m+1}, and let z be any other convex combination of y_0 and y_{m+1}. Then for a certain unique $i \in \{0, \dots, m\}$, the point z is a convex combination of the points y_i and y_{i+1}.*

Let us introduce one more notion. Let S be the idempotent semifield or the semiring $R_{\max\min}$. The set of pairwise distinct points $y_1, \dots, y_{m+1} \in S^n$ will be called the *vertex sequence of the convex hull* of the points y_1 and y_{m+1} if it meets the following three requirements:

(1) all y_i belong to the convex hull of y_1 and y_{m+1};
(2) for any $i \in \{1, \dots, m\}$, the points y_i and y_{i+1} belong to the line specified by the equations $y^j = c^j = \text{const}$ for $j \in I_i$ and $y^j = \lambda \odot y_i^j$ for $j \notin I_i$ if S is the idempotent semifield, and the equations $y^j = c^j = \text{const}$ for $j \in I_i$ and $y^j = \lambda$ for $j \notin I_i$ if S is the semiring $R_{\max\min}$;
(3) for any $i \in \{1, \dots, m-1\}$, the sets I_i and I_{i+1} do not coincide.

Lemma 2. *The vertex sequence $y_1, \dots, y_{m+1} \in S^n$ is monotone and unique.*

Proof. If $m = 1, 2$, then the statement is trivial. Now suppose that $m > 2$. Let us prove by induction that the vertex sequence is monotone. The base of induction is also trivial: $y_1 \preceq_{y_1,y_{m+1}} y_2$. Now suppose that $y_1 \preceq_{y_1,y_{m+1}} \cdots \preceq_{y_1,y_{m+1}} y_k$. Let us prove that $y_k \preceq_{y_1,y_{m+1}} y_{k+1}$ by contradiction. Then the vertex y_{k+1}, by Lemma 1, belongs to either $\{y_{k-1}, y_k\}$ (then the third requirement of the definition is not true) or $\{y_{i-1}, y_i\}$, where $i < k$ (then the second requirement of the definition is not true). Thus, $y_k \preceq_{y_1,y_{m+1}} y_{k+1}$, and the vertex sequence is monotone by induction.

Let us prove the uniqueness of the sequence of distinct vertices. Suppose that this is not true, and their exists another vertex sequence $z_1, \dots, z_s$. Then this sequence of points cannot be a subsequence of $y_1, \dots, y_m$, because in that case, for $z_l = y_i$ and $z_{l+1} = y_{i+k}$, where $k > 1$, the second requirement of the definition is violated. It remains to assume that, e.g., the point z_j does not coincide with any of the vertices y. Then, by Lemma 1, this point belongs to the convex hull of one of the pairs $\{y_i, y_{i+1}\}$. The points z_{j-1} and z_{j+1} must belong to the same convex hull, otherwise they do not satisfy the second requirement of the definition. But in this case the points z_{j-1}, z_j, and z_{j+1} do not satisfy the third requirement of the definition. Hence the sequence $z_1, \dots, z_s$ coincides with $y_1, \dots, y_k$. $\square$

Let us construct the vertex sequence of the convex hull of the points x_1 and x_2. We shall consider two cases: $x_1 \prec x_2$ and the case in which x_1 and x_2 are incomparable. Notice that without loss of generality we can assume that $x_1^i \neq x_2^i$ for all i.

In the case $x_1 \prec x_2$, we shall prove the following theorem:

Theorem 1. *Suppose that $x_1, x_2 \in S^n$ and $x_1^i \prec x_2^i$ for all i. Then the vertex sequence y_i of the convex hull of x_1 and x_2 exists, and the following statements hold up to coincident vertices:*

(1) if S is the linearly ordered idempotent semifield, then $i = 1, \dots, n+1$, $y_{n+1} = x_2$, and $y_i = x_1 \oplus \lambda_i x_2$ for $i \neq n+1$, where λ_i is the ith term of the sequence

$$x_1^{l_1} \odot (x_2^{l_1})^{-1} \preceq \dots \preceq x_1^{l_n} \odot (x_2^{l_n})^{-1},$$

and $\{l_1, \dots, l_n\}$ is the suitable permutation of $\{1, \dots, n\}$;

(2) if S is the semiring $R_{\max\min}$, then $i = 1, \dots, 2n$ and $y_i = x_1 \oplus \lambda_i x_2$, where λ_i is the ith term of the sequence

$$x_1^{l(1)} \preceq \dots \preceq x_{u_i}^{l(i)} \preceq \dots \preceq x_2^{l(2n)}$$

and $i \mapsto (u_i, l(i))$ is the suitable bijection of $\{1, \dots, 2n\}$ onto $\{1, 2\} \times \{1, \dots, n\}$.

Proof. By choosing a representative from each equivalence class of coincident convex combinations of y_j and arranging them in order of increasing parameter λ, we obtain a sequence $z_1, \dots, z_{l+1}$, where $z_1 = x_1$, $z_{l+1} = x_2$, and $z_j = y_{r(j)}$, where $r(j)$ is the corresponding map of the set $\{1, \dots, l+1\}$ to the set $\{1, \dots, n+1\}$ in the case of a semifield and to the set $\{1, \dots, 2n\}$ in the case of a semiring $R_{\max\min}$. Now the statement of the theorem reduces to the fact that the sequence $z_1, \dots, z_{l+1}$ is the vertex sequence of the convex hull of $x_1 = z_1$ and $x_2 = z_{l+1}$.

Let us prove this statement in the case of a semimodule over the idempotent semifield.

First, since $x_1^i \prec x_2^i$ for any i, we have $\lambda_i \prec \mathbf{1}$, and so all y_i and z_i are convex combinations of the points x_1 and x_2.

Let us prove the second and third requirements of the definition of a vertex sequence. We introduce the index sets

$$N^+(\lambda) = \{i \mid x_1^i \odot (x_2^i)^{-1} \succ \lambda\},$$
$$N^0(\lambda) = \{i \mid x_1^i \odot (x_2^i)^{-1} = \lambda\},$$
$$N^-(\lambda) = \{i \mid x_1^i \odot (x_2^i)^{-1} \prec \lambda\},$$

and set $y(\lambda) = x_1 \oplus \lambda x_2$. Then for $i \in N^+(\lambda) \cup N^0(\lambda)$, we have $y^i(\lambda) = x_1^i$, and for $i \in N^-(\lambda) \cup N^0(\lambda)$, we have $y^i(\lambda) = \lambda \odot x_1^i$. This yields the second requirement of the definition with $I_j = N^+(\lambda_{r(j)+1}) \cup N^0(\lambda_{r(j)+1})$ for any $j = 1, \dots, l$. Notice that the index $l_{r(j)}$ belongs to I_{j-1}, but does not belong to I_j; therefore, the third requirement is also satisfied.

Let us prove the statement of the theorem for a semimodule over the semiring $R_{\max\min}$.

Since none of the elements of the semiring $R_{\max\min}$ exceeds $\mathbf{1}$, the first requirement of the definition of vertices of the convex hull is trivially true.

Let us verify the second and third requirements. We introduce the index sets

$$N_1^+(\lambda) = \{i \mid x_1^i \succ \lambda\}, \qquad N_1^0(\lambda) = \{i \mid x_1^i = \lambda\}, \qquad N_1^-(\lambda) = \{i \mid x_1^i \prec \lambda\},$$
$$N_2^+(\lambda) = \{i \mid x_2^i \succ \lambda\}, \qquad N_2^0(\lambda) = \{i \mid x_2^i = \lambda\}, \qquad N_2^-(\lambda) = \{i \mid x_2^i \prec \lambda\}.$$

Then for $i \in N_2^-(\lambda) \cup N_2^0(\lambda)$, we have $y^i(\lambda) = x_2^i$; for $i \in (N_2^+(\lambda) \cup N_2^0(\lambda)) \cap (N_1^+(\lambda) \cup N_1^0(\lambda))$, we have $y^i(\lambda) = x_1^j$; and for $j \in (N_2^+(\lambda) \cup N_2^0(\lambda)) \cap (N_1^-(\lambda) \cup N_1^0(\lambda))$, we have $y^i(\lambda) = \lambda$. This yields the second requirement with

$$I_j = (N_2^-(\lambda_{r(j)}) \cup N_2^0(\lambda_{r(j)})) \cup ((N_2^+(\lambda_{r(j)+1}) \cup N_2^0(\lambda_{r(j)+1})) \cap (N_1^+(\lambda_{r(j)+1}) \cup N_1^0(\lambda_{r(j)+1})))$$

for any j=1, … ,l. The index $l(r(j))$ cannot belong to both $I(j-1)$ and $I(j)$; hence the third requirement is true as well.

Lemma 2 ensures the uniqueness of the sequence $z_1, \dots, z_{l+1}$. $\square$

Now we shall prove a similar theorem for the second case. We shall use the notation

$$M_0 = \{i \mid x_1^i = x_2^i\}, \qquad M_1 = \{i \mid x_2^i \prec x_1^i\}, \qquad M_2 = \{i \mid x_1^i \prec x_2^i\}.$$

Theorem 2. *Suppose that points $x_1, x_2 \in S^n$ are incomparable and $M_0 = \varnothing$. Then the vertex sequence of their convex hull exists and is unique.*

Proof. In the case $x_1 \prec x_2$, the statement has already been proved. It remains to prove it for the case in which x_1 and x_2 are incomparable. In this case, $\{1, \dots, n\} = M_1 \cup M_2$. Notice that the point $x_1 \oplus x_2$ is a convex combination of the points x_1 and x_2 distinct from them. By Lemma 1, all convex combinations of the points x_1 and x_2 belong either to the convex hull of $\{x_1, x_1 \oplus x_2\}$ or to the convex hull of $\{x_1 \oplus x_2, x_2\}$. Since $x_1 \prec x_1 \oplus x_2$ and $x_2 \prec x_1 \oplus x_2$, for each of the two convex hulls there exists the sequence of intermediate vertices. Joining the two vertex sequences together by the point $x_1 \oplus x_2$, we obtain a vertex sequence of the convex hull of x_1 and x_2. All the vertices except $x_1 \oplus x_2$ automatically satisfy all the requirements of the definition, and the point $x_1 \oplus x_2$ automatically satisfies the first two requirements. Let us verify the third requirement for this point. The convex hull of $\{x_1, x_1 \oplus x_2\}$ belongs to the plane $x^i = x_1^i$, $i \in M_1$, and the convex hull of $\{x_1 \oplus x_2, x_2\}$ belongs to the plane $x^i = x_2^i$, $i \in M_2$; in addition, $M_1 \cap M_2 = 0$. This means that the sets I for the point $x_1 \oplus x_2$ and its neighboring vertices coincide.

The uniqueness is guaranteed by Lemma 2. □

In what follows, using Theorems 1 and 2, we propose algorithms for the construction of the sequence of distinct vertices of the convex hull of points x_1 and x_2 in the semimodule S^n over the idempotent semifield and over the semiring $R_{\max\min}$.

Both algorithms begin with dividing the initial index set $\{1, \dots, n\}$ into the three subsets M_1, M_2, and M_0. Further, if S is a semifield, then for the indices i from M_1, we sort the products $x_1^i \odot (x_2^i)^{-1}$ in ascending order to obtain the sequence $\lambda_1 \preceq \dots \preceq \lambda_m$, where $\lambda_i = x_1^{l_i} \odot (x_2^{l_i})^{-1}$. For the i from M_2, we sort $x_2^i \odot (x_1^i)^{-1}$ to obtain the sequence $\mu_1 \preceq \dots \preceq \mu_k$, where $\mu_i = x_2^{m_i} \odot (x_1^{l_i})^{-1}$. If S is the semiring $R_{\max\min}$, then for the indices i from M_1, we perform simultaneous ascending sorting of x_1^i and x_2^i, which results into the sequence $\lambda_1 \preceq \dots \preceq \lambda_{2m}$, where $\lambda_i = x_{u_i}^{l(i)}$. For the i from M_2, we perform similar sorting to obtain the sequence $\mu_1 \preceq \dots \preceq \mu_{2k}$, where $\mu_i = x_{s_i}^{m(i)}$. Then we construct the vertex sequences of the convex hulls of $\{x_1, x_1 \oplus x_2\}$ and $\{x_1 \oplus x_2, x_2\}$. We shall give the construction algorithms for the vertex sequence of the first of these convex hulls. The construction algorithm for the second one can be obtained by replacing x_1 by x_2, M_1 by M_2, and (in the case of the semiring $R_{\max\min}$) $N_1^{+,-,0}$ by $N_2^{+,-,0}$.

CONSTRUCTION ALGORITHM FOR THE VERTEX SEQUENCE OF THE CONVEX HULL OF x_1 AND $x_1 \oplus x_2$ IN A SEMIMODULE OVER THE IDEMPOTENT SEMIFIELD

State of the computation process. The set $\{1, \dots, n\}$ is partitioned into disjoint sets M_1, M_2, and M_0, the first of which is partitioned into disjoint sets N^-, N^0, and N^+. The coordinates of vertices are computed depending on the membership of the corresponding indices to these sets.

Initial state: $N^0 = \{l_1\}$, $N^+ = M_1 \backslash \{l_1\}$.

Standard step of the algorithm. If $\lambda_{i+1} \succ \lambda_i$, then the index l_{i+1} is placed in N^0, the index l_i is placed in N^-, and the coordinates of the vertices y_{i+1} are computed by the following rules: $y_{i+1}^j = \lambda_{i+1} \odot x_2^j$ if $j \in (N^- \cup N^0)$, and $y_{i+1}^j = x_1^j$ if $j \in N^+ \cup M_2 \cup M^0$.

If $\lambda_{i+1} = \lambda_i$, then the index l_{i+1} is placed in N^0 and no other actions are performed.

CONSTRUCTION ALGORITHM FOR THE VERTEX SEQUENCE OF THE CONVEX HULL OF x_1 AND $x_1 \oplus x_2$ IN A SEMIMODULE OVER THE SEMIRING $R_{\max\min}$

State of the computation process. The set $\{1, \dots, n\}$ is partitioned into the disjoint sets M_1, M_2, and M_0, the first of which is partitioned into disjoint sets N_1^-, N_1^0, and N_1^+, as well as into N_2^-, N_2^0, and N_2^+. The coordinates of vertices are computed depending on the membership of the corresponding indices to these sets.

Initial state: $N_1^0 = \{l_1\}$, $N_1^+ = M_1 \backslash \{l_1\}$, $N_2^0 = 0$, $N_2^+ = M_1$.

Standard step of the algorithm. If $\lambda_{i+1} \succ \lambda_i$, then the index $l(i+1)$ is placed in $N_{u_{i+1}}^0$, the index $l(i)$ is placed in $N_{u_{i+1}}^-$, and the coordinates of the vertex y_{i+1} are computed by the following rules: $y_{i+1}^j = \lambda_{i+1}$ if $j \in (N_1^- \cup N_1^0) \cap (N_2^+)$, $y_{i+1}^j = x_2^j$ if $j \in (N_2^- \cup N_2^0)$, and $y_{i+1}^j = x_1^j$ if $j \in (N_2^+ \cap N_1^+) \cup M_2 \cup M_0$.

If $\lambda_{i+1} = \lambda_i$, then the index $l(i+1)$ is placed in $N_{u_{i+1}}^0$ and no other actions are performed.

The algorithms thus constructed are of computational complexity n^2. This follows from the fact that the number of vertices (not counting the initial points) can be as large as $n-1$ for the semifield and $2n-2$ for $R_{\max\min}$.

In conclusion, let us mention the connection between the vertex sequence of a convex hull and the theory of (idempotently) linear functionals in idempotent semimodules developed in [7]. In this paper, the functional defined by the formula

$$y \longmapsto x^*(y) = \inf\{k \in S \mid kx \succeq y\},$$

where S is an idempotent semiring, and x, y are elements of a semimodule over this semiring, is called the *x-functional*. It is proved that if S is an idempotent semifield and the semimodule satisfies some natural conditions, then the x-functional is linear and any linear functional is representable in the form of an x-functional. Now we notice that in the semimodule S^n, where S is an idempotent semifield, we have $x^*(y) = \bigoplus_i (x^i)^{-1} \odot y^i$. Then, if $x_1^i \prec x_2^i$ for any i, then $y_n = x_1 \oplus x_2^*(x_1)x_2 = x_2^*(x_1)x_2$. One of the coordinates of y_n coincides with the corresponding coordinate of x_1; hence all the other vertices inherit it automatically. Considering x_1 and y_n as points on the $(n-1)$-dimensional plane, we find that $y_{n-1} = y_n^*(x_1)y_n$. All other vertices are found similarly.

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REFERENCES

1. V. P. Maslov, *Asymptotic Methods of Solution of Pseudodifferential Equations* [in Russian], Nauka, Moscow, 1987.
2. F. L. Bacelli, G. Cohen, G. J. Olsder, and J. P. Quadrat, *Synchronization and Linearity: an Algebra for Discrete Event Systems*, John Wiley & Sons Publ., New York, 1992.
3. V. P. Maslov and V. N. Kolokol′tsov, *Idempotent Analysis and Its Application in Optimal Control* [in Russian], Nauka, Moscow, 1994.
4. V. P. Maslov and S. N. Samborskii (editors), *Idempotent Analysis*, Adv. Sov. Math, vol. 13, Amer. Math. Soc., Providence, RI, 1992.
5. J. Gunawardena (ed.), *Idempotency*, Publ. of the Newton Institute, Cambridge Univ. Press, Cambridge, 1998.
6. G. L. Litvinov and V. P. Maslov, *Correspondence Principle for Idempotent Calculus and Some Computer Applications*, (IHES/M/95/33), Institut des Hautes Études Scientifiques, Bures-Sur-Yvette, 1995.
7. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, "Idempotent Functional Analysis. Algebraic Approach," *Mat. Zametki* [*Math. Notes*], **69** (2001), no. 5, 758–797.
8. K. Zimmermann, "A general separation theorem in extremal algebras," *Ekonomicko-matematicky obzor (Prague)*, **13** (1977), no. 2, 179–201.

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