mathematics

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Algorithms for Approximating Solutions of Split Variational Inclusion and Fixed-Point Problems

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Abstract: In this paper, the split fixed point and variational inclusion problem is considered. With the help of fixed point technique, Tseng-type splitting method and self-adaptive rule, an iterative algorithm is proposed for solving this split problem in which the involved operators S and T are demicontractive operators and g is plain monotone. Strong convergence theorem is proved under some mild conditions.

Keywords: split problem; fixed point; variational inclusion; demicontractive operator; monotone operator

MSC: 47H10; 47J25; 65K15; 90C25

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Citation: Zhu, L.-J.; Yao, Y. Algorithms for Approximating Solutions of Split Variational Inclusion and Fixed-Point Problems. *Mathematics* **2023**, *11*, 641. https://doi.org/10.3390/math11030641

Academic Editors: Tomás Domínguez Benavides, Shin-ya Matsushita and Manuel Sanchis

Received: 16 December 2022
Revised: 20 January 2023
Accepted: 24 January 2023
Published: 27 January 2023

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces with inner $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $f : H_1 \to 2^{H_1}$ be a multi-valued operator. Let $g, T : H_1 \to H_1$ and $S : H_2 \to H_2$ be three single-valued non-linear operators. Let $A : H_1 \to H_2$ be a non-zero bounded linear operator and A^* be the adjoint operator of A.

In this paper, we investigate the following split problem which aims to find a point $q \in H_1$, such that

$$q \in \mathrm{Fix}(T) \cap (f+g)^{-1}(0) \text{ and } Aq \in \mathrm{Fix}(S), \tag{1}$$

where $\mathrm{Fix}(S) := \{x \in H_2 : x = S(x)\}$ and $\mathrm{Fix}(T) := \{x \in H_1 : y = T(y)\}$ stand for the fixed point sets of S and T, respectively, and $(f+g)^{-1}(0)$ denotes the solution set of the variational inclusion of finding a point $q \in H_1$, such that

$$0 \in (f+g)(q). \tag{2}$$

In what follows, we use Γ to denote the solution set of (1), i.e.,

$$\Gamma := \{z \in H_1 : z \in \mathrm{Fix}(T) \cap (f+g)^{-1}(0) \text{ and } Az \in \mathrm{Fix}(S)\}.$$

A special case of (1) is the split fixed point problem of finding a point $q \in H_1$, such that

$$q \in \mathrm{Fix}(T) \text{ and } Aq \in \mathrm{Fix}(S), \tag{3}$$

which generalizes the convex feasibility problem and the two-sets split feasibility problem arising in the intensity-modulated radiation therapy [1].

There are various ways to solve the split problems, see [2–13]. To solve (3), a remarkable channel brought up by Censor and Segal [14] is of the manner:

$$z_0 \in H_1,\ z_{n+1} = T(z_n - \nu A^*(I-S)Az_n), \quad n \geq 0, \tag{4}$$

where S, T are directed operators.

A relaxation version of the (4) proposed by Moudafi [15] is defined by

$$z_0 \in H_1,\ z_{n+1} = (1-\gamma_n)z_n + \gamma_n T[z_n - \nu A^*(I-S)Az_n], \quad n \geq 0,$$

where S, T are demicontractive operators.

Note that solving (3) is equivalent to solve the fixed point equation $x = T(x) - \nu A^*(I - S)Ax(\nu > 0)$. By using this equivalent relation, Zheng et al. [16] suggested the following algorithm for solving (3)

$$z_0 \in H_1,\ z_{n+1} = (1-\sigma_n)z_n + \sigma_n[T(z_n) - \nu A^*(I-S)Az_n],\ n \geq 0,$$

where S, T are demicontractive operators.

At the same time, variational inclusion affords a powerful tool for exploring all kinds of problems appearing in natural science and engineering applications ([17,18]). In particular, the variational inclusion theory is a natural development of the variational principle. A variety of approaches have been proposed for solving variational inclusion (2), see [19–25]. A valuable approach to solve (2) is the well-known forward–backward algorithm ([26]) defined by

$$z_0 \in H_1,\ z_{n+1} = (I + \gamma_n f)^{-1}(I - \gamma_n g)(z_n),\ n \geq 0,$$

where the operator g is generally (inverse) strongly monotone.

To relax the strong monotonicity condition imposed on g, Tseng [27] proposed a modified forward–backward algorithm defined by

$$\begin{cases} y_n = (I + \gamma_n f)^{-1}(I - \gamma_n g)(z_n), \\ z_{n+1} = y_n - \mu_n(g(y_n) - g(z_n)), \end{cases}$$

where g is a monotone operator.

With the help of self-adaptive rule, Cholamjiak et. al. [28] suggested the following Tseng-type algorithm to solve (2):

$$\begin{cases} y_n = (I + \gamma_n f)^{-1}(I - \gamma_n g)(z_n), \\ z_{n+1} = (1-\theta_n)z_n + \theta_n y_n + \theta_n \mu_n (g(z_n) - g(y_n)), \\ \mu_{n+1} = \min\left\{\mu_n, \frac{\lambda_n \|y_n - z_n\|}{\|g(y_n) - g(z_n)\|}\right\}. \end{cases}$$

Motivated and inspired by the works in this field, the main purpose of this paper is to construct an iterative algorithm for finding a solution of the split problem (1) in which S and T are two demicontractive operators and g is a plain monotone operator. The used method consists of forward–backward method, fixed point method and self-adaptive method. Strong convergence analysis of the sequence generated by the algorithm is proved provided the involved parameter satisfy some basic assumptions.

2. Preliminaries

Let H be a real Hilbert space. Let $\{z_n\}$ be a sequence in H and $u \in H$ be a point.

- $z_n \to u$ indicates that z_n converges strongly to u as $n \to +\infty$;
- $z_n \rightharpoonup u$ indicates that z_n converges weakly to u as $n \to +\infty$;
- $\omega_w(z_n)$ denotes the the set of the weak cluster points of $\{z_n\}$ in H, i.e.,

$$\omega_w(z_n) := \{u \in H : \text{there exists a subsequence } \{z_{n_i}\} \text{ of } \{z_n\} \text{ such that } z_{n_i} \rightharpoonup u(i \to \infty)\}.$$

Let $g\colon H \to H$ be an operator and g is called:

(i) Strongly monotone if for some constant $\gamma > 0$, the following inequality holds

$$\langle g(x) - g(y), x - y \rangle \geq \gamma \|x - y\|^2, \ \forall x, y \in H.$$

(ii) Inverse strongly monotone if for some constant $\gamma > 0$ we have

$$\langle g(x) - g(y), x - y \rangle \geq \gamma \|g(x) - g(y)\|^2, \ \forall x, y \in H.$$

(iii) Monotone if the following result holds

$$\langle g(x) - g(y), x - y \rangle \geq 0, \ \forall x, y \in H.$$

Let $f\colon H \to 2^H$ be an operator. The graph $\mathrm{Graph}(f)$ of f is defined by

$$\mathrm{Graph}(f) := \{(x, y) \in H \times H : y \in f(x)\}.$$

Recall that f is said to be:

(i) Monotone if the set $\mathrm{Graph}(f)$ is monotone, namely,

$$\langle s_1 - s_2, t_1 - t_2 \rangle \geq 0, \ \forall (s_i, t_i) \in \mathrm{Graph}(f), i = 1, 2.$$

(ii) Maximal monotone if and only if f is a monotone operator and the following relation holds

$$(s, t) \in H \times H, \langle s - u, t - v \rangle \geq 0, \forall (u, v) \in \mathrm{Graph}(f) \Rightarrow (s, t) \in \mathrm{Graph}(f). \tag{5}$$

Let $T : H \to H$ be an operator and T is called

(i) Demiclosed if

$$\left.\begin{array}{r} z_n \rightharpoonup z(n \to \infty) \\ T(z_n) \to y(n \to \infty) \end{array}\right\} \Rightarrow y = T(z).$$

(ii) Lipschitz if for some constant $\mu > 0$, we have

$$\|T(x) - T(y)\| \leq \mu \|x - y\|, \ \forall x, y \in H.$$

(iii) Directed if $\forall x \in H, \forall y \in \mathrm{Fix}(T)$ we have

$$\|T(x) - y\|^2 \leq \|x - y\|^2 - \|x - T(x)\|^2.$$

(iv) Demicontractive if for some constant $\delta \in [0, 1)$, there holds

$$\|T(x) - y\|^2 \leq \|x - y\|^2 + \delta \|x - T(x)\|^2, \ \forall x \in H, \forall y \in \mathrm{Fix}(T),$$

or

$$\langle x - T(x), x - y \rangle \geq \frac{1 - \delta}{2} \|x - T(x)\|^2, \ \forall x \in H, \forall y \in \mathrm{Fix}(T). \tag{6}$$

In this case, T is said to be a δ-demicontractive operator.

Remark 1. *It is clearly from (iii) and (iv) that the demicontractive operator includes the directed operator. Demicontractive operators have many applications, for instance, demicontractive operators in terms of admissible perturbation are used during the construction phase of the matrix of ants artificial pheromone (*[29]*).*

Let Γ be a non-empty closed convex subset of H. Let proj_Γ be the metric projection from H onto Γ, i.e.,

$$\mathrm{proj}_\Gamma(x^\dagger) := \arg\min_{x \in \Gamma} \|x - x^\dagger\|, \ x^\dagger \in H.$$

It is well known that the following result holds: for $x^\dagger \in H$,

$$\langle x^\dagger - \text{proj}_\Gamma(x^\dagger), x - \text{proj}_\Gamma(x^\dagger)\rangle \geq 0,\ \forall x \in \Gamma. \tag{7}$$

The following lemma is well-known.

Lemma 1. *Let H be a real Hilbert space. Then, for all $x, y \in H$, we have*

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \tag{8}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{9}$$

and

$$\|\beta x + (1-\beta)y\|^2 = \beta\|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x-y\|^2,\ \forall \beta \in \mathbf{R}. \tag{10}$$

Lemma 2 ([30])**.** *Let H be a real Hilbert space. Let $f : H \to 2^H$ be a maximal monotone operator. Let the operator $g : H \to H$ be monotone and Lipschitz continuous. Then $f + g$ is a maximal monotone operator.*

Lemma 3 ([16])**.** *Let H_1 and H_2 be two real Hilbert spaces. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two demicontractive operators. Let $A : H_1 \to H_2$ be a non-zero bounded linear operator. Then, $x \in \text{Fix}(T), Ax \in \text{Fix}(S) \Leftrightarrow x \in \text{Fix}(T - \nu A^*(I - S)A)(\forall \nu > 0)$.*

Lemma 4 ([31])**.** *Let $\{r_n\}$, $\{s_n\}$ and $\{\mu_n\}$ be three sequences in $\mathbf{R}$. Assume that*

(i) $r_{n+1} \leq (1-\mu_n)r_n + \mu_n s_n$ *for all* $n \geq 0$*;*
(ii) $r_n \geq 0$ *and* $\mu_n \in [0, 1]$ *for all* $n \geq 0$*;*
(iii) $\sum_{n=0}^{\infty} \mu_n = +\infty$ *and* $\limsup_{n\to\infty} s_n \leq 0$.

Then $\lim_{n\to\infty} r_n = 0$.

3. Main Results

Assume H_1 and H_2 are two real Hilbert spaces. Assume the involved operators fulfil the following conditions:

- $f : H_1 \to 2^{H_1}$ is a maximal monotone operator;
- $g : H_1 \to H_1$ is an L-Lipschitz monotone operator;
- $T : H_1 \to H_1$ is a σ-demicontractive operator;
- $S : H_2 \to H_2$ is a ς-demicontractive operator;
- $A : H_1 \to H_2$ is a non-zero bounded linear operator and A^* is the adjoint operator of A.

Suppose that the involved parameters fulfil the following conditions:

- $\alpha, \beta, \delta, \gamma$, and ν are five constants, such that $\alpha \in (0, 1), \beta \in (0, 1), \delta \in (0, 1), \gamma \in (0, \frac{1-\sigma}{2})$ and $\nu \in (0, \frac{1-\varsigma}{2\gamma\|A\|^2})$;
- $\{\lambda_n\}$ and $\{\mu_n\}$ are two sequences in $[0, 1]$ satisfying $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n \leq 1$, $\lim_{n\to\infty} \mu_n = 0$, and $\sum_n \mu_n = \infty$.

In the sequel, suppose $\Gamma \neq \emptyset$. Next, we first introduce an iterative algorithm for solving the split problem (1).

To obtain our main theorem, we first show several propositions.

Proposition 1. *The sequences $\{z_n\}$, $\{u_n\}$ and $\{t_n\}$ are bounded.*

Proof. Let p be a fixed point in Γ. Then, $p = T(p)$, $Ap = S(Ap)$ and $p \in (f+g)^{-1}(0)$. Based on Lemma 3, we have $T(p) - \nu A^*(I - S)Ap = p$ for all $\nu > 0$.

By (36), we have $z_n - v_n = z_n - T(z_n) + \nu A^*(I - S)Az_n$. Then, we obtain

$$\begin{aligned} \|z_n - v_n\|^2 &= \|z_n - T(z_n) + \nu A^*(I - S)Az_n\|^2 \\ &\le (\|z_n - T(z_n)\| + \nu\|A\|\|(I - S)Az_n\|)^2 \\ &\le 2\|z_n - T(z_n)\|^2 + 2\nu^2\|A\|^2\|(I - S)Az_n\|^2, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \langle z_n - v_n, z_n - p\rangle &= \langle z_n - T(z_n), z_n - p\rangle + \langle \nu A^*(I - S)Az_n, z_n - p\rangle \\ &= \langle z_n - T(z_n), z_n - p\rangle + \nu\langle (I - S)Az_n, Az_n - Ap\rangle. \end{aligned} \tag{12}$$

Since S, T are ς-demicontractive and σ-demicontractive, respectively, from (6), we deduce

$$\begin{aligned} \langle (I - S)Az_n, Az_n - Ap\rangle &= \langle (I - S)Az_n - (I - S)Ap, Az_n - Ap\rangle \\ &\ge \frac{1 - \varsigma}{2}\|(I - S)Az_n\|^2, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \langle z_n - T(z_n), z_n - p\rangle &= \langle (I - T)z_n - (I - T)p, z_n - p\rangle \\ &\ge \frac{1 - \sigma}{2}\|z_n - T(z_n)\|^2. \end{aligned} \tag{14}$$

So, from (12)–(14), we get

$$\langle z_n - v_n, z_n - p\rangle \ge \frac{1 - \sigma}{2}\|z_n - T(z_n)\|^2 + \frac{\nu(1 - \varsigma)}{2}\|(I - S)Az_n\|^2. \tag{15}$$

By virtue of (8) and (37), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|z_n - p - \gamma(z_n - v_n)\|^2 \\ &= \|z_n - p\|^2 - 2\gamma\langle z_n - p, z_n - v_n\rangle + \gamma^2\|z_n - v_n\|^2. \end{aligned} \tag{16}$$

Substituting (11) and (15) into (16) to deduce

$$\begin{aligned} \|u_n - p\|^2 \le{}& \|z_n - p\|^2 - \gamma(1 - \sigma)\|z_n - T(z_n)\|^2 - \gamma\nu(1 - \varsigma)\|(I - S)Az_n\|^2 \\ &+ 2\gamma^2\|z_n - T(z_n)\|^2 + 2\gamma^2\nu^2\|A\|^2\|(I - S)Az_n\|^2 \\ ={}& \|z_n - p\|^2 - \gamma(1 - \sigma - 2\gamma)\|z_n - T(z_n)\|^2 \\ &- \gamma\nu(1 - \varsigma - 2\gamma\nu\|A\|^2)\|(I - S)Az_n\|^2 \\ \le{}& \|z_n - p\|^2. \end{aligned} \tag{17}$$

Since $\gamma \in (0, \frac{1-\sigma}{2})$ and $\nu \in (0, \frac{1-\varsigma}{2\gamma\|A\|^2})$, it follows from (17) that $\|u_n - p\| \le \|z_n - p\|$. Note that

$$\begin{aligned} &\|w_n - p + \alpha\rho_n(g(u_n) - g(w_n))\|^2 \\ &\quad= \|w_n - p\|^2 + 2\alpha\rho_n\langle g(u_n) - g(w_n), w_n - p\rangle \\ &\qquad+ \alpha^2\rho_n^2\|g(u_n) - g(w_n)\|^2 \\ &\quad= \|u_n - p\|^2 + \|w_n - u_n\|^2 + 2\langle w_n - u_n, u_n - w_n\rangle \\ &\qquad+ 2\langle w_n - u_n, w_n - p\rangle + 2\alpha\rho_n\langle g(u_n) - g(w_n), w_n - p\rangle \\ &\qquad+ \alpha^2\rho_n^2\|g(u_n) - g(w_n)\|^2 \\ &\quad= \|u_n - p\|^2 + 2\langle w_n - u_n + \alpha\rho_n(g(u_n) - g(w_n)), w_n - p\rangle \\ &\qquad- \|w_n - u_n\|^2 + \alpha^2\rho_n^2\|g(u_n) - g(w_n)\|^2. \end{aligned} \tag{18}$$

Thanks to (38), we gain

$$u_n - w_n - \alpha\rho_n(g(u_n) - g(w_n)) \in \alpha\rho_n(f+g)(w_n). \tag{19}$$

Since $0 \in \alpha\rho_n(f+g)(p)$, it follows from (19) and the monotonicity of $\alpha\rho_n(f+g)$ that

$$\langle w_n - u_n + \alpha\rho_n(g(u_n) - g(w_n)), w_n - p\rangle \le 0. \tag{20}$$

Combining (40), (18) and (20), we have

$$\begin{aligned}\|w_n - p + \alpha\rho_n(g(u_n) - g(w_n))\|^2 &\le \|u_n - p\|^2 - \|w_n - u_n\|^2 + \alpha^2\rho_n^2\|g(u_n) - g(w_n)\|^2 \\ &\le \|u_n - p\|^2 - \|w_n - u_n\|^2 + \beta^2\|u_n - w_n\|^2 \\ &= \|u_n - p\|^2 - (1-\beta^2)\|u_n - w_n\|^2.\end{aligned} \tag{21}$$

In light of (10) and (39), we derive

$$\begin{aligned}\|t_n - p\|^2 &= \|(1-\lambda_n)(u_n - p) + \lambda_n[w_n - p + \alpha\rho_n(g(u_n) - g(w_n))]\|^2 \\ &= (1-\lambda_n)\|u_n - p\|^2 + \lambda_n\|w_n - p + \alpha\rho_n(g(u_n) - g(w_n))\|^2 \\ &\quad - \lambda_n(1-\lambda_n)\|w_n - u_n + \alpha\rho_n(g(u_n) - g(w_n))\|^2,\end{aligned}$$

which together with (21) implies that

$$\begin{aligned}\|t_n - p\|^2 &\le (1-\lambda_n)\|u_n - p\|^2 + \lambda_n(\|u_n - p\|^2 - (1-\beta^2)\|u_n - w_n\|^2) \\ &\quad - \lambda_n(1-\lambda_n)\|w_n - u_n + \alpha\rho_n(g(u_n) - g(w_n))\|^2 \\ &= \|u_n - p\|^2 - \lambda_n(1-\beta^2)\|u_n - w_n\|^2 \\ &\quad - \lambda_n(1-\lambda_n)\|w_n - u_n + \alpha\rho_n(g(u_n) - g(w_n))\|^2 \\ &\le \|u_n - p\|^2.\end{aligned} \tag{22}$$

From (41), we obtain

$$\begin{aligned}\|z_{n+1} - p\| &= \|\mu_n(u-p) + (1-\mu_n)(t_n - p)\| \\ &\le \mu_n\|u-p\| + (1-\mu_n)\|t_n - p\| \\ &\le \mu_n\|u-p\| + (1-\mu_n)\|z_n - p\| \\ &\le \cdots \\ &\le \max\{\|u-p\|, \|z_0 - p\|\}.\end{aligned}$$

Then, $\{z_n\}$ is bounded and so are $\{u_n\}$ (by (17)) and $\{t_n\}$ (by (22)). □

Proposition 2. $-1 \le \limsup_{n\to\infty} s_n < +\infty$, *where*

$$\begin{aligned}s_n = &-(1-\mu_n)\gamma\nu(1-\varsigma-2\gamma\nu\|A\|^2)\frac{\|(I-S)Az_n\|^2}{\mu_n} - (1-\mu_n)\lambda_n(1-\beta^2)\frac{\|u_n - w_n\|^2}{\mu_n} \\ &- (1-\mu_n)\gamma(1-\sigma-2\gamma)\frac{\|z_n - T(z_n)\|^2}{\mu_n} + 2\langle u-p, z_{n+1} - p\rangle \\ &- (1-\mu_n)\lambda_n(1-\lambda_n)\frac{\|w_n - u_n + \alpha\rho_n(g(u_n) - g(w_n))\|^2}{\mu_n},\end{aligned}$$

for all $n \ge 0$.

Proof. From (9) and (41), we acquire

$$\begin{aligned}\|z_{n+1}-p\|^2 &= \|\mu_n(u-p)+(1-\mu_n)(t_n-p)\|^2\\ &\le (1-\mu_n)\|t_n-p\|^2+2\mu_n\langle u-p, z_{n+1}-p\rangle.\end{aligned} \tag{23}$$

Taking into account (17) and (22), we attain

$$\begin{aligned}\|t_n-p\|^2 \le \|z_n-p\|^2 &- \gamma(1-\sigma-2\gamma)\|z_n-T(z_n)\|^2-\lambda_n(1-\beta^2)\|u_n-w_n\|^2\\ &-\lambda_n(1-\lambda_n)\|w_n-u_n+\alpha\rho_n(g(u_n)-g(w_n))\|^2\\ &-\gamma\nu(1-\varsigma-2\gamma\nu\|A\|^2)\|(I-S)Az_n\|^2.\end{aligned} \tag{24}$$

By (23) and (24), we have

$$\begin{aligned}\|z_{n+1}-p\|^2 &\le (1-\mu_n)\|z_n-p\|^2-(1-\mu_n)\gamma(1-\sigma-2\gamma)\|z_n-T(z_n)\|^2\\ &-(1-\mu_n)\lambda_n(1-\lambda_n)\|w_n-u_n+\alpha\rho_n(g(u_n)-g(w_n))\|^2\\ &-(1-\mu_n)\gamma\nu(1-\varsigma-2\gamma\nu\|A\|^2)\|(I-S)Az_n\|^2\\ &-(1-\mu_n)\lambda_n(1-\beta^2)\|u_n-w_n\|^2+2\mu_n\langle u-p,z_{n+1}-p\rangle\\ &=(1-\mu_n)\|z_n-p\|^2+\mu_n\Big\{-(1-\mu_n)\gamma(1-\sigma-2\gamma)\frac{\|z_n-T(z_n)\|^2}{\mu_n}\\ &-(1-\mu_n)\lambda_n(1-\lambda_n)\frac{\|w_n-u_n+\alpha\rho_n(g(u_n)-g(w_n))\|^2}{\mu_n}\\ &-(1-\mu_n)\lambda_n(1-\beta^2)\frac{\|u_n-w_n\|^2}{\mu_n}+2\langle u-p,z_{n+1}-p\rangle\\ &-(1-\mu_n)\gamma\nu(1-\varsigma-2\gamma\nu\|A\|^2)\frac{\|(I-S)Az_n\|^2}{\mu_n}\Big\}.\end{aligned} \tag{25}$$

Set $r_n=\|z_n-p\|^2$ and

$$\begin{aligned}s_n=&-(1-\mu_n)\gamma\nu(1-\varsigma-2\gamma\nu\|A\|^2)\frac{\|(I-S)Az_n\|^2}{\mu_n}-(1-\mu_n)\lambda_n(1-\beta^2)\frac{\|u_n-w_n\|^2}{\mu_n}\\ &-(1-\mu_n)\gamma(1-\sigma-2\gamma)\frac{\|z_n-T(z_n)\|^2}{\mu_n}+2\langle u-p,z_{n+1}-p\rangle\\ &-(1-\mu_n)\lambda_n(1-\lambda_n)\frac{\|w_n-u_n+\alpha\rho_n(g(u_n)-g(w_n))\|^2}{\mu_n},\end{aligned} \tag{26}$$

for all $n\ge 0$.

On account of (25), we obtain

$$r_{n+1}\le(1-\mu_n)r_n+\mu_n s_n,\ n\ge 0. \tag{27}$$

From (26), we have

$$s_n\le 2\langle u-p,z_{n+1}-p\rangle\le 2\|u-p\|\|z_{n+1}-p\|$$

which leads to $\limsup_{n\to\infty}s_n<+\infty$.

Now, we show $\limsup_{n\to\infty}s_n\ge-1$. If $\limsup_{n\to\infty}s_n<-1$, then there is an integer $m\in\mathbf{N}$ fulfilling $s_n<-1,\forall n\ge m$. As a result of (27), we get $r_{n+1}\le r_n-\mu_n$ when $n\ge m$. It follows that $r_{n+1}\le r_m-\sum_{i=m}^{n}\mu_i$ and so $\limsup_{n\to\infty}r_{n+1}\le r_m-\limsup_{n\to\infty}\sum_{i=m}^{n}\mu_i=-\infty$ which is a contradiction because $r_{n+1}=\|z_{n+1}-p\|^2\ge 0$. Therefore, $-1\le\limsup_{n\to\infty}s_n<+\infty$. □

Proposition 3. *Suppose that $I-S$ and $I-T$ are demiclosed at origin. Then, $\omega_w(z_n)\subset\Gamma$.*

Proof. Thanks to Propositions 1 and 2, we conclude that $\{z_n\}$ and $\limsup_{n\to\infty} s_n$ are bounded. Choose any $p^\dagger \in \omega_w(z_n)$. Thus, there exist $\{z_{n_k}\} \subset \{z_n\}$ and $\{s_{n_k}\} \subset \{s_n\}$ satisfying $z_{n_k} \rightharpoonup p^\dagger (k \to \infty)$ and

$$\begin{aligned}\limsup_{n\to\infty} s_n &= \lim_{k\to\infty} s_{n_k} \\ &= \lim_{k\to\infty} \Big\{ -(1-\mu_{n_k})\lambda_{n_k}(1-\lambda_{n_k})\frac{\|w_{n_k}-u_{n_k}+\alpha\rho_{n_k}(g(u_{n_k})-g(w_{n_k}))\|^2}{\mu_{n_k}} \\ &\quad -(1-\mu_{n_k})\gamma\nu(1-\varsigma-2\gamma\nu\|A\|^2)\frac{\|(I-S)Az_{n_k}\|^2}{\mu_{n_k}} + 2\langle u-p, z_{n_k+1}-p\rangle \\ &\quad -(1-\mu_{n_k})\gamma(1-\sigma-2\gamma)\frac{\|z_{n_k}-T(z_{n_k})\|^2}{\mu_{n_k}} \\ &\quad -(1-\mu_{n_k})\lambda_{n_k}(1-\beta^2)\frac{\|u_{n_k}-w_{n_k}\|^2}{\mu_{n_k}}\Big\}.\end{aligned} \tag{28}$$

Since $\{z_{n_k+1}\}$ is bounded, without loss of generality, we assume that $\lim_{k\to\infty}\langle u - p, z_{n_k+1} - p\rangle$ exists. This together with (28) implies that

$$\begin{aligned}\lim_{k\to\infty} \Big\{ &-(1-\mu_{n_k})\gamma(1-\sigma-2\gamma)\frac{\|z_{n_k}-T(z_{n_k})\|^2}{\mu_{n_k}} - (1-\mu_{n_i})\lambda_{n_k}(1-\beta^2)\frac{\|u_{n_k}-w_{n_k}\|^2}{\mu_{n_k}} \\ &-(1-\mu_{n_k})\lambda_{n_k}(1-\lambda_{n_k})\frac{\|w_{n_k}-u_{n_k}+\rho_{n_k}(g(u_{n_k})-g(w_{n_k}))\|^2}{\mu_{n_k}} \\ &-(1-\mu_{n_k})\gamma\nu(1-\varsigma-2\gamma\nu\|A\|^2)\frac{\|(I-S)Az_{n_k}\|^2}{\mu_{n_k}}\Big\} \text{ exists.}\end{aligned}$$

Hence,

$$\lim_{k\to\infty} \|z_{n_k} - T(z_{n_k})\| = 0, \tag{29}$$

$$\lim_{k\to\infty} \|(I-S)Az_{n_k}\| = 0, \tag{30}$$

$$\lim_{k\to\infty} \|u_{n_k} - w_{n_k}\| = 0, \tag{31}$$

and

$$\lim_{k\to\infty} \|w_{n_k} - u_{n_k} + \rho_{n_k}(g(u_{n_k}) - g(w_{n_k}))\| = 0. \tag{32}$$

By (36), $\|z_{n_k} - v_{n_k}\| \le \|z_{n_k} - T(z_n)\| + \nu\|A\|\|(I-S)Az_{n_k}\|$. It follows from (29) and (30) that $\lim_{k\to\infty}\|z_{n_k} - v_{n_k}\| = 0$. This together with (37) implies that

$$\lim_{k\to\infty}\|u_{n_k} - z_{n_k}\| = 0 \text{ and } u_{n_k} \rightharpoonup p^\dagger (k\to\infty). \tag{33}$$

Since $Az_{n_k} \rightharpoonup Ap^\dagger (k\to\infty)$ and $I - S$ is demiclosed at the origin, by (30), we obtain $Ap^\dagger \in \mathrm{Fix}(S)$. Since $z_{n_k} \rightharpoonup p^\dagger (k\to\infty)$ and $I - T$ is demiclosed at the origin, from (29), we deduce that $p^\dagger \in \mathrm{Fix}(T)$.

Finally, we show $p^\dagger \in (f+g)^{-1}(0)$. Let $(u, v) \in \mathrm{Graph}(f+g)$. Thus, $v - g(u) \in f(u)$. Owing to $\frac{u_{n_k} - w_{n_k}}{\alpha\rho_{n_k}} - g(u_{n_k}) \in f(w_{n_k})$, by the monotonicity of f, we deduce

$$\langle v - g(u) - \frac{u_{n_k} - w_{n_k}}{\alpha\rho_{n_k}} + g(u_{n_k}), u - w_{n_k}\rangle \ge 0.$$

It follows that

$$\begin{aligned}\langle v, u - w_{n_k}\rangle &\geq \langle g(u) - g(u_{n_k}) + \frac{u_{n_k} - w_{n_k}}{\alpha\rho_{n_k}}, u - w_{n_k}\rangle \\ &= \langle g(u) - g(w_{n_k}), u - w_{n_k}\rangle + \langle g(w_{n_k}) - g(u_{n_k}), u - w_{n_k}\rangle \\ &\quad + \langle \frac{u_{n_k} - w_{n_k}}{\alpha\rho_{n_k}}, u - w_{n_k}\rangle.\end{aligned} \tag{34}$$

At the same time, by the monotonicity of g, we have $\langle g(u) - g(w_{n_k}), u - w_{n_k}\rangle \geq 0$. This together with (34) implies that

$$\langle v, u - w_{n_k}\rangle \geq \langle g(w_{n_k}) - g(u_{n_k}), u - w_{n_k}\rangle + \langle \frac{u_{n_k} - w_{n_k}}{\alpha\rho_{n_k}}, u - w_{n_k}\rangle. \tag{35}$$

By (31) and the Lipschitz continuity of g, we deduce $\|g(w_{n_k}) - g(u_{n_k})\| \to 0 (k \to \infty)$. It follows from (35) that $\langle v, u - p^\dagger\rangle \geq 0$. Taking into account Lemma 2 and (5), we conclude that $p^\dagger \in (f+g)^{-1}(0)$. Therefore, $p^\dagger \in \Gamma$ and $\omega_w(z_n) \subset \Gamma$. □

Finally, according to Propositions 1–3, we show our main theorem.

Theorem 1. *If $I - T$ and $I - S$ are demiclosed at the origin, then the sequence $\{z_n\}$ generated by Algorithm 1 converges strongly to* $\text{proj}_\Gamma(u)$.

Algorithm 1: Tseng-type method I.

Let $u \in H_1$ be a fixed point and $z_0 \in H_1, \rho_0 > 0$ be two initial points.
Step 1. Assume that z_n is given. Compute

$$v_n = T(z_n) - \nu A^*(I - S)Az_n, \tag{36}$$

and

$$u_n = z_n - \gamma(z_n - v_n). \tag{37}$$

Step 2. Compute

$$w_n = (I + \alpha\rho_n f)^{-1}(u_n - \alpha\rho_n g(u_n)), \tag{38}$$

and

$$t_n = (1 - \lambda_n)u_n + \lambda_n[w_n + \alpha\rho_n(g(u_n) - g(w_n))], \tag{39}$$

where $\rho_n = \max\{1, \delta, \delta^2, \cdots\}$ such that

$$\alpha\rho_n\|g(u_n) - g(w_n)\| \leq \beta\|u_n - w_n\|. \tag{40}$$

Step 3. Compute

$$z_{n+1} = \mu_n u + (1 - \mu_n)t_n. \tag{41}$$

Set $n := n + 1$ and return to Step 1.

Proof. First, by (39) and (41), We have

$$\begin{aligned}\|z_{n_k+1} - z_{n_k}\| &= \|\mu_{n_k}(u - z_{n_k}) + (1 - \mu_{n_k})(t_{n_k} - z_{n_k})\| \\ &\leq \mu_{n_k}\|u - z_{n_k}\| + (1 - \mu_{n_k})(1 - \lambda_{n_k})\|u_{n_k} - z_{n_k}\| \\ &\quad + (1 - \mu_{n_k})\lambda_{n_k}\|w_{n_k} - u_{n_k} + \alpha\rho_{n_k}(g(u_{n_k}) - g(w_{n_k}))\|.\end{aligned} \tag{42}$$

Combining (32), (33), and (42), we obtain $\|z_{n_k+1} - z_{n_k}\| \to 0 (k \to \infty)$ and $z_{n_k+1} \rightharpoonup p^\dagger \in \Gamma (k \to \infty)$.

Select $p = \text{proj}_\Gamma(u)$. Take into account of (28), we get

$$\limsup_{n\to\infty} s_n \leq \lim_{k\to\infty} 2\langle u - \text{proj}_\Gamma(u), z_{n_k+1} - \text{proj}_\Gamma(u)\rangle = 2\langle u - \text{proj}_\Gamma(u), p^\dagger - \text{proj}_\Gamma(u)\rangle.$$

This together with (7) implies that

$$\limsup_{n\to\infty} s_n \leq 0.$$

By virtue of (25), we acquire

$$\|z_{n+1} - \text{proj}_\Gamma(u)\|^2 \leq (1-\mu_n)\|z_n - \text{proj}_\Gamma(u)\|^2 + 2\mu_n\langle u - \text{proj}_\Gamma(u), z_{n+1} - \text{proj}_\Gamma(u)\rangle. \quad (43)$$

In view of (43) and Lemma 4, we conclude that $z_n \to \text{proj}_\Gamma(u)$ as $n \to \infty$. □

Setting $T = I$ in Algorithm 2 and Theorem 1, we obtain the following algorithm and corollary.

Algorithm 2: Tseng-type method II.

Let $u \in H_1$ be a fixed point and $z_0 \in H_1, \rho_0 > 0$ be two initial points.
Step 1. Assume that z_n is given. Compute

$$v_n = z_n - \nu A^*(I - S)Az_n,$$

and

$$u_n = z_n - \gamma(z_n - v_n).$$

Step 2. Compute

$$w_n = (I + \alpha\rho_n f)^{-1}(u_n - \alpha\rho_n g(u_n)),$$

and

$$t_n = (1-\lambda_n)u_n + \lambda_n[w_n + \alpha\rho_n(g(u_n) - g(w_n))],$$

where $\rho_n = \max\{1, \delta, \delta^2, \cdots\}$ such that

$$\alpha\rho_n\|g(u_n) - g(w_n)\| \leq \beta\|u_n - w_n\|.$$

Step 3. Compute

$$z_{n+1} = \mu_n u + (1-\mu_n)t_n.$$

Set $n := n + 1$ and return to Step 1.

Corollary 1. *If $I - S$ is demiclosed at the origin, then the sequence $\{z_n\}$ generated by Algorithm 2 converges strongly to $\text{proj}_{\Gamma_1}(u)$, where $\Gamma_1 := \{x \in (f+g)^{-1}(0), Ax \in \text{Fix}(S)\}$.*

4. Conclusions

In this paper, we investigate iterative algorithms for solving the split fixed points and variational inclusion in Hilbert spaces. We propose an iterative method which consists of fixed point method, Tseng-type splitting method and self-adaptive method for finding a solution of the considered split problem in which the involved fixed point operators S and T are all demicontractive and another operator g is plain monotone. We show that the sequence generated the constructed algorithm converges strongly to a solution of the investigated split problem provided some additional conditions are satisfied.

Author Contributions: Writing—original draft preparation, L.-J.Z. and Y.Y.; writing—review, L.-J.Z. and Y.Y. All the authors have contributed equally to this paper. All authors have read and agreed to the published version of the manuscript.

Funding: Li-Jun Zhu was supported in part by the National Natural Science Foundation of China (grant number 11861003), the Natural Science Foundation of Ningxia province (grant number NZ17015), the Major Research Projects of NingXia (grant numbers 2021BEG03049) and Major Scientific and Technological Innovation Projects of YinChuan (grant numbers 2022RKX03 and NXYLXK2017B09).

Data Availability Statement: Not Applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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