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ALGORITHMS FOR LINEAR INTERPOLATOR AND INTERPOLATION ERROR FOR MINIMAL STATIONARY STOCHASTIC PROCESSES[1]

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Algorithms for linear interpolator and interpolation error for a minimal univariate weakly stationary stochastic process with discrete multiparameter are derived. The Fourier coefficients of the inverse of the spectral density play an important role in the determination of these algorithms.

Introduction. Let Z be the set of all integers, $n \geqslant 1$, and Z^n be the Cartesian product of Z with itself n times. Endowed with the usual addition operation, Z^n is a discrete abelian group. An important problem in the theory of q-variate, $q \geqslant 1$, weakly stationary stochastic process indexed by elements of Z^n is to obtain formulas for the linear interpolator and interpolation error matrix. This problem seems to have potential application to many diverse areas of physical, natural and social sciences. In each case, the values of a stochastic process representing a particular phenomena either are missing at some points of Z^n or it is not possible to obtain direct measurements at these points. The natural thing then is to try to interpolate these missing values from the known values of the process. In [3] Masani considered a full-rank minimal q-variate process (the missing value is at one point) over Z and obtained an explicit expression for the interpolation error matrix in terms of the spectral density of the process, thereby extending the $q = 1$ result due to Kolmogorov [1]. Using J. von Neumann's alternating projections theorem, an infinite series expansion for the linear interpolator of a q-variate process over Z was obtained in [12]. The expression obtained in [12] depends on the optimal factor of the spectral density, the reciprocal of the optimal factor, and the innovations of the past and future of the process via the alternating projections theorem. As a result, the expression obtained for the interpolator does not assume a compact form. In addition for processes over Z^n, $n \geqslant 2$, a new difficulty arises, namely the determination of the optimal factor of the spectral density which unlike the case $n = 1$ is not available in the literature. The main purpose of this paper is to obtain a closed form expression for the interpolator and interpolation error for a univariate minimal stationary process over the discrete group Z^n with n being an arbitrary positive integer.

For the benefit of the reader we add that the general theory of q-variate weakly stationary stochastic processes over any locally compact abelian group has been

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developed over the past few years [2], [13]. Materials regarding the interpolation error matrix for a q-variate process, not necessarily of full rank, over Z, Z^n ($n \geqslant 1$), or any discrete locally compact abelian group may be found in [4a], [9], [11], [13]. Under the assumption of minimality Theorem 2 of Yaglom given in [14] may be reduced to our Theorem 2 for the $n = 1$ case. The problem of linear interpolator is discussed by Rozanov in [5] and [6] for the case $n = 1$. Combining [5] and [6] one can deduce our Theorem 4 for the special case $n = 1$ from Rozanov's work. However, our proof is direct, short and our result is given for the general case of arbitrary n. We might point out here that the extension of Theorem 4(a) of this paper to the nonminimal case remains open. Also the problem of determination of the linear interpolator and interpolation error for processes indexed by the set of real numbers are unresolved [10]. For this case the solution will undoubtedly lead to complicated integral equations which would be difficult to handle. For rational spectral density this problem is discussed in [7]. An explicit solution for the linear interpolator and interpolation error matrix for a general q-variate stationary process over Z^n or more generally over any locally compact abelian group would be a significant contribution to the field. Another problem which is worthy of research is the relaxation of assumption of Theorem 4. Could the algorithm of Theorem 4(a) be obtained by replacing the square integrability of the inverse of the spectral density by its mere summability?

Let x_k, $k \in Z^n$, be a univariate weakly stationary stochastic process taking values in a Hilbert space $\mathcal{H}$. For the following notations we refer the reader to [3], [4] and [13].

1. Notation. T will denote the topological dual of Z, i.e., T is the unit circle in the complex plane which is identified with $[0, 2\pi]$. T_n the dual of Z^n is simply the Cartesian product of T with itself n times (T_n is the n-dimensional torus). For k in Z^n with components $k_1, \cdots, k_n$ and θ in T_n with components $\theta_1, \cdots, \theta_n$ we will write (k, θ) for $k_1\theta_1 + \cdots + k_n\theta_n$. F will denote the spectral distribution of x_k, $k \in Z^n$; f will denote the spectral density of x_k, $k \in Z^n$. Both F and f are defined on T_n. γ will stand for the correlation function of x_k, $k \in Z^n$. γ and F are related by $\gamma(k) = (1/2\pi)^n \int_{T_n} e^{-i(k,\theta)}\, dF$, $k \in Z^n$. (This is Bochner's theorem.) For absolutely continuous F with density f this is $\gamma(k) = \int_T e^{-i(k,\theta)} f\, d\sigma$, where σ is the normalized Haar measure on T_n. $L_{2,F}$, or in the absolutely continuous case $L_{2,f}$, will denote the usual Hilbert space of square integrable functions on T_n with respect to (w.r.t.) F. We recall that the space $L_{2,F}$ is isometric to $\mathcal{G}(x)$, the subspace generated by x_k, $k \in Z^n$, under the isomorphism map $x_k \leftrightarrow e^{-i(k,\theta)}$ [13]. As usual this map between the time and spectral domains will play an important role in this paper. We will simply refer to it as the isomorphism map.

For a set T contained in Z^n, $\mathcal{G}_T$ will denote the subspace of $\mathcal{H}$ spanned by the elements x_k, k not in T. For T consisting of 0, the zero elements in Z^n, we write $\mathcal{G}_0$ instead of $\mathcal{G}_T$. P_T will denote the orthogonal projection operator onto $\mathcal{G}_T$; P_0 is used when $T = \{0\}$. σ_T^2 denotes the error quantity $\|x_0 - P_T x_0\|^2$. When $T = \{0\}$, we simply write σ_0^2. When f^{-1}, the inverse of the spectral density, exists and is integrable w.r.t. σ without loss of generality ([2], [13]), we may assume that the spectral distribution F is absolutely continuous. Moreover, it is known that

$$\sigma_0^2 = \left[\int_{T_n} f^{-1}\, d\sigma\right]^{-1},$$

which is originally given by Kolmogorov in 1941 in [1] for the case $n = 1$.

L_1, L_2 and L_∞ will stand for the usual equivalence classes of functions on T_n w.r.t. the Haar measure σ on T_n. Let $T \subseteq Z^n$ contain finitely many elements. Let $k \in T$. In the following theorems we obtain formulas for the determination of σ_T^2 and $P_T x_k$. Because of stationarity of x_k, $k \in Z^n$, without loss of generality we may assume that $0 \in T$ and $k = 0$. T^c denotes $Z^n \setminus T$.

2. Theorem. *Let x_k, $k \in Z^n$, be a weakly stationary stochastic process whose spectral distribution F is absolutely continuous* w.r.t. σ *with f denoting its density. Let f^{-1} exist almost everywhere* (a.e.) w.r.t. σ *and $f^{-1} \in L_1$. Then*

$$\sigma_T^2 = \left[\int_{T_n} \left|\sum_{k\in T} d_k e^{i(k,\theta)}\right|^2 f^{-1}\, d\sigma\right]^{-1} \tag{1}$$

where the coefficients d_k are obtained uniquely from the relations:

$$\begin{aligned} d_0 &= 1 \\ \sum_{k\in T} d_k c_{l-k} &= 0, \quad l \in T\setminus 0, \end{aligned} \tag{2}$$

with $c_k = \int_{T_n} e^{-i(k,\theta)} f^{-1}\, d\sigma$.

PROOF. We first note that the coefficient matrix $[c_{l-k}]_{l,k\in T}$ involved in equation (2) is the Gramian of the vectors $e^{i(k,\theta)}f^{-1}$, $k \in T$ in the space $L_{2,f}$. We claim that $e^{i(k,\theta)}f^{-1}$, $k \in T$, are linearly independent in $L_{2,f}$, because otherwise there would exist a set of constants, α_k, $k \in T$, not all zero, such that $\sum_{k\in T}\alpha_k e^{i(k,\theta)}f^{-1} = 0$ in $L_{2,f}$. This means that $|\sum_{k\in T}\alpha_k e^{i(k,\theta)}|^2 f^{-1}(\theta) = 0$ a.e. σ. But by 8.4.2 ([8]) $|\sum_{k\in T}\alpha_k e^{i(k,\theta)}|^2$ cannot vanish on a set of positive σ measure. Since also $f \neq 0$ a.e. we arrive at a contradiction. Thus the matrix $[c_{l-k}]_{l,k\in T}$ is invertible. Therefore the coefficients d_k, $d \in T$, are uniquely determined from (2). Let $\hat{x}_0 = x_0 - P_T x_0$. As noted in [14], $\|\hat{x}_0\|^2$ is the maximum of all $\|x\|^2$ such that $x \perp \mathcal{G}_T$ and $(x_0, x) = \|x\|^2$ ($\perp$ stands for the orthogonality sign). This is just the characterization of a projection in a Hilbert space. Hence from the isomorphism map between $\mathcal{G}(x)$ and $L_{2,f}$ we conclude that

$$\|\hat{x}_0\|^2 = \max_\varphi \int_{T_n} |\varphi|^2 f\, d\sigma, \qquad \varphi \in L_{2,f}, \tag{3}$$

where

$$\begin{aligned} \int_{T_n} e^{i(k,\theta)}\varphi f\, d\sigma &= 0, \qquad k \in T^c \\ \int_{T_n} |\varphi|^2 f\, d\sigma &= \int_{T_n} \varphi f\, d\sigma. \end{aligned} \tag{4}$$

From (4) we have $\varphi f = \sum_{k\in T}\alpha_k e^{-i(k,\theta)}$ [a.e. σ, where $\alpha_k = \int_{T_n} e^{i(k,\theta)}]\, \varphi f\, d\sigma$, $k \in T$.

But $\int_{T_n}|\varphi|^2 f\, d\sigma = \int_{T_n}|\varphi f|^2 f^{-1}\, d\sigma$. Therefore by (3)

$$\|\hat{x}_0\|^2 = \max \int_{T_n}|\Sigma_{k\in T}\alpha_k e^{-i(k,\theta)}|^2 f^{-1}\, d\sigma,$$

and by (4)

$$\alpha_0 = \|\hat{x}_0\|^2.$$

That is,

$$\|\hat{x}_0\|^2 = \left[\min \int_{T_n}|1 + \Sigma_{k\in T\setminus 0}\alpha_k e^{-i(k,\theta)}|^2 f^{-1}\, d\sigma\right]^{-1},$$

where max, min are taken over sequences of α_k, $k \in T \setminus 0$. Note that $\|\hat{x}_0\|^2 \neq 0$ because of the minimality condition [13]. But the min $\int_{T_n}|1 + \Sigma_{k\in T\setminus 0}\alpha_k e^{-i(k,\theta)}|^2 f^{-1}\, d\sigma$ is merely the squared distance of 1 from the finite dimensional subspace spanned by $e^{-i(k,\theta)}$, $k \in T \setminus 0$, in $L_{2,f^{-1}}$. So this minimum is obtained precisely when we have

$$1 + \Sigma_{k\in T\setminus 0}\alpha_k e^{-i(k,\theta)} \perp e^{-i(l,\theta)} \text{ in } L_{2,f^{-1}},\ l \in T \setminus 0.$$

That is, with $\alpha_0 = 1$, $\Sigma_{k\in T}\bar{\alpha}_k \int_{T_n} e^{-i(l-k,\theta)} f^{-1}\, d\sigma = 0$, $l \in T \setminus 0$ or $\Sigma_{k\in T}\bar{\alpha}_k c_{l-k} = 0$, $l \in T \setminus 0$ with $\alpha_0 = 1$ and $c_k = \int_{T_n} e^{-i(k,\theta)} f^{-1}\, d\sigma$. Then the last expression for $\|\hat{x}_0\|^2$ reduces to (1) and this completes the proof.

3. Remark. In [13] it is shown that

$$\|\hat{x}_0\|^2 = \left[\int_{T_n}\left|\Sigma_{k\in T}\frac{(\hat{x}_0, x_k)}{(\hat{x}_0, \hat{x}_0)} e^{-i(k,\theta)}\right|^2 f^{-1}\, d\sigma\right]^{-1}.$$

From the above discussion and the work in [13] it follows that the quantities $(\hat{x}_0, x_k)/(\hat{x}_0, \hat{x}_0)$ are precisely the unique solutions of the system (2). We also note that if the coefficients d_k, $k \in T$ are defined through the system $\Sigma_{k\in T} d_k c_k = 1$; $\Sigma_{k\in T} d_k c_{k-l} = 0$, $l \in T \setminus 0$, then $\sigma_T^2 = [\int_{T_n}|\Sigma_{k\in T} d_k e^{-i(k,\theta)}|^2 f^{-1}\, d\sigma]^{-1}$, which is more consistent with the work of several authors in the field.

To obtain an explicit expression for the linear interpolator we are forced to impose a stronger assumption on f.

4. Theorem. *Let x_k, $k \in Z^n$, be a weakly stationary stochastic process whose spectral distribution F is absolutely continuous with the spectral density f.*

(a) *If $f \in L_\infty$ and $f^{-1} \in L_2$, then $P_T x_0$ can be represented in the form*

$$P_T x_0 = \Sigma_{k\in T^c}\bar{\beta}_k x_k, \tag{5}$$

$$\beta_k = -\Sigma_{l\in T} d_l c_{k-l}, \qquad k \in T^c, \tag{6}$$

with $c_k = \int_{T_n} e^{-i(k,\theta)} f^{-1}\, d\sigma$, and where d_k, $k \in T$, are obtained uniquely from the system of equations:

$$\Sigma_{k\in T} d_k c_{l-k} = 1, \qquad l = 0 \tag{7}$$

$$\Sigma_{k\in T} d_k c_{l-k} = 0, \qquad l \in T \setminus 0.$$

(b) *Merely under the condition $f^{-1} \in L_1$, the interpolation error σ_T^2 may be computed by*

$$\sigma_T^2 = \left[\int_{T_n} |\Sigma_{k \in T} d_k e^{i(k,\theta)}|^2 f^{-1}\, d\sigma \right] = d_0. \tag{8}$$

(Compare formulas (8) and (1). For (8) we solve a system of r equations in r unknowns, r = cardinality of T, and then d_0 gives σ_T^2. In (1) we solve a system of $(r-1)$ equations in $r-1$ unknowns, but then it is necessary to perform one integration operation.)

PROOF. Similar to the first paragraph in proof of Theorem 2 we can show the uniqueness of the solution of system of equation (7).

Let β_k's be defined through (6), and the polynomial P be defined by $P(\theta) = \Sigma_{k \in T} \bar{d}_k e^{-i(k,\theta)}$. Since $f^{-1} \in L_2$ and since T has finitely many points, we have $\Sigma_{k \in T^c} |\beta_k|^2 < \infty$. Hence, β_k, $k \in T^c$, uniquely determine the function $\Sigma_{k \in T^c} \beta_k e^{-i(k,\theta)}$ in L_2. Moreover, from (6) and (7) it then follows that

$$f^{-1}(\theta)P(\theta) = 1 - \Sigma_{k \in T^c} \bar{\beta}_k e^{-i(k,\theta)} \quad \text{a.e. } \sigma \tag{9}$$

or equivalently

$$P(\theta) = f(\theta)\left[1 - \Sigma_{k \in T^c} \bar{\beta}_k e^{-i(k,\theta)}\right] \quad \text{a.e. } \sigma. \tag{10}$$

The infinite series $1 - \Sigma_{k \in T^c} \bar{\beta}_k e^{-i(k,\theta)}$ converges in L_2. Since f is in L_∞ then $1 - \Sigma_{k \in T^c} \bar{\beta}_k e^{-i(k,\theta)}$ converges in $L_{2,f}$. From the isomorphism between the time and spectral domains it follows that $x_0 - \Sigma_{k \in T^c} \bar{\beta}_k x_k$, which corresponds to $1 - \Sigma_{k \in T^c} \bar{\beta}_k e^{-i(k,\theta)}$, converges in $\mathcal{G}(x)$. Obviously, $\Sigma_{k \in T^c} \bar{\beta}_k x_k \in \mathcal{G}_T$. Moreover, for $l \in T^c$, using (10) we have

$$0 = \int_{T_n} \bar{P}(\theta) e^{-i(l,\theta)}\, d\sigma = \int_{T_n} \left[1 - \Sigma_{k \in T^c} \beta_k e^{i(k,\theta)}\right] f(\theta) e^{-i(l,\theta)}\, d\sigma.$$

Hence,

$$\int_{T_n} \left[1 - \Sigma_{k \in T^c} \bar{\beta}_k e^{-i(k,\theta)}\right] f(\theta) e^{i(l,\theta)}\, d\sigma = 0, \qquad l \in T^c.$$

This implies that $x_0 - \Sigma_{k \in T^c} \bar{\beta}_k x_k \perp x_l$ in $\mathcal{G}(x)$. Hence, $P_T x_0 = \Sigma_{k \in T^c} \bar{\beta}_k x_k$.

(b) The series $\Sigma_{k \in T^c} \bar{\beta}_k e^{-i(k,\theta)}$ may not converge under the mere assumption $f^{-1} \in L_1$, and hence formula (9) may not be correct. However, $f^{-1}\bar{P}$ is in L_1 and its kth Fourier coefficient vanishes for $k \in T \setminus 0$. This is because by (7) for $l \in T$ we have

$$\begin{aligned}
\int_{T_n} f^{-1} \bar{P} e^{-i(l,\theta)}\, d\sigma &= \Sigma_{k \in T} d_k \int_{T_n} f^{-1} e^{-i(k,\theta)} e^{-i(l,\theta)}\, d\sigma \\
&= \Sigma_{k \in T} d_k \int_{T_n} f^{-1} e^{-i(l-k,\theta)}\, d\sigma \\
&= \Sigma_{k \in T} d_k c_{l-k} = 1, l = 0 \\
&\qquad\qquad = 0, l \in T \setminus 0.
\end{aligned}$$

To indicate this fact, as it is customary, we will write

$$f^{-1}\bar{P} \simeq 1 - \Sigma_{k \in T^c} \beta_k e^{i(k,\theta)}. \tag{11}$$

Note that $f^{-1}\bar{P} \in L_{2,f}$. Moreover, $\int_{T_n} \bar{P} f^{-1} f e^{-i(l,\theta)}\, d\sigma = \Sigma_{k\in T} d_k \int_{T_n} e^{i(k-l,\theta)}\, d\sigma = 0$ if $l \in T^c$, i.e., $\bar{P} f^{-1} \perp e^{i(l,\theta)}$ in $L_{2,f}$ for $l \in T^c$ or equivalently

$$(12) \qquad Pf^{-1} \perp e^{-i(l,\theta)} \quad \text{in} \quad L_{2,f} \quad \text{for} \quad l \in T^c.$$

Let $\mathfrak{M}_T$ = the space spanned by $e^{-i(l,\theta)}$, $l \in T^c$, in $L_{2,f}$. From [13] we have that $\mathfrak{M}_T^{\perp} = \{Q/f,\ Q = \Sigma_{k\in T}\alpha_k e^{-i(k,\theta)}\}$. From this or relation (12) it follows that

$$(13) \qquad Pf^{-1} \in \mathfrak{M}_T^{\perp}.$$

We claim that $1 - Pf^{-1} \perp \mathfrak{M}_T^{\perp}$. For this it suffices to show that $1 - Pf^{-1} \perp Q/f$ in $L_{2,f}$ with $Q = \Sigma_{k\in T}\alpha_k e^{-i(k,\theta)}$. But

$$\int_{T_n} \overline{(1 - Pf^{-1})}\, f\left(\frac{Q}{f}\right) d\sigma = \int_{T_n} \overline{(1 - Pf^{-1})}\, Q\, d\sigma$$

$$= \Sigma_{k\in T}\alpha_k \int_{T_n} \left(1 - \bar{P} f^{-1}\right) e^{-i(k,\theta)}\, d\sigma = 0$$

by (11). Hence, $1 - Pf^{-1} \perp \mathfrak{M}_T^{\perp}$. Since $\mathfrak{M}_T$ is closed this implies

$$(14) \qquad 1 - Pf^{-1} \in \mathfrak{M}_T.$$

By (13), (14) and the isomorphism theorem between the time and spectral domains, we conclude that

$$\|\hat{x}_0\|^2 = \int_{T_n} \left(\frac{P}{f}\right) f \left(\frac{\bar{P}}{f}\right) d\sigma = \int_{T_n} \frac{|P|^2}{f}\, d\sigma$$

$$= \int_{T_n} \frac{\bar{P}}{f} P\, d\sigma = \Sigma_{k\in T}\bar{d}_k \int_{T_n} \frac{\bar{P}}{f} e^{-i(k,\theta)}\, d\sigma$$

$$= \bar{d}_0 = d_0,$$

where in the fifth step relation (11) is used. This completes the proof.

If in the above theorem α_k, $k \in T$ are defined $\Sigma_{k\in T} d_k c_k = 1$, $\Sigma_{k\in T} d_k c_{k-l}$ for $l \in T \setminus 0$, and if we set $\beta_k = -\Sigma_{l\in T} d_l c_{l-k}$ for $l \in T^c$, then $\sigma_T^2 = \int_{T_n} |\Sigma_{k\in T} d_k e^{-i(k,\theta)}|^2 f^{-1}\, d\sigma = d_0$ and $P_T x_0 = \Sigma_{k\in T^c}\beta_k x_k$, which conforms more closely to the work of several authors.

In case $T = \{0\}$ Theorem 4 takes a simple form which we state below.

5. Corollary. *With our earlier notations let T consist of only the element* 0.

(a) *if $f \in L_\infty$ and $f^{-1} \in L_2$ then*

$$(15) \qquad P_0 x_0 = -\sigma_0^2 \Sigma_{k\neq 0} \bar{c}_k x_k,$$

with

$$c_k = \int_{T_n} e^{-i(k,\theta)} f^{-1}\, d\theta.$$

(b) *If we merely assume that $f^{-1} \in L_1$ then*

$$(16) \qquad \sigma_0^2 = \left[\int_{T_n} f^{-1}\, d\sigma\right]^{-1}.$$

PROOF. $\beta_k = -\sum_{l \in T} d_l c_{k-l} = -d_0 c_k$, so $\bar{\beta}_k = -d_0 \bar{c}_k$. Moreover the system (7) reduces to $d_0 c_0 = 1$. Hence, $\sigma_0^2 = d_0 = \dfrac{1}{c_0} = [\int_{T_n} f^{-1}\, d\sigma]^{-1}$ and $\bar{\beta}_k = -\sigma_0^2 \bar{c}_k$.

6. Remark. We briefly remark on the history of formula (16) and its extensions. Formula (16) for the $n = 1$ case is due to Kolmogorov. Again for $n = 1$, but for q-variate processes of full rank the analogue of (16) was obtained by Masani [3]. For nonfull rank q-variate processes the extension of (16) is contained in [9] for $n = 1$; [11] for $1 \leqslant n < \infty$; [13] for q-variate processes over any locally compact abelian group. For the $n = 1$ case Yaglom [14] extended (16) to cover the nonminimal univariate processes. Also for the $n = 1$ case Rozanov [5], [6], [7] has studied the question of best linear interpolation problem. Rozanov obtains an expression for the functions in $L_{2,f}$ which corresponds to $P_T x_0$ and $x_0 - P_T x_0$. However, explicit results such as our (5) or (8) are not contained in his work.

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