Algorithms for polynomial spectral factorization and bounded-real balanced state space representations — Source link

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ORIGINAL ARTICLE

Algorithms for polynomial spectral factorization and bounded-real balanced state space representations

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Abstract We illustrate an algorithm that starting from the image representation of a strictly bounded-real system computes a minimal balanced state variable, from which a minimal balanced state realization is readily obtained. The algorithm stems from an iterative procedure to compute a storage function, based on a technique to solve a generalization of the Nevanlinna interpolation problem.

Keywords Nevanlinna interpolation problem · Model reduction · Balanced state space representation · Quadratic differential forms · Two-variable polynomial matrices

1 Introduction

Computing a balanced state representation for a system constitutes the first step of various model order reduction methods, see for example [5,12–14,16,19,21,22]. These model reduction procedures take as their starting point a given state-space

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representation; however, usually in engineering practice a state-space model is not given a priori, but it is *derived* from the equations describing the dynamics of the system. These are usually higher-order differential equations obtained from a tearing-and-zooming modeling procedure (see [35]), which may include algebraic constraints among the variables. Instrumental in the computation of a state-space model from such a higher-order model is the concept of *state map* introduced in [28] and further studied in [8,36]. A state map is a polynomial differential operator which, when applied to the variables of a system, induces a state variable: if w is the system variable, and $X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is the state map, then $x := X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w$ is a state variable, to which corresponds, for example an input-state-output representation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}x &= Ax + Bu \\ y = Cx + Du, &\quad w = \begin{bmatrix} u \\ y \end{bmatrix}.\end{aligned} \tag{1}$$

When considering behavior representations involving latent variables ℓ besides the external variables w (see Appendix A), it is possible (see section 7 of [28]) to define state maps also as acting on the full trajectories (w, ℓ) to produce state variables for the external behavior associated with the hybrid representation, i.e. $x = X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \begin{bmatrix} w \\ \ell \end{bmatrix}$.

Algorithms for computing a state map from the equations describing a system have been given in [28], where the problem of determining the matrices involved in a description (1) is also considered (on this issue see also [7,30]).

In this paper, we illustrate a procedure to compute *directly* from the higher-order equations describing a system a *balanced* state map, defined as one that induces an i/s/o representation (1) such that the minimal and the maximal solutions of the associated algebraic Riccati equation are diagonal and the inverse of each other. Our procedure for the computation of a balanced state map is based on an *iterative algorithm for the spectral factorization of a polynomial matrix*, which is of independent interest. This algorithm is based on recursive computations to solve a Nevanlinna-type interpolation problem associated with the spectral zeroes of the system and the associated directions (see [18,23,29] for applications in other contexts). Since there is a one–one correspondence between spectral factorizations and dissipation functions (see section 5 of [37]), and also a one–one correspondence between dissipation functions and storage functions, our algorithm leads in a straightforward manner to the computation of a storage function. Exploiting the close relation between storage functions corresponding to constant supply rates and state variables (see [33]), a balanced state map can then be obtained using standard linear algebra computations. From this balanced state map, a balanced state representation (1) is computed in a straightforward manner.

The results presented in this paper arise from previous work in different areas, most prominently behavioral system theory, quadratic differential forms (QDFs) and dissipativity theory; to make the paper more readable we have gathered the background material necessary in order to follow the exposition in an Appendix A, to which we frequently refer in the rest of the paper; Appendix A also contains a notation section. We state the problem in Sect. 2; in Sect. 3.1, we introduce the Pick matrix associated with a set of trajectories of a system; in Sect. 3.2, we illustrate the concept of Σ-unitary kernel representation. In Sect. 4, we describe our interpolation-based procedure for spectral factorization; we exploit this result in Sect. 5, where we state an algorithm to compute a balanced state map.

2 Problem statement

We are given a controllable behavior $\mathfrak{B}$ (see [25] for a definition) with external variable $w = \mathrm{col}(u, y)$ with u an input and y an output variable with, respectively, $\mathtt{u}$ and $\mathtt{y}$ components. $\mathfrak{B}$ is bounded-real, i.e. half-line dissipative with respect to the supply rate

$$Q_\Sigma(u, y) := \begin{bmatrix} u^\top & y^\top \end{bmatrix} \underbrace{\begin{bmatrix} I_{\mathtt{u}} & 0 \\ 0 & -I_{\mathtt{y}} \end{bmatrix}}_{=:\Sigma} \begin{bmatrix} u^\top & y^\top \end{bmatrix} = \|u\|^2 - \|y\|^2. \tag{2}$$

Note that since the matrix Σ in (2) is constant, it follows from Proposition 9 of Appendix A that if Q_Ψ is a storage function, then it is a quadratic function of the state, in the following sense. For every X inducing a state map for $\mathfrak{B}$, for example one acting on the external variables of the system, there exists a real symmetric matrix K such that $Q_\Psi(w) = \left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w\right)^\top K \left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w\right)$. Moreover, since the number $\sigma_+(\Sigma)$ of positive eigenvalues of the matrix Σ in (2) equals the number $\mathtt{u} = \mathtt{m}(\mathfrak{B})$ of input variables of the system, it follows from Proposition 12 of Appendix A that all the storage functions are positive, and consequently if X is minimal, then $K > 0$.

A balanced state map is defined as follows.

Definition 1 Let Σ be defined as in (2). Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ be Σ-dissipative on $\mathbb{R}^-$. A minimal state map $X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ induced by $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\bullet}[\xi]$ is *balanced* if the maximal and minimal storage functions for $\mathfrak{B}$ can be written as

$$\begin{aligned} \Psi_+(\zeta, \eta) &= X(\zeta)^\top \Delta X(\eta) \\ \Psi_-(\zeta, \eta) &= X(\zeta)^\top \Delta^{-1} X(\eta) \end{aligned}$$

for some diagonal matrix $\Delta \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$.

In Sect. 5, we prove that if X is a balanced state map then the corresponding i/s/o representation (1) is balanced in the classical sense, i.e. the minimal and the maximal solutions of the algebraic Riccati equation associated with the cost functional (2) are diagonal and the inverse of each other.

In the following without loss of generality we work with an observable image representation $\mathfrak{B} = \mathrm{im} M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, induced by a matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{u}}[\xi]$ such that rank $M(\lambda) = \mathtt{u}$ for all $\lambda \in \mathbb{C}$; moreover, we assume $M = \mathrm{col}(D, N)$, with $D \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ nonsingular, $N \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$, and ND^{-1} strictly proper. Note that in this case (see section 7 of [28]) it can be shown that state maps act on the latent variable ℓ. The problem we set ourselves to solve in the rest of this paper is to find a minimal balanced state map $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{u}}[\xi]$ and a corresponding minimal balanced i/s/o representation (1).

To solve this problem, we use an approach based on an algorithm for the spectral factorization of the polynomial matrix $M(-\xi)^\top \Sigma M(\xi)$ associated with the image representation of $\mathfrak{B}$ and the supply rate. A fundamental ingredient in our spectral factorization algorithm is the modeling of vector-exponential time series associated with the zeroes of $\det(M(-\xi)^\top \Sigma M(\xi))$ and the associated directions, a topic which we devote to the next section.

Remark 1 Several results presented in this paper hold also without any modification for any supply rate induced by a matrix $\Sigma = \Sigma^\top$ such that $\Sigma^2 = I$ (i.e. Σ represents an involution) and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. For simplicity of exposition and because of the importance of bounded-real systems, however, in the rest of this paper we only consider this special case.

3 Pick matrices and Σ-unitary kernel representations

The purpose of this section is to introduce the notion of Σ-unitary model of a finite set of vector-exponential trajectories; in order to do this, we need to define the Pick matrix associated with such a set, and to discuss some of its properties.

3.1 Pick matrices

Pick matrices were introduced in [24] in the context of interpolation problems; they also arise in many other areas of mathematics and engineering. It is unrealistic to try to summarize all their applications here; we refer to the monograph [1] for an exhaustive survey of rational interpolation theory, which is closer to the area considered here. The definition of Pick matrix is the following.

Definition 2 Let Σ be as in (2). Let $\lambda_i \in \mathbb{C}$, $\mathcal{V}_i \subset \mathbb{C}^{\mathtt{w}}$ be linear subspaces, $i = 1, \ldots, \mathtt{k}$. Denote $\dim(\mathcal{V}_i) =: \mathtt{n}_i$. Let $V_i \in \mathbb{C}^{\mathtt{w} \times \mathtt{n}_i}$ be full column rank matrices such that $\mathrm{im}(V_i) = \mathcal{V}_i$, $i = 1, \ldots, \mathtt{k}$. Denote $\sum_{i=1}^{\mathtt{k}} \mathtt{n}_i =: \mathtt{n}$.

The Pick matrix associated with $\{(\mathcal{V}_i, \lambda_i)\}_{i=1,\ldots,\mathtt{k}}$ is the $\mathtt{n} \times \mathtt{n}$ matrix

$$\left[T_{\{(\mathcal{V}_i,\lambda_i)\}_{i=1,\ldots,\mathtt{k}}}\right]_{i,j=1,\ldots,\mathtt{k}} := \left[\frac{V_i^* \Sigma V_j}{\overline{\lambda_i} + \lambda_j}\right]_{i,j=1,\ldots,\mathtt{k}}. \tag{3}$$

Remark 2 Note that the Pick matrix is Hermitian. Note also that since the definition of the basis matrices V_i for the subspaces $\mathcal{V}_i$ is not unique, $T_{\{(\mathcal{V}_i,\lambda_i)\}_{i=1,\ldots,\mathtt{k}}}$ also depends on the particular choice of the V_i, $i = 1, \ldots, \mathtt{k}$. However, since this nonuniqueness is of no consequence for our uses of Pick matrices, we will continue to talk about "the" Pick matrix, and denote it as $T_{\{(\mathcal{V}_i,\lambda_i)\}_{i=1,\ldots,\mathtt{k}}}$.

Pick matrices arise also when considering trajectories of a behavior; this point of view is especially important in this paper, and we now elaborate on it in detail. In order to do this we must first introduce the notion of Λ-set of a *para-Hermitian polynomial matrix*, i.e. a matrix $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $\Gamma(-\xi)^\top = \Gamma(\xi)$.

Definition 3 Let $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be para-Hermitian and such that $\det \Gamma(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$. A subset $S \subset \mathbb{C}$ is a Λ-set of Γ if:

1. there exists a factorization $c \cdot p(-\xi) \cdot p(\xi)$ of $\det(\Gamma(\xi))$ with $c \in \mathbb{R}$ and $p \in \mathbb{R}[\xi]$ such that the set of roots (counting multiplicities) of p equals S;
2. $\{\lambda \in S\} \Longrightarrow \{-\lambda \notin S\}$.

Remark 3 Notions analogous to that of Λ-set appear in the work of several authors concerning (J)-spectral factorization, the mixed solutions of the algebraic Riccati equation, etc. It is impossible to quote all of the relevant references here; we refer to the work of Gohberg, Lancaster, and their collaborators, and to that of Callier, Dym, and Faibusovich.

The determinant of a para-Hermitian matrix has always even degree, say $2\mathtt{n}$; it follows from Definition 3 that every Λ-set has exactly $\mathtt{n}$ elements. The number of distinct elements in a given Λ-set S is called the *effective cardinality* of S. Observe that if S is a Λ-set, then also the set

$$\bar{S} := \{\lambda \in \mathbb{C} \mid \det(\Gamma(\lambda)) = 0 \text{ and } \lambda \notin S\}$$

is a Λ-set; we call $\bar{S}$ the *complementary* Λ-set of S.

The following well-known result connects Λ-sets of a para-Hermitian matrix and spectral factorizations.

Proposition 1 *Let* $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *be a para-Hermitian matrix such that* $\det(\Gamma(i\omega)) > 0$ *for all* $\omega \in \mathbb{R}$. *Let* S *be a* Λ*-set for* Γ. *Then there exists* $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *such that* $\Gamma(\xi) = F(-\xi)^\top F(\xi)$ *and the set of roots of* $\det(F)$ *equals* S.

Proof See [3]. □

In the following, a matrix F as in Proposition 1 will be called a S-spectral factor of Γ.

The para-Hermitian matrices Γ we work with in this paper arise from an observable image representation of a behavior $\mathfrak{B}$ induced by a matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$, and the matrix Σ in (2), as $\Gamma(\xi) = M(-\xi)^\top \Sigma M(\xi)$. In the following, for simplicity of exposition we only consider *semisimple* matrices Γ, defined as follows.

Definition 4 Let $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. Γ is *semisimple* if for every $\lambda \in \mathbb{C}$, the multiplicity of λ as a root of $\det(\Gamma)$ equals the dimension of the subspace $\ker \Gamma(\lambda)$ of $\mathbb{C}^{\mathtt{w}}$.

Thus, Γ is semisimple if the algebraic multiplicity of $\lambda \in \mathbb{C}$, i.e. its multiplicity as a zero of $\det(\Gamma)$, equals its geometric multiplicity, i.e. the dimension of $\ker \Gamma(\lambda)$. Note that the property of semisimplicity is generic among para-Hermitian matrices.

In the semisimple case, the number of elements in every Λ-set for $M(-\xi)^\top \Sigma M(\xi)$ is directly related to the McMillan degree of $\mathfrak{B} = \operatorname{im} M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$.

Proposition 2 *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ *be strictly* Σ*-dissipative, with* Σ *defined as in* (2). *Let* $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}(\mathfrak{B})}[\xi]$ *induce an observable image representation of* $\mathfrak{B}$. *Assume that* $M(-\xi)^\top \Sigma M(\xi)$ *is semisimple. Then every* Λ*-set has* $\mathtt{n}(\mathfrak{B})$ *elements, counting multiplicities.*

Proof By Proposition 1 conclude that corresponding to each Λ-set S of $M(-\xi)^\top \Sigma$ $M(\xi)$ there exists a factorization $M(-\xi)^\top \Sigma M(\xi) = F(-\xi)^\top F(\xi)$ where S is the set consisting of all roots of $\det(F)$. Conclude from this that the cardinality of S equals $\frac{\deg(\det(M(-\xi)^\top \Sigma M(\xi)))}{2}$. Once we prove that $\deg(\det(M(-\xi)^\top \Sigma M(\xi))) = 2\mathtt{n}(\mathfrak{B})$ the claim of the Proposition follows. In order to do this, recall that $\mathfrak{B}$ is strictly dissipative, and consequently there exists $\epsilon > 0$ such that for every $\omega \in \mathbb{R}$

$$M(-i\omega)^\top \Sigma M(i\omega) \geq \epsilon M(-i\omega)^\top M(i\omega). \tag{4}$$

Now let $D \in \mathbb{R}^{\mathtt{m}(\mathfrak{B})\times\mathtt{m}(\mathfrak{B})}[\xi]$ be a nonsingular submatrix of M of maximal determinantal degree. Multiply the inequality (4) by $D(-i\omega)^{-\top}$ on the left and by $D(i\omega)^{-1}$ on the right, obtaining

$$D(-i\omega)^{-\top} M(-i\omega)^\top \Sigma M(i\omega) D(i\omega)^{-1} \geq \epsilon D(-i\omega)^{-\top} M(-i\omega)^\top M(i\omega) D(i\omega)^{-1}.$$

Take the limit for $\omega \to \infty$, and observe that since D is a maximal determinantal degree submatrix of M, $\lim_{\omega\to\infty} M(i\omega)D(i\omega)^{-1}$ has full column rank, and that consequently $\epsilon D(-i\omega)^{-\top} M(-i\omega)^\top M(i\omega) D(i\omega)^{-1}$ is nonsingular. Conclude from this that $D(-i\omega)^{-\top} M(-i\omega)^\top \Sigma M(i\omega) D(i\omega)^{-1}$ is also nonsingular. It is easily seen that this implies that $D(-\xi)^{-\top} M(-\xi)^\top \Sigma M(\xi) D(\xi)^{-1}$ has a proper inverse. It follows that $\deg(\det(M(-\xi)^\top \Sigma M(\xi)))$ equals $2\deg(\det(D))$. Since D is a maximal determinantal degree submatrix of M, it follows that $\deg(\det(D)) = \mathtt{n}(\mathfrak{B})$. This proves the claim. □

Now let Σ be as in (2), and let $\mathfrak{B} = \mathrm{im}\ M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ be an observable image representation of $\mathfrak{B}$, with $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}(\mathfrak{B})}[\xi]$. Assume that $\Gamma(\xi) := M(-\xi)^\top \Sigma M(\xi)$ is semisimple and that $\det(\Gamma)$ has no roots on the imaginary axis. Let $S = \{\lambda_i\}_{i=1,\ldots,\mathtt{n}}$ be a Λ-set of Γ with effective cardinality $\mathtt{k}$, and denote with $\lambda_1, \ldots, \lambda_{\mathtt{k}}$ the distinct elements of S. Denote with $\mathtt{n}_i$ the multiplicity of λ_i as a root of $\det(\Gamma)$. Let $V_i \in \mathbb{C}^{\mathtt{m}\times\mathtt{n}_i}$, $i = 1, \ldots, \mathtt{k}$, be full column rank matrices such that $\ker(\Gamma(\lambda_i)) = \mathrm{im}(V_i)$. Then we can associate a Pick matrix to the Λ-set S and the subspaces $\mathrm{im}\ M(\lambda_i)V_i$ as

$$T_{\{(\mathrm{im}\ M(\lambda_i)V_i,\lambda_i)\}_{i=1,\ldots,\mathtt{k}}} := \left[\frac{V_i^* M(\overline{\lambda_i})^\top \Sigma M(\lambda_j) V_j}{\overline{\lambda_i} + \lambda_j}\right]_{i,j=1,\ldots,\mathtt{k}}.$$

It follows from Proposition 2 that the Pick matrix defined in this way has dimension $\mathtt{n}(\mathfrak{B}) \times \mathtt{n}(\mathfrak{B})$. Similar considerations as those made in Remark 2 hold; we will not repeat them.

Remark 4 Assume that $M(-\xi)^\top \Sigma M(\xi)$ has no singularities on the imaginary axis, and consider the special important case of the two Λ-sets consisting of all open left half-plane, respectively, open right half-plane zeroes of $\det\ M(-\xi)^\top \Sigma M(\xi)$. Then the Pick matrix is the Gramian associated with the indefinite inner product $\langle w_1, w_2\rangle_{+,\Sigma} := \int_0^{+\infty} w_1(t)^\top \Sigma w_2(t)dt$, respectively, $\langle w_1, w_2\rangle_{-,\Sigma} := \int_{-\infty}^0 w_1(t)^\top \Sigma w_2(t)dt$ on the subspaces spanned by the trajectories in $\bigcup_{i=1,\ldots,\mathtt{k}} \mathrm{im}\ M(\lambda_i)V_i \exp_{\lambda_i}$, where if $\mathcal{V} \subset \mathbb{C}^{\mathtt{w}}$ is a linear subspace and $\lambda \in \mathbb{C}$, we define $\mathcal{V}\exp_\lambda := \{v \exp_\lambda \mid v \in \mathcal{V}\}$.

3.2 Σ-unitary kernel representations

Let Σ be as in (2); $R \in \mathbb{C}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is Σ-unitary if there exists $p \in \mathbb{C}[\xi]$, $p \neq 0$, such that

$$R\Sigma R^{\sim} = R^{\sim}\Sigma R = pp^{\sim}\Sigma. \tag{5}$$

Now let $\mathcal{V}_i \subset \mathbb{C}^{\mathtt{w}}$ be linear subspaces, $V_i \in \mathbb{C}^{\mathtt{w}}$ be full column rank matrices such that $\mathrm{im}(V_i) = \mathcal{V}_i$, and λ_i be distinct complex numbers not lying on the imaginary axis, $i = 1, \ldots, \mathtt{k}$. Let $\mathfrak{B} \subset \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{C}^{\mathtt{w}})$ be the autonomous complex behavior

$$\mathfrak{B} := \mathrm{span}\left(\bigcup_{i=1,\ldots,\mathtt{k}} \mathcal{V}_i \exp_{\lambda_i} \;\cup\; \mathcal{V}_i^{\perp_\Sigma} \exp_{-\overline{\lambda_i}}\right), \tag{6}$$

where $\mathcal{V}_i \exp_{\lambda_i} := \{v \exp_{\lambda_i} \mid v \in \mathcal{V}_i\}$, and $\mathcal{V}_i^{\perp_\Sigma} := \{v \in \mathbb{C}^{\mathtt{w}} \mid v^*\Sigma v' = 0 \text{for all } v' \in \mathcal{V}_i\}$. The following result shows that $\mathfrak{B}$ has a Σ-unitary kernel representation.

Theorem 1 *Assume that the Pick matrix* $T := \left[\frac{V_i^*\Sigma V_j}{\overline{\lambda_i}+\lambda_j}\right]_{i,j=1,\ldots,\mathtt{k}}$ *is nonsingular. Then* $\mathfrak{B}$ *defined in* (6) *admits a* Σ*-unitary kernel representation.*

Proof We prove the claim by induction on i.

For $i = 1$, consider the $\mathtt{w} \times \mathtt{w}$ polynomial matrix with complex coefficients

$$R_1(\xi) := (\xi + \overline{\lambda_1})I_{\mathtt{w}} - V_1\left(\frac{V_1^*\Sigma V_1}{\overline{\lambda_1} + \lambda_1}\right)^{-1} V_1^*\Sigma. \tag{7}$$

It is easily verified that $\ker(R_1(\frac{\mathrm{d}}{\mathrm{d}t})) \supseteq \mathcal{V}_1 \exp_{\lambda_1} \cup \mathcal{V}_1^{\perp} \exp_{-\overline{\lambda_1}}$. Since $\frac{V_1^*\Sigma V_1}{\overline{\lambda_1}+\lambda_1}$ is nonsingular, it holds that $\dim(\mathcal{V}_1) + \dim(\mathcal{V}_1^{\perp_\Sigma}) = \mathtt{w}$ and consequently

$$\dim(\mathrm{span}(\mathcal{V}_1 \exp_{\lambda_1} \cup \mathcal{V}_1^{\perp} \exp_{-\overline{\lambda_1}})) = \mathtt{w}.$$

Since $\deg(\det(R_1)) = \mathtt{w}$, it follows that $\ker(R_1(\frac{\mathrm{d}}{\mathrm{d}t}))$ is a kernel representation for span $(\mathcal{V}_1 \exp_{\lambda_1} \cup \mathcal{V}_1^{\perp} \exp_{-\overline{\lambda_1}})$. It is a matter of straightforward verification to see that R_1 is Σ-unitary. This proves the case $i = 1$.

We now assume that the claim holds for $i \leq \mathtt{k} - 1$; we prove it for $i = \mathtt{k}$. We first show that the Pick matrix associated with Σ, the subspaces im $R_1(\lambda_i)V_i$, and the numbers λ_i, $2 \leq i \leq \mathtt{k}$, is nonsingular. Observe first that

$$V_i' := R_1(\lambda_i)V_i = (\lambda_i + \overline{\lambda_1})V_i - V_1\left(T_{V_1,\lambda_1}\right)^{-1} V_1^{\top}\Sigma V_i$$

for $i = 2, \ldots, \mathtt{k}$. In order to prove that the Pick matrix T' associated with im $R_1(\lambda_i)V_i$, and the λ_is, $2 \leq i \leq \mathtt{k}$, is nonsingular, first verify that the $(i-1, j-1)$-block of T' is

$$\frac{V_i'^T \Sigma V_j'}{\lambda_i + \bar{\lambda_i}} = \frac{(\lambda_1 + \bar{\lambda_i})(\lambda_j + \overline{\lambda_1})}{\lambda_j + \bar{\lambda_i}} V_i^T \Sigma V_j - V_i^T \Sigma V_1 \left(T_{\{(V_1,\lambda_1)\}}\right)^{-1} V_1^\top \Sigma V_j$$

for $i, j = 1, \ldots, \mathtt{k}$. Now write $T = \begin{bmatrix} T_{\{(V_1,\lambda_1)\}} & b^\top \\ b & T'' \end{bmatrix}$, with $b := \text{col}\left(\frac{V_i^\top \Sigma V_1}{\overline{\lambda_i} + \lambda_1}\right)$. Let $\Delta := \text{diag}(\bar{\lambda_i} + \lambda_1)_{i=2,\ldots,\mathtt{k}}$. Observe that

$$\begin{bmatrix} I_{\dim(\mathcal{V}_1)} & 0 \\ -\Delta b \left(T_{\{(V_1,\lambda_1)\}}\right)^{-1} & \Delta \end{bmatrix} T \begin{bmatrix} I_{\dim(\mathcal{V}_1)} & -\left(T_{\{(V_1,\lambda_1)\}}\right)^{-1} b^\top \Delta \\ 0 & \Delta \end{bmatrix}$$
$$= \begin{bmatrix} T_{\{(V_1,\lambda_1)\}} & 0 \\ 0 & \Delta T'' \Delta - \Delta b \left(T_{\{(V_1,\lambda_1)\}}\right)^{-1} b^\top \Delta \end{bmatrix}.$$

It is a matter of straightforward verification to see that the (i, j)th block of $\Delta T'' \Delta - \Delta b \left(T_{\{(V_1,\lambda_1)\}}\right)^{-1} b^\top \Delta$ equals the $(i-1, j-1)$-block of T', $2 \leq i, j \leq \mathtt{k}$. This implies that the matrix T' is nonsingular, since T is nonsingular by assumption.

Conclude from the induction hypothesis and from the nonsingularity of T'' that there exists a Σ-unitary kernel representation for

$$\text{span}\left(\bigcup_{2 \leq i \leq \mathtt{k}} V_i' \exp_{\lambda_i} \cup V_i'^{\perp_\Sigma} \exp_{-\overline{\lambda_i}}\right).$$

Denote this kernel representation with R'. It is immediate to verify that a kernel representation for the linear subspace span$(\bigcup_{1 \leq i \leq \mathtt{k}} V_i \exp_{\lambda_i} \cup V_i^{\perp_\Sigma} \exp_{-\overline{\lambda_i}})$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{C}^{\mathtt{w}})$ is given by $R' R_1$. Observe that the Σ-unitariness of R_1 and of R' implies that $R_1 R'$ is also Σ-unitary. This concludes the proof of the claim. □

In general, the polynomial matrix R inducing a kernel representation of $\mathfrak{B}$ defined in (6) has complex coefficients; however, if the set $S := \{\lambda_i\}_{i=1,\ldots,\mathtt{k}}$ and the associated subspaces $\mathcal{V}_i$ and basis matrices V_i are such that

$$\begin{aligned} [\lambda_i \in S] &\Longrightarrow \left[\overline{\lambda_i} \in S\right] \\ [V_i \text{ is associated with } \lambda_i] &\Longrightarrow \left[\overline{V_i} \text{ is associated with } \overline{\lambda_i}\right], \end{aligned} \tag{8}$$

then there exists a kernel representation of $\mathfrak{B}$ with *real* coefficients, as we presently show; consequently, in that case p in (5) has real coefficients. Note for future reference that (8) holds if S is a Λ-set of a para-Hermitian matrix $\Gamma(\xi) = M(-\xi)^\top \Sigma M(\xi)$ with $M \in \mathbb{R}^{\mathtt{w} \times \mathtt{m}}[\xi]$; indeed, then det $\Gamma(\xi)$ is a polynomial with real coefficients and consequently if λ is a zero, then also $\bar{\lambda}$ is; moreover, the associated directions satisfy the second relation in (8) since $\overline{\text{im } M(\lambda_i)} = \text{im } \overline{M(\lambda_i)} = \text{im } M(\overline{\lambda_i})$ and $\overline{\ker \Gamma(\lambda_i))} = \ker \overline{\Gamma(\lambda_i)} = \ker \Gamma(\overline{\lambda_i})$.

Proposition 3 *Let $\mathcal{V}_i \subset \mathbb{C}^{\mathtt{w}}$ be linear subspaces, $V_i \in \mathbb{C}^{\mathtt{w}}$ be full column rank matrices such that* $\mathrm{im}(V_i) = \mathcal{V}_i$*, and λ_i be distinct complex numbers not lying on the imaginary axis, $i = 1, \ldots, \mathtt{k}$. Assume that properties* (8) *hold for $\{(\mathcal{V}_i, \lambda_i)\}_{i=1,\ldots,\mathtt{k}}$, and that the Pick matrix is nonsingular. Then the behavior $\mathfrak{B}$ defined in* (6) *has a kernel representation induced by a polynomial matrix R with real coefficients. Consequently, the polynomial p in* (5) *has real coefficients.*

Proof We examine the case of two pairs $(\mathcal{V}, \lambda)$ and $(\overline{\mathcal{V}}, \overline{\lambda})$, with associated basis matrices V and $\overline{V}$, respectively; the general case follows in a straightforward manner.

We construct a Σ-unitary kernel representation for $\mathfrak{B}$ as in the proof of Theorem 1, by first constructing a kernel representation for $\mathcal{V}\exp_\lambda$ and its orthogonal $\overline{\mathcal{V}}^{\perp_\Sigma} \exp_{-\lambda}$, and then one for $\overline{\mathcal{V}}\exp_{\overline{\lambda}}$ and its orthogonal $\mathcal{V}^{\perp_\Sigma}\exp_{-\overline{\lambda}}$, as in (7). Denote with R the polynomial matrix inducing the kernel representation computed in this way, and with $\overline{R}$ the matrix obtained from R by conjugating its coefficients. We now show that $\overline{R} = R$.

Observe that $R(\lambda)V = 0$ since $\ker R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \supset \mathcal{V}\exp_\lambda$; and that $R(\overline{\lambda})\overline{V} = 0$ since $\ker R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \supset \overline{\mathcal{V}}\exp_{\overline{\lambda}}$; analogous results hold for the orthogonal trajectories. Moreover, it can be verified by taking conjugates on the left of the double implication arrows that $[R(\lambda)V = 0] \iff \left[\overline{R}(\overline{\lambda})\overline{V} = 0\right]$, $\left[R(\overline{\lambda})\overline{V} = 0\right] \iff \left[\overline{R}(\lambda)V = 0\right]$, and moreover $\left[R(-\overline{\lambda})V^{\perp_\Sigma} = 0\right] \iff \left[\overline{R}(-\lambda)\overline{V}^{\perp_\Sigma} = 0\right]$, $\left[R(-\lambda)\overline{V}^{\perp_\Sigma} = 0\right] \iff \left[\overline{R}(-\overline{\lambda})V^{\perp_\Sigma} = 0\right]$. These implications show that also $\ker \overline{R}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ contains $\mathcal{V}\exp_\lambda$, $\overline{\mathcal{V}}\exp_{\overline{\lambda}}$, and their orthogonals. Since $\deg(\det(R)) = \deg(\det(\overline{R}))$, we conclude that $\ker R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \ker \overline{R}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$. Consequently, there exists a unimodular $U \in \mathbb{C}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $R = U\overline{R}$. We now prove that $U = I_{\mathtt{w}}$. From (7) we see that the highest order term of R is $\xi^2 I_{\mathtt{w}}$; equating the two highest order terms on both sides of $R = U\overline{R}$ yields that U is the identity. The claim on p follows straightforwardly. This proves the claim. $\square$

The concept of Σ-unitary kernel representation and the constructive proof of Theorem 1 play an important role in our algorithm for spectral factorization which we present in the next section.

4 An iterative algorithm for S-spectral factorization

In this section, we illustrate an iterative procedure for the computation of polynomial spectral factors with zeroes in a pre-specified Λ-set S (in the following "S-spectral factorization"). Our procedure is germane to that for J-spectral factorization presented in [23], which in turn is related to the work of Georgiou and collaborators (see [9–11]) in the context of rational spectral factorization and the solution to the Riccati equation; see also [2] for an interpolation approach to spectral factorization. The algorithm presented only involves operations on polynomial matrices, and does not require rational matrices or realizations of these. It generates a polynomial spectral factor in finitely many steps:

Theorem 2 *Let Σ be as in (2), and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ be strictly Σ-dissipative. Let $M = \mathrm{col}(D, N) \in \mathbb{R}^{\mathtt{w}\times\mathtt{u}}[\xi]$ induce an observable image representation of $\mathfrak{B}$. Assume that $M(-\xi)^\top \Sigma M(\xi)$ is semisimple, and that ND^{-1} is strictly proper. Let S be a Λ-set of $M(-\xi)^\top \Sigma M(\xi)$ with effective cardinality $\mathtt{k}$; denote with λ_i, $i = 1, \ldots, \mathtt{k}$, the distinct elements of S. Assume that the Pick matrix $\left[\frac{M(\overline{\lambda_i})^\top \Sigma M(\lambda_j)}{\overline{\lambda_i}+\lambda_j}\right]_{i,j=1,\ldots,\mathtt{k}}$ is nonsingular.*

Define $K_0(\xi) := M(\xi)$, and consider the following recursion for $i = 1, \ldots, \mathtt{k}$:

1. $V_i :=$ *full column-rank matrix such that* $\mathrm{im}(V_i) = \mathrm{im}(K_{i-1}(\lambda_i))$;
2. $R_i(\xi) := (\xi + \overline{\lambda_i})I_{\mathtt{w}} - V_i \left(\frac{V_i^* \Sigma V_i}{\overline{\lambda_i}+\lambda_i}\right)^{-1} V_i^* \Sigma$;
3. $K_i(\xi) := \frac{R_i(\xi)K_{i-1}(\xi)}{\xi - \lambda_i}$;

Then:

1. $\frac{V_i^* \Sigma V_i}{\overline{\lambda_i}+\lambda_i}$ *is nonsingular for* $i = 1, \ldots, \mathtt{k}$;
2. $K_i(\xi)$ *is a polynomial matrix for* $i = 0, \ldots, \mathtt{k}$;
3. $K_i(-\xi)^* \Sigma K_i(\xi) = K_{i-1}(-\xi)^* \Sigma K_{i-1}(\xi)$ *for* $i = 1, \ldots, \mathtt{k}$;
4. $K_{\mathtt{k}}(\xi) = \mathrm{col}(H(\xi), 0)$, *with* $H \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ *such that* $M(-\xi)^\top \Sigma M(\xi) = H(-\xi)^\top H(\xi)$ *and the set of zeroes of* $\det(H)$ *is the complementary Λ-set $\bar{S}$.*

Observe that the recursions 1–3 in the statement of Theorem 2 involve matrices with complex coefficients, since in general the S-spectral zeroes λ_i, $i = 1, \ldots, \mathtt{k}$ have nonzero imaginary part, and consequently the matrices R_i and K_i in general have complex coefficients. However, the S-spectral factor H obtained after $\mathtt{k}$ iterations is a polynomial matrix with *real* coefficients.

Proof It follows from Theorem 1 that $R_i(\xi)$ induces a Σ-unitary kernel representation for span $(V_i \exp_{\lambda_i} \cup V_i^{\perp_\Sigma} \exp_{-\overline{\lambda_i}})$.

1. The nonsingularity of the Pick matrix $\frac{V_i^* \Sigma V_i}{\overline{\lambda_i}+\lambda_i}$ at the ith stage follows from the same argument used in Theorem 1 for proving that the Pick matrix associated with Σ and $\{(\mathrm{im}\ R_1(\lambda_i)V_i, \lambda_i)\}_{i=2,\ldots,\mathtt{k}}$ is nonsingular.
2. In order to prove that K_i, $i = 0, \ldots, \mathtt{k}$, is polynomial, we use induction. The claim is true for $i = 0$. Now assume that the claim is true for i, and observe that since at the ith step $\ker R_i(\lambda_i) = \mathrm{im}\ K_{i-1}(\lambda_i)$, $(\xi - \lambda_i)$ must be a factor of $R_i(\xi)K_{i-1}(\xi)$. This implies that the matrix $\frac{R_i(\xi)K_{i-1}(\xi)}{\xi-\lambda_i} = K_i(\xi)$ is polynomial.
3. Using (2) in this Theorem and the Σ-unitariness of R_i, it is easy to prove that $K_i(-\xi)^* \Sigma K_i(\xi) = \frac{K_{i-1}(-\xi)^*}{-\xi-\overline{\lambda_i}} R_i(-\xi)^* \Sigma R_i(\xi) \frac{K_{i-1}(\xi)^*}{\xi-\lambda_i} = K_{i-1}(-\xi)^* \Sigma$ $K_{i-1}(\xi)$.
4. The proof of the last statement is based on the following lemma, which uses the notion of *greatest right divisor* (*GRD*) of a polynomial matrix. Let $K \in \mathbb{C}^{\bullet\times\bullet}[\xi]$ be a full column rank polynomial matrix. Then F is a GRD of K if there exists $K' \in \mathbb{C}^{\bullet\times\bullet}[\xi]$ such that $K = K'F$, and moreover whenever $F' \in \mathbb{C}^{\bullet\times\bullet}[\xi]$ is such that $K = K''F'$ for some $K'' \in \mathbb{C}^{\bullet\times\bullet}[\xi]$, then there exists $V \in \mathbb{C}^{\bullet\times\bullet}[\xi]$ such that $F = VF'$. All GRD are nonsingular, and they differ by unimodular left factors (see sect. 6.3 of [17]). □

Lemma 1 *Partition* $K_i := \mathrm{col}(G_i, F_i)$, $i = 1, \ldots, \mathtt{k}$, *with* $G_i \in \mathbb{C}^{\mathtt{u}\times\mathtt{u}}[\xi]$ *and* $F_i \in \mathbb{C}^{\mathtt{y}\times\mathtt{u}}[\xi]$. *Then:*

(i) G_i *is nonsingular for* $i = 1, \ldots, \mathtt{k}$.
(ii) $F_i G_i^{-1}$ *is strictly proper for* $i = 1, \ldots, \mathtt{k}$.
(iii) $\deg(\det(G_i)) = \deg(\det(G_{i+1}))$ *for* $i = 0, \ldots, \mathtt{k}-1$.
(iv) *Denote* $\gamma_i := \dim \ker M(\lambda_i)^* \Sigma M(-\overline{\lambda_i})$. *Let* $U_{-i} \in \mathbb{C}^{(\mathtt{u}+\mathtt{y})\times\gamma_i}$, $i = 1, \ldots, \mathtt{k}$ *with full column rank be such that* $\mathrm{im}\, U_{-i} = \ker M(\lambda_i)^* \Sigma M(-\overline{\lambda_i})$. *Then* $K_i(-\overline{\lambda_j})U_{-j} = 0$ *for all* $j \leq i$.
(v) *If* K_i *is partitioned as* $K_i =: \mathrm{col}(G_i', F_i')E_i$, *with* $\mathrm{col}(G_i', F_i') \in \mathbb{R}^{(\mathtt{u}+\mathtt{y})\times\mathtt{u}}[\xi]$ *right prime, and* $E_i \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ *a greatest right divisor of* K_i, *then* $\deg(\det(E_i)) = \gamma_1 + \cdots + \gamma_i$ *and* $\deg(\det(G_i')) = \mathtt{n}(\mathfrak{B}) - \deg(\det(E_i))$ *for* $i = 0, \ldots, \mathtt{k}$.
(vi) *Let* $\mathtt{n}(K_i)$ *be the McMillan degree of* $\mathrm{im}\, K_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$. *Then,*

$$\mathtt{n}(K_i) = \deg(\det(G_i')) = \mathtt{n}(\mathfrak{B}) - (\gamma_1 + \cdots + \gamma_i) \quad i = 1, \ldots, \mathtt{k}.$$

Proof (i) We prove the claim by induction. The statement is true by assumption for $j = 0$. Assume it true for $j < i$; we prove it for $j = i$. Partition the kernel representation at the ith iteration compatibly with the matrix Σ in (2) as

$$R_i := \begin{bmatrix} D_i^{\sim} & N_i^{\sim} \\ Q_i & -P_i \end{bmatrix}. \tag{9}$$

G_i can be expressed in terms of the block-elements of R_i and K_{i-1} as

$$G_i = \frac{D_i^{\sim} G_{i-1} + N_i^{\sim} F_{i-1}}{\xi - \lambda_i} = D_i^{\sim}[I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1}] G_{i-1} \frac{1}{\xi - \lambda_i}. \tag{10}$$

Observe that $D_i^{\sim}$ is nonsingular, and that G_{i-1} is nonsingular by inductive assumption. We now prove that $[I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1}]$ is also nonsingular. Note that from the Σ-unitariness of R_i it follows that $\|N_i D_i^{-1}\|_\infty < 1$; from statement (3) of Theorem 2 and the strict-dissipativeness of $\mathfrak{B}$ it also follows that $\|F_{i-1} G_{i-1}^{-1}\|_\infty < 1$. Consequently, $I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1}$ is nonsingular on the imaginary axis and consequently also as a polynomial matrix. Conclude from (10) that G_i is also nonsingular.

(ii) We prove the claim by induction. The statement is true by assumption for $j = 0$. Assume that it true for $j < i$; we prove it for $j = i$. Recall that

$$R_i \begin{bmatrix} G_{i-1} \\ F_{i-1} \end{bmatrix} = \begin{bmatrix} D_i^{\sim} & N_i^{\sim} \\ Q_i & -P_i \end{bmatrix} \begin{bmatrix} G_{i-1} \\ F_{i-1} \end{bmatrix} = \begin{bmatrix} D_i^{\sim} G_{i-1} + N_i^{\sim} F_{i-1} \\ Q_i G_{i-1} - P_i F_{i-1} \end{bmatrix}$$

From the expression for R_i in Step 2 of the recursion of the Theorem it follows that D_i and P_i are nonsingular, and that their determinant has degree $\mathtt{u}$, respectively, $\mathtt{y}$. Now observe that $F_i G_i^{-1} = (Q_i G_{i-1} - P_i F_{i-1})(D_i^{\sim} G_{i-1} + N_i^{\sim} F_{i-1})^{-1}$. Write

$$
\begin{aligned}
&(Q_i G_{i-1} - P_i F_{i-1})(D_i^{\sim} G_{i-1} + N_i^{\sim} F_{i-1})^{-1} \\
&\quad = P_i (P_i^{-1} Q_i - F_{i-1} G_{i-1}^{-1}) G_{i-1} G_{i-1}^{-1} (I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1})^{-1} (D_i^{\sim})^{-1} \\
&\quad = P_i (P_i^{-1} Q_i - F_{i-1} G_{i-1}^{-1})(I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1})^{-1} (D_i^{\sim})^{-1}.
\end{aligned}
$$

$D_i^{\sim}$ is column proper, and consequently $(D^{\sim})^{-1}$ is strictly proper; moreover, $(I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1})^{-1}$ is bi-proper. Conclude that

$$
(I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1})^{-1} (D_i^{\sim})^{-1}
$$

is a strictly proper rational function. It follows from the expression for R_i that $P_i^{-1} Q_i$ is strictly proper; moreover, $F_{i-1} G_{i-1}^{-1}$ is strictly proper by induction hypothesis. Consequently, $P_i^{-1} Q_i - F_{i-1} G_{i-1}^{-1}$ is a matrix of strictly proper rational functions. Conclude that $(P_i^{-1} Q_i - F_{i-1} G_{i-1}^{-1})(I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1})^{-1} (D_i^{\sim})^{-1}$ is also strictly proper. Since every entry of $P_i(\xi)$ has degree at most one, the claim is proved.

(iii) From (10) it follows that $\frac{\det(G_i)}{\det(G_{i-1})} = \frac{\det(D_i^{\sim})}{\det((\xi - \lambda_i) I_{\mathrm{u}})} \det([I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1}])$. It is easy to verify that $\deg(\det(D_i^{\sim})) = \mathtt{u} = \deg(\det((\xi + \overline{\lambda_i}) I_{\mathrm{u}})$, and consequently $\frac{\det(D_{i-1}^{\sim})}{\det((\xi - \lambda_i) I_{\mathrm{u}})}$ is bi-proper. The same argument used in proving (ii) yields that $I + (N_i D_i^{-1})^{\sim} F_{i-1} G_{i-1}^{-1}$ is a matrix of bi-proper rational functions. Conclude that $\frac{\det(G_i)}{\det(G_{i-1})}$ is bi-proper. This concludes the proof of $\deg(\det(G_i)) = \deg(\det(G_{i-1}))$.

(iv) We proceed by induction. For $i = 1$, since im $U_{-1} = \ker M(\lambda_1)^* \Sigma M(-\overline{\lambda_1})$, and since $M(\lambda_1)^* \Sigma M(-\overline{\lambda_1})$ is Hermitian, it follows that $U_{-1}^* M(-\overline{\lambda_1})^* \Sigma M(\lambda_1) = 0$. Consequently, $\mathrm{im}(M(-\overline{\lambda_1}) U_{-1}) \subseteq (\mathrm{im} V_1)^{\perp_\Sigma}$. Now recall that $\ker R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ contains $(\mathrm{im} V_1)^{\perp_\Sigma} \exp_{-\overline{\lambda_1}}$, and consequently, $K_1(-\overline{\lambda_1}) U_{-1} = \frac{1}{-\overline{\lambda_1} - \lambda_1} R_1(-\overline{\lambda_1}) M(-\overline{\lambda_1}) U_{-1} = 0$. Next, we prove the induction step. Again note that $U_{-i}^* M(-\overline{\lambda_i})^* \Sigma M(\lambda_i) = 0$. Use statement (3) to obtain $U_{-i}^* M(-\overline{\lambda_i})^* \Sigma M(\lambda_i) = U_{-i}^* K_{i-1}(-\overline{\lambda_i})^* \Sigma K_{i-1}(\lambda_i) = 0$, from which conclude $\mathrm{im}(K_{i-1}(-\overline{\lambda_i}) U_{-i}) \subseteq (\mathrm{im} V_i)^{\perp_\Sigma}$. Recall that $\ker R_{i-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ contains $(\mathrm{im} V_i)^{\perp_\Sigma} \exp_{-\overline{\lambda_i}}$; consequently, $K_i(-\overline{\lambda_i}) U_{-i} = \frac{1}{-\overline{\lambda_i} - \lambda_i} R_i(-\overline{\lambda_i}) K_{i-1}(-\overline{\lambda_i}) U_{-i} = 0$.

(v) The statement $\deg(\det(E_i)) = \gamma_1 + \cdots + \gamma_i$ is an immediate consequence of (iv), of the observability of M, and of the fact that the only zeroes of $\det(R_i)$ are in λ_i and $-\overline{\lambda_i}$. In order to prove $\deg(\det(G_i')) = \mathtt{n} - \deg(\det(E_i))$, we use item (iii) in this Lemma to conclude that $\mathtt{n}(\mathfrak{B}) = \deg \det(G_0) = \deg(\det(G_i)) = \deg(\det(G_i' E_i)) = \deg(\det(G_i')) + \deg(\det(E_i))$. This yields the claim *(v)*.

(vi) Follows from Propositions 3.5.1 and 3.3.5 in [27]. □

Statement (iv) of Lemma 1 has two important consequences. First, γ_i elements of the complementary Λ-set $\bar{S}$ "accumulate" at every iteration as singularities of every greatest right factor of K_i. Secondly, that $\deg(\det(G_{\mathtt{k}}')) = 0$, i.e. that $G_{\mathtt{k}}'$ is unimodular. Since $F_{\mathtt{k}} G_{\mathtt{k}}^{-1}$ is strictly proper by item (i) of Lemma 1, it follows that $F_{\mathtt{k}} = 0_{\mathtt{y} \times \mathtt{u}}$. This proves that $K_{\mathtt{k}} = \mathrm{col}(H, 0)$, and together with $K_i(-\xi)^* \Sigma K_i(\xi) = M(-\xi)^* \Sigma M(\xi)$ for $i = 1, \ldots, \mathtt{k}$, implies that $K_{\mathtt{k}}(-\xi)^* \Sigma K_k(\xi) = G_{\mathtt{k}}(-\xi)^* G_{\mathtt{k}}(\xi) = M(-\xi)^* \Sigma$

$M(\xi)$. Since $G_{\mathrm{k}} = G'_k E_{\mathrm{k}}$ with G'_{k} unimodular and E_{k} having all its zeroes in $\bar{S}$, G_{k} is a $\bar{S}$-spectral factor; we now show it is a real polynomial matrix. Now $\mathrm{col}(G_{\mathrm{k}}, 0) = K_{\mathrm{k}}(\xi) = R_{\mathrm{k}}(\xi) \cdots R_1(\xi) M(\xi) =: R(\xi)M(\xi)$, with R a kernel representation of the trajectories in the span of the subspaces $\mathrm{im} M(\lambda_i)$ and their orthogonals, $i = 1, \ldots, \mathrm{k}$. Argue as in Proposition 3 that there exists a kernel representation for these trajectories induced by a real polynomial matrix R'. R differs from R' by a unimodular left factor, which again using the argument of Proposition 3 is shown to be the identity. □

5 An off-line algorithm for the computation of balanced state maps

It follows from the one–one correspondence between storage functions and dissipation rates stated in Proposition 8 of Appendix A that once a $\bar{S}$-spectral factor H is obtained following the iterations 1–3 in Theorem 2, the storage function corresponding to $\bar{S}$ (in the following called the $\overline{S}$-storage function) can be computed as:

$$\Psi_{\bar{S}}(\zeta, \eta) = \frac{M(\zeta)^{\top} \Sigma M(\eta) - H(\zeta)^{\top} H(\eta)}{\zeta + \eta}. \tag{11}$$

We now show how to compute a *minimal diagonalizing state map* for $\Psi_{\overline{S}}(\zeta, \eta)$, i.e. a polynomial matrix $X \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}) \times \mathrm{u}}[\xi]$ such that

1. $X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is a minimal state map acting on the latent variable of $\mathfrak{B} = \mathrm{im}\ M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$;
2. $\Psi_{\overline{S}}(\zeta, \eta) = X(\zeta)^{\top} X(\eta)$.

Note that minimal diagonalizing state maps are not unique: if $X(\xi)$ is one such matrix then $TX(\xi)$ also is, for every unitary matrix T.

Computing a minimal diagonalizing state map can be accomplished by factorization of the coefficient matrix of $\Psi_{\bar{S}}(\zeta, \eta)$, as the following result shows.

Proposition 4 *Let $\mathfrak{B} = \mathrm{im}\ M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ be Σ-strictly dissipative on $\mathbb{R}^{-}$, with Σ defined as in (2); assume $M \in \mathbb{R}^{\mathrm{w} \times \mathrm{u}}[\xi]$ observable. Let S be any Λ-set for $M(-\xi)^{\top} \Sigma M(\xi)$. Then:*

1. *The coefficient matrix $\mathrm{mat}(\Psi_{\overline{S}})$ of $\Psi_{\overline{S}}$ has rank $\mathrm{n}(\mathfrak{B})$, the McMillan degree of $\mathfrak{B}$;*
2. *Let $\tilde{X} \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}) \times \infty}$ be such that $\mathrm{mat}(\Psi_{\overline{S}}) = \tilde{X}^{\top} \tilde{X}$; then the polynomial matrix $X(\xi) := \tilde{X} \mathrm{col}(I_{\mathrm{u}}, I_{\mathrm{u}}\xi, \ldots)$ induces a minimal diagonalizing state map for $\mathfrak{B}$.*

Proof (1) It follows from Proposition 9 of Appendix A that for every state map $X \in \mathbb{R}^{\bullet \times \mathrm{u}}[\xi]$ there exists $K = K^{\top} \in \mathbb{R}^{\bullet \times \bullet}$ such that $\Psi_{\overline{S}}(\zeta, \eta) = X(\zeta)^{\top} K X(\eta)$. Now let X be minimal, and use the fact that in the bounded real case $\sigma_{+}(\Sigma) = \mathrm{m}(\mathfrak{B}) = \mathrm{u}$ and Proposition 12 to conclude that $K > 0$. From $\Psi_{\overline{S}}(\zeta, \eta) = X(\zeta)^{\top} K X(\eta)$ now follows that $\mathrm{mat}(\Psi_{\overline{S}}) = \tilde{X}^{\top} K \tilde{X}$, with $\tilde{X} \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}) \times \infty}$ the coefficient matrix of $X(\xi)$. From this factorization and the positive-definiteness of K the claim follows immediately.

(2) Let $X' \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}) \times \mathrm{u}}[\xi]$ be a minimal state map, and write $\Psi_{\overline{S}}(\zeta, \eta) = X'(\zeta)^{\top} K X'(\eta)$ for some $K = K^{\top} \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}) \times \mathrm{n}(\mathfrak{B})}$. Evidently, $X''(\xi) := K^{\frac{1}{2}} X'(\xi)$ is such that $\Psi_{\overline{S}}(\zeta, \eta) = X''(\zeta)^{\top} X''(\eta)$, and consequently $\mathrm{mat}(\Psi_{\overline{S}}) = \tilde{X''}^{\top} \tilde{X''} = \tilde{X}^{\top} \tilde{X}$, with $\tilde{X''}$,

$\tilde{X}$ the coefficient matrices of X'' and X, respectively. Any two factorizations $\tilde{X''}^{\top}\tilde{X''} = \tilde{X}^{\top}\tilde{X}$ of $\mathrm{mat}(\Psi_{\overline{S}})$ differ by a unitary matrix, i.e. there exists $T \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ such that $T^{\top}T = I_{\mathtt{n}(\mathfrak{B})}$ and $\tilde{X} = T\tilde{X''}$. This implies the minimality of $X(\xi)$; that $X(\xi)$ is diagonalizing is immediate. □

Example 1 Consider the system with one input ($\mathtt{m}(\mathfrak{B}) = \mathtt{u} = 1$) and 2 outputs described in observable image form by

$$M(\xi) = \begin{bmatrix} \frac{14}{3} + \frac{23}{3}\xi + \xi^2 \\ \frac{2\sqrt{2}}{3}(4+\xi) \\ \frac{2\sqrt{2}}{3}(-2+7\xi) \end{bmatrix};$$

$\mathrm{im}\, M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is strictly R_{-}-dissipative with respect to the supply rate induced by $\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The spectral zeroes of $M(-\xi)\Sigma M(\xi) = 4 - 5\xi^2 + \xi^4$ in the right half-plane are $\lambda_1 = 1$ and $\lambda_2 = 2$. A spectral factorization of $M(-\xi)\Sigma M(\xi)$ can be computed directly as $M(-\xi)\Sigma M(\xi) = (\xi+1)(\xi+2)(\xi-1)(\xi-2)$. Let the Λ-set $S := \{1, -2\}$; then the storage function for $\overline{S}$ is

$$\Psi_{\overline{S}}(\zeta,\eta) = \frac{2}{3}(64 + 10\eta + 10\zeta + 13\eta\zeta) = \begin{bmatrix} 1 & \zeta & \zeta^2 & \ldots \end{bmatrix} \begin{bmatrix} \frac{128}{3} & \frac{20}{3} & 0 & \ldots \\ \frac{20}{3} & \frac{26}{3} & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ \eta \\ \eta^2 \\ \vdots \end{bmatrix}.$$

It is readily verified that the coefficient matrix $\mathrm{mat}(\Psi_{\overline{S}})$ can be factored as

$$\begin{bmatrix} \frac{128}{3} & \frac{20}{3} & 0 & \ldots \\ \frac{20}{3} & \frac{26}{3} & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 8\sqrt{\frac{2}{3}} & \frac{5}{2\sqrt{6}} & 0 & \ldots \\ 0 & \frac{\sqrt{\frac{61}{2}}}{2} & 0 & \ldots \end{bmatrix}^{\top} \begin{bmatrix} 8\sqrt{\frac{2}{3}} & \frac{5}{2\sqrt{6}} & 0 & \ldots \\ 0 & \frac{\sqrt{\frac{61}{2}}}{2} & 0 & \ldots \end{bmatrix}.$$

$\mathrm{mat}(\Psi_{\overline{S}})$ has rank 2, equal to the McMillan degree of $\mathfrak{B}$. Now define

$$X(\xi) := \begin{bmatrix} 8\sqrt{\frac{2}{3}} & \frac{5}{2\sqrt{6}} & 0 & \ldots \\ 0 & \frac{\sqrt{\frac{61}{2}}}{2} & 0 & \ldots \end{bmatrix} \begin{bmatrix} 1 \\ \xi \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{32+5\xi}{2\sqrt{6}} \\ \frac{1}{2}\sqrt{\frac{61}{2}}\xi \end{bmatrix};$$

it is readily verified that $\Psi_{\overline{S}}(\zeta,\eta) = X(\zeta)^{\top}X(\eta)$. □

It follows from statement (2) of Proposition 4 that every symmetric factorization of the coefficient matrix of (11) yields a minimal diagonalizing state map X. We now show how to compute from such an X a balanced state map. In order to do this, we need to introduce the definition of V-matrix.

Definition 5 Let $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{u}}[\xi]$ be such that rank $M(\lambda) = \mathtt{u}$ for all $\lambda \in \mathbb{C}$. Let $\Sigma = \Sigma^\top$ be nonsingular. Assume that $M(-\xi)^\top \Sigma M(\xi)$ is semisimple and that $\mathfrak{B} = \mathrm{im}\ M(\frac{\mathrm{d}}{\mathrm{d}t})$ is strictly Σ-dissipative. Let $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{m}}[\xi]$ be a minimal state map for $\mathfrak{B}$ acting on the latent variable. Let $S = \{\lambda_i\}_{i=1,\ldots,\mathtt{n}}$ be a Λ-set of $M(-\xi)^\top \Sigma M(\xi)$; denote its effective cardinality with $\mathtt{k}$. Let $V_i \in \mathbb{C}^{\mathtt{w}\times\mathtt{n}_i}$, $i = 1,\ldots,\mathtt{k}$, be full column rank matrices such that ker $M(-\lambda_i)^\top \Sigma M(\lambda_i) = \mathrm{im}\ V_i$. The V-matrix associated with S and X is the $\mathtt{n}(\mathfrak{B}) \times \mathtt{n}(\mathfrak{B})$ matrix

$$V := \left[X(\lambda_1)V_1 \ \ldots \ X(\lambda_\mathtt{k})V_\mathtt{k}\right].$$

It can be shown (see Theorem 7.1 of of [32]) that the V-matrix associated with S and X is nonsingular.

The following result relates storage functions with Pick matrices and V-matrices.

Proposition 5 *Let $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{u}}[\xi]$ be such that* rank $M(\lambda) = \mathtt{u}$ *for all $\lambda \in \mathbb{C}$. Let $\Sigma = \Sigma^\top$ be nonsingular. Assume that $M(-\xi)^\top \Sigma M(\xi)$ is semisimple and that $\mathfrak{B} = \mathrm{im}\ M(\frac{\mathrm{d}}{\mathrm{d}t})$ is strictly Σ-dissipative. Let $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{m}}[\xi]$ be a minimal state map for $\mathfrak{B}$ acting on the latent variable. Let $K \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ be symmetric, and let $S = \{\lambda_i\}_{i=1,\ldots,\mathtt{n}}$ be a Λ-set for $M(-\xi)^\top \Sigma M(\xi)$; denote its effective cardinality with $\mathtt{k}$. Let $V_i \in \mathbb{C}^{\mathtt{w}\times\mathtt{n}_i}$, $i = 1,\ldots,\mathtt{k}$, be full column rank matrices such that* ker $M(-\lambda_i)^\top \Sigma M(\lambda_i) = \mathrm{im}\ V_i$. *The following two statements are equivalent:*

1. *$X^\top(\zeta)KX(\eta)$ is the storage function of $M(\zeta)^\top \Sigma M(\eta)$ corresponding to the dissipation rate $F(\zeta)^\top F(\eta)$, with F an S-spectral factor of $M(-\xi)^\top \Sigma M(\xi)$;*
2. *$K = (V^*)^{-1} T_{\{(\mathrm{im}\ V_i,\lambda_i)\}_{i=1,\ldots,\mathtt{k}}} V^{-1}$, with V the V-matrix of (S, X) and with $T_{\{(\mathrm{im}\ V_i,\lambda_i)\}_{i=1,\ldots,\mathtt{k}}}$ the Pick matrix of $\{(\mathrm{im}\ V_i, \lambda_i)\}_{i=1,\ldots,\mathtt{k}}$.*

Proof See Theorem 7.1 of [32]. □

If $\mathtt{u} = \mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, as happens for Σ defined in (2), then Proposition 12 of Appendix A and Proposition 5 imply the following result.

Proposition 6 *Let $\mathfrak{B} = \mathrm{im}\ M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, with M observable, and $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathfrak{B}$ is strictly Σ-dissipative, and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Assume that $M(-\xi)^\top \Sigma M(\xi)$ is semisimple. Let S be any Λ-set. Then the Pick matrix associated with S is positive definite.*

In the rest of this section we work with the Λ-set consisting of all right half-plane zeroes of $M(-\xi)^\top \Sigma M(\xi)$:

$$S_+ := \{\lambda \in \mathbb{C}_+ \mid \det\ M(-\lambda)\Sigma M(\lambda) = 0\}.$$

The complementary Λ-set $\overline{S_+}$ consists of all left half-plane zeroes of det $M(-\xi)^\top \Sigma M(\xi)$; we denote it with S_-. It follows from Proposition 11 of Appendix A that with this choice of spectral zeroes, the storage function $\Psi_{\overline{S_+}}$ equals Ψ_-, the smallest storage function for $\mathfrak{B}$ with respect to the supply rate Σ. Now consider the following algorithm.

Algorithm 1

Input: $\mathfrak{B} = \text{im } M(\frac{d}{dt})$ *strictly bounded-real, with* $M = \text{col}(D, N)$ *observable and such that* ND^{-1} *is strictly proper.*

Output: *A balanced state map* X_b *for* $\mathfrak{B}$ *in the sense of Definition 1.*

Step 1: *Compute* $S = \{\lambda_i\}_{i=1,\ldots,\mathtt{k}}$*, the* Λ*-set of* $M(-\xi)\Sigma M(\xi)$ *in* $\mathbb{C}_+$*; and full column rank matrices* U_i *such that* $\ker(M(-\lambda_i)^\top \Sigma M(\lambda_i)) = \text{im}(U_i)$*,* $i = 1, \ldots, \mathtt{k}$*;*

Step 2: *Use* $(1)-(3)$ *of Th. 2 to compute a Hurwitz spectral factor of* $M(-\xi)^\top \Sigma M(\xi)$*;*

Step 3: *Compute the smallest storage function* $\Psi_-(\zeta, \eta)$ *from (11);*

Step 4: *Factorize* $\text{mat}(\Psi_-) = \tilde{X}^\top \tilde{X}$ *and define* $X(\xi) := \tilde{X}\text{col}(I_\mathtt{u}, I_\mathtt{u}\xi, \ldots)$*;*

Step 5: *Compute* $Z_+ := [X(\lambda_1)U_1 \ldots X(\lambda_\mathtt{k})U_\mathtt{k}]$*;*

Step 6: *Compute the Pick matrix* $T_+ = \left[\frac{U_i^* M(\lambda_i)^* \Sigma M(\lambda_j) U_j}{\overline{\lambda_i} + \lambda_j}\right]_{i,j=1,\ldots,\mathtt{k}}$*;*

Step 7: *Compute* $K_+ = (Z_+^*)^{-1} T_+ Z_+^{-1}$*;*

Step 8: *Compute a singular value decomposition* $S_2 S_3 S_2^*$ *of* K_+^{-1}*;*

Step 9: *Define* $T := S_3^{-\frac{1}{4}} S_2^{-1}$*;*

Step 10: *Return* $X_b(\xi) = TX(\xi)$*.*

The following result holds.

Proposition 7 *The state map* X_b *returned by Algorithm* 1 *is balanced in the sense of Definition* 1*; moreover,* $\Psi_-(\zeta, \eta) = X_b(\zeta)^\top S_3^{-\frac{1}{2}} X_b(\eta)$ *and* $\Psi_+(\zeta, \eta) = X_b(\zeta)^\top S_3^{\frac{1}{2}} X_b(\eta)$*.*

Proof X is a minimal diagonalizing state map for the smallest storage function; consequently the smallest and largest storage functions for $\mathfrak{B}$ with respect to Σ can be written as $X(\zeta)^\top I_\mathrm{n} X(\eta)$ and $X(\zeta)^\top K_+ X(\eta)$, respectively (see Proposition 6), where K_+ is some symmetric matrix. Steps 4–6 of Algorithm 1 compute a diagonalizing congruence transformation matrix T between $K_- = I_\mathrm{n}$ and K_+ following the algorithm of [34]. Since $TX(\xi)$ is also a minimal state map, the claim of the Proposition follows. □

We now discuss how to obtain a realization from a balanced state map, and moreover we also show that this realization is balanced *in the classical sense*. Let $\mathfrak{B} = \text{im } M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, with $M = \text{col}(D, N)$ such that ND^{-1} is proper. It follows from the material in [28] that if X is a state map for $\text{im}\left(M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, then there exist matrices $A, B, C, G \in \mathbb{R}^{\bullet \times \bullet}$ such that

$$\begin{aligned} \xi X(\xi) &= AX(\xi) + BD(\xi) \\ N(\xi) &= CX(\xi) + GD(\xi). \end{aligned} \tag{12}$$

The computation of the matrices A, B, C, G can be efficiently performed, see [4].

Now assume that X is a balanced state map; we show that the realization associated with the matrices (A, B, C, G) satisfying (12) is balanced in the classical sense, i.e. that the minimal and maximal solutions to the Riccati equation are diagonal and one the inverse of the other.

Theorem 3 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$, and let $\Sigma = \Sigma^\top$ be nonsingular. Assume that $\mathfrak{B}$ is strictly Σ-dissipative. Let X be a minimal balanced state map for $\mathfrak{B}$. Let A, B, C, G be such that (12) holds. Then there exists a diagonal matrix Δ such that the minimal and maximal solutions of the ARE satisfy*

$$K_- = K_+^{-1} = \Delta$$

Proof Use (12) and rewrite the supply rate $Q_\Sigma(w) = \begin{bmatrix} u^\top & y^\top \end{bmatrix} \Sigma \begin{bmatrix} u \\ y \end{bmatrix}$ as a function of $x = X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ and $u = D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$, obtaining

$$\begin{bmatrix} \left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell\right)^\top & \left(D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell\right)^\top \end{bmatrix} \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \\ D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \end{bmatrix},$$

where $Q = Q^\top \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, $S \in \mathbb{R}^{\mathrm{u}\times\mathrm{n}}$, $R = R^\top \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}$ are suitable matrices. Now define $\Phi(\zeta,\eta) := \begin{bmatrix} X(\zeta)^\top & D(\zeta)^\top \end{bmatrix} \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} X(\eta) \\ D(\eta) \end{bmatrix}$; from the fact that $\mathfrak{B}$ is strictly Σ-dissipative, it follows that the behavior represented in image form by

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \ell \tag{13}$$

is strictly Σ'-dissipative, with Σ' defined by

$$\Sigma' := \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix}. \tag{14}$$

It is also easy to see that $\Psi(\zeta,\eta)$ induces a storage function for the behavior (13) and the supply rate defined by (14) if and only if it induces a storage function for $\mathfrak{B}$ with respect to the supply rate Σ. Multiply the equality

$$\Phi(-\xi,\xi) = X(-\xi)^\top Q X(\xi) + X(-\xi)^\top S^\top D(\xi) + D(-\xi)^\top S X(\xi) + D(-\xi)^\top R D(\xi)$$

by $D(-\xi)^{-\top}$ on the left and $D(\xi)^{-1}$ on the right, obtaining

$$D(-\xi)^{-\top}\Phi(-\xi,\xi)D(\xi)^{-1} = D(-\xi)^{-\top}X(-\xi)^{\top}QX(\xi)D(\xi)^{-1} \\ +D(-\xi)^{-\top}X(-\xi)^{\top}S^{\top} + SX(\xi)D(\xi)^{-1} + R.$$

Substitute $\xi = i\omega$ and use the strict dissipativity of $\mathfrak{B}$ to conclude that $R > 0$.

In Theorem 11 of [31] it has been shown that there exists a bijection between S-spectral factors (and corresponding storage functions) for the supply rate induced by Σ' defined in (14) on (13), and solutions of the algebraic Riccati equation associated with (A, B, C, G) of (12) and the cost-functional induced by (14). Recall that the minimal and maximal storage functions for $\mathfrak{B}$ with respect to Σ can be written as $\Psi_-(\zeta,\eta) = X(\zeta)^{\top}K_-X(\eta)$, $\Psi_+(\zeta,\eta) = X(\zeta)^{\top}K_+X(\eta)$ for some symmetric matrices K_- and K_+; from the previous discussion it follows that K_- and K_+ are the minimal, respectively, maximal, solution of the ARE. Now use the fact that X is balanced in the sense of Definition 1 in order to conclude that $K_- = K_+^{-1} = \Delta$. This implies that the realization associated with X is balanced in the classical sense. □

Example 2 We consider the system of Example 1 again. A minimal diagonalizing state map corresponding to the smallest storage function is $X(\xi) = \begin{bmatrix} \frac{58+4\xi}{\sqrt{87}} \\ \sqrt{\frac{130}{29}}\xi \end{bmatrix}$ (notice it is different from the one computed in Example 1). The V-matrix constructed as in Step 2 of Algorithm 1 is

$$Z_+ = \begin{bmatrix} \frac{62}{\sqrt{87}} & 22\sqrt{\frac{3}{29}} \\ \sqrt{\frac{130}{29}} & 2\sqrt{\frac{130}{29}} \end{bmatrix}.$$

The Pick matrix of Step 3 is $T_+ = \begin{bmatrix} \frac{200}{3} & 80 \\ 80 & 104 \end{bmatrix}$. We compute K_+ as in Step 4, obtaining $K_+ = \begin{bmatrix} 1.31034 & -0.06286 \\ -0.06286 & 2.35119 \end{bmatrix}$. An SVD of K_+ is obtained with

$$S_2 = \begin{bmatrix} -0.99819 & -0.06006 \\ -0.06006 & 0.99819 \end{bmatrix} \text{ and } S_3 = \begin{bmatrix} 0.76537 & 0 \\ 0 & 0.42463 \end{bmatrix}.$$

The transformation matrix T is $T = \begin{bmatrix} -1.0672 & -0.06422 \\ -0.07441 & 1.23655 \end{bmatrix}$ and the balanced state map is $X_b(\xi) = \begin{bmatrix} -6.63615 - 0.59363\xi \\ -0.46269 + 2.58618\xi \end{bmatrix}$.

Remark 5 Steps 1–4 of Algorithm 1 require first the computation of a Hurwitz spectral factorization, and then the factorization of the coefficient matrix of the minimal storage function, from which a diagonalizing state map is obtained. It can be shown that a minimal diagonalizing state map can also be obtained *iteratively and directly*, i.e. without computing the $\bar{S}$-spectral factor H explicitly. This brings the computational complexity down: while the factorization of Step 4 of Algorithm 1 requires $O(\mathtt{n}^3)$

operations, the modified algorithm computes the diagonalizing state map with $O(\mathtt{w}^2\mathtt{n})$ ops. Details can be found in section 5.5 of [15].

6 Conclusions

We have illustrated an interpolation-based approach to the computation of storage functions and of balanced state maps. Our main result is an algorithm that starting from the image representation of a half-line bounded-real system computes a balanced state map X, i.e. one such that the minimal and maximal storage functions can be written as $X(\zeta)^\top \Delta X(\eta)$ and $X(\zeta)^\top \Delta^{-1} X(\eta)$, respectively, with Δ a constant diagonal matrix. From this balanced state map an input-state-output realization balanced in the classical sense can be obtained in a straightforward way.

Appendix A: Notation and background material

A.1 Notation

The space of $\mathtt{n}$ dimensional real, respectively, complex, vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, respectively, $\mathbb{C}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real, respectively, complex, matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, respectively, $\mathbb{C}^{\mathtt{m}\times\mathtt{n}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; for example, $\mathbb{R}^{\bullet\times\mathtt{w}}$ denotes the set of matrices with $\mathtt{w}$ columns and with an arbitrary finite number of rows. Given two column vectors x and y, we denote with $\mathrm{col}(x, y)$ the vector obtained by stacking x over y; a similar convention holds for the stacking of matrices with the same number of columns. If $A \in \mathbb{C}^{\mathtt{p}\times\mathtt{m}}$, then $A^* \in \mathbb{C}^{\mathtt{m}\times\mathtt{p}}$ denotes its complex conjugate transpose. If $S = S^\top$, then we denote with $\sigma_+(S)$ the number of positive eigenvalues of S.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$. To a polynomial matrix $P(\xi) = \sum_{k\in\mathbb{Z}_+} P_k \xi^k$, we associate its *coefficient matrix*, defined as the block-column matrix $\mathrm{mat}(P) := \begin{bmatrix} P_0 & P_1 & \dots & P_N & \dots \end{bmatrix}$. Observe that $\mathrm{mat}(P)$ has only a finite number of nonzero entries; moreover, $P(\xi) = \mathrm{mat}(P)\mathrm{col}(I_{\mathtt{w}}, I_{\mathtt{w}}\xi, \dots)$. If $F \in \mathbb{C}^{\bullet\times\bullet}[\xi]$, we define $F^\sim(\xi) := F(-\xi)^*$.

We denote with $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. The set of infinitely differentiable functions with compact support is denoted with $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. The exponential function whose value at t is $e^{\lambda t}$ is denoted with $\exp_\lambda$.

A.2 Linear differential systems and their representations

A subspace $\mathfrak{B}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is a *linear differential behavior* if it consists of the solutions of a system of linear, constant-coefficient differential equations; equivalently, if there exists a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}.$$

We denote with $\mathfrak{L}^{\mathtt{w}}$ the set of linear differential systems with $\mathtt{w}$ external variables. In this paper, we also consider complex behaviors, i.e. subspaces of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{C}^{\mathtt{w}})$ described by polynomial matrices with complex coefficients; the definitions and results that follow can be adapted with obvious modifications to this case.

The representation $\mathfrak{B} = \ker R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ is called a *kernel representation* of $\mathfrak{B}$. If $\mathfrak{B}$ is controllable (for a definition, see [25]) then it also admits an *image representation*, i.e. $\mathfrak{B} = \operatorname{im} M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, where $M \in \mathbb{R}^{\mathtt{w} \times \mathtt{l}}[\xi]$; equivalently,

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \text{ s.t. } w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell \right\}. \tag{15}$$

The variable ℓ is called the *latent variable* of the system. In the following we denote the set of controllable behaviors with $\mathtt{w}$ external variables by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Given an image representation induced by a polynomial matrix M, there exists a permutation matrix Π such that $\Pi M = \operatorname{col}(D, N)$ with D nonsingular and ND^{-1} proper. The partition of the external variables associated with the permutation Π is then called an *input–output partition* for $\mathfrak{B} = \operatorname{im} M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ (see [25]).

The representation (15) is a special case of a *hybrid* or *latent variable representation*

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{l}}) \text{ s.t. } R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\} \tag{16}$$

where $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$, $M \in \mathbb{R}^{\bullet \times \mathtt{l}}[\xi]$. We call the behavior

$$\mathfrak{B}_{\mathrm{full}} = \left\{ (w, \ell) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{l}}) \mid R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\}$$

the *full behavior* of the hybrid representation.

A state system is a special type of latent variable system, in which the latent variable, typically denoted with x, satisfies the *axiom of state*, stated as follows. Given full trajectories (w_i, x_i), $i = 1, 2$, define their concatenation at zero as the trajectory

$$(w_1, x_1) \wedge (w_2, x_2)(t) := \begin{cases} (w_1, x_1)(t) & \text{for } t < 0 \\ (w_2, x_2)(t) & \text{for } t \geq 0 \end{cases}.$$

Then x is a state variable (and $\mathfrak{B}_{\mathrm{full}}$ a state system) if

$$\begin{aligned} &[(w_i, x_i) \in \mathfrak{B}_{\mathrm{full}}, i = 1, 2] \text{ and } [x_1, x_2 \text{continuous at} 0] \text{ and } [x_1(0) = x_2(0)] \\ &\Longrightarrow \left[(w_1, x_1) \wedge (w_2, x_2) \in \overline{\mathfrak{B}_{\mathrm{full}}}\right] \end{aligned}$$

with $\overline{\mathfrak{B}_{\mathrm{full}}}$ being the closure (in the topology of $\mathfrak{L}_1^{loc}$) of $\mathfrak{B}_{\mathrm{full}}$.

A state system is said to be *minimal* if the state variable has minimal number of components among all state representations that have the same manifest behavior.

In [28] it was shown that a state variable (and in particular, a minimal one) for $\mathfrak{B}$ can be obtained from the external- or full trajectories by applying to them a state map, defined as follows. Let $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ be such that the subspace $\left\{(w, X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w) \mid w \in \mathfrak{B}\right\}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{n}})$ is a state system; then $X(\frac{\mathrm{d}}{\mathrm{d}t})$ is called a state map for $\mathfrak{B}$, and $X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w$ is a state variable for $\mathfrak{B}$. In this paper, we consider state maps for systems in image form; in this case it can be shown (see [28]) that a state map can be chosen acting on the latent variable ℓ alone, and we consider state systems $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$, $x = X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$, with x a state variable. The definition of minimal state map follows in a straightforward manner. In [28], algorithms are stated to construct a state map from the equations describing the system.

There are a number of important *integer invariants* associated with a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$: the *input cardinality* denoted $\mathtt{m}(\mathfrak{B})$; the output cardinality, denoted $\mathtt{p}(\mathfrak{B})$; and the dimension of any minimal state variable for $\mathfrak{B}$, also called the *McMillan degree* of $\mathfrak{B}$, and denoted with $\mathtt{n}(\mathfrak{B})$. Observe that the number of external variables $\mathtt{w}$ equals $\mathtt{m}(\mathfrak{B}) + \mathtt{p}(\mathfrak{B})$. If $\mathtt{m}(\mathfrak{B}) = 0$, the behavior is said to be *autonomous*; it can be proved that in this case $\mathfrak{B}$ is finite-dimensional, and consists of vector polynomial-exponential trajectories, see [25]. Moreover, it can be shown that $\mathtt{m}(\mathfrak{B})$ is the number of columns of the matrix M in any *observable* image representation $\mathfrak{B} = \operatorname{im} M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, i.e. one such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. It can also be shown (see, for example sections 8–9 of [28]) that if $M = \operatorname{col}(D, N)$ with D nonsingular and of maximal determinantal degree, then $\deg(\det(D)) = \mathtt{n}(\mathfrak{B})$, the McMillan degree of $\mathfrak{B}$.

A.3 Quadratic differential forms

Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, written out in terms of its coefficient matrices $\Phi_{k,\ell}$ as the (finite) sum $\Phi(\zeta, \eta) = \sum_{k,\ell\in\mathbb{Z}_+} \Phi_{k,\ell}\zeta^k\eta^\ell$. It induces the map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by $Q_\Phi(w) = \sum_{k,\ell\in\mathbb{Z}_+}(\frac{d^k}{dt^k}w)^\top \Phi_{k,\ell}(\frac{d^\ell}{dt^\ell}w)$. This map is called the *quadratic differential form* (QDF) induced by Φ. When considering QDFs, we can without loss of generality assume that Φ is symmetric, i.e. $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. We denote the set of real symmetric $\mathtt{w}$-dimensional two-variable polynomial matrices with $\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$.

We associate with $\Phi(\zeta, \eta) = \sum_{k,\ell\in\mathbb{Z}_+} \Phi_{k,\ell}\zeta^k\eta^\ell \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ its *coefficient matrix*, defined as the infinite block-matrix:

$$\operatorname{mat}(\Phi) := \begin{bmatrix} \Phi_{0,0} & \cdots & \Phi_{0,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \Phi_{N,0} & \cdots & \Phi_{N,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Observe that $\operatorname{mat}(\Phi)$ has only a finite number of nonzero entries, and that $\Phi(\zeta, \eta) = \operatorname{col}(I_{\mathtt{w}}, I_{\mathtt{w}}\zeta, \ldots, I_{\mathtt{w}}\zeta^k, \ldots)^\top \operatorname{mat}(\Phi) \operatorname{col}(I_{\mathtt{w}}, I_{\mathtt{w}}\eta, \ldots, I_{\mathtt{w}}\eta^k, \ldots)$.

It is easy to see that Φ is symmetric if and only if $\mathrm{mat}(\Phi) = (\mathrm{mat}(\Phi))^\top$; in this case, we can factor $\mathrm{mat}(\Phi) = \tilde{M}^\top \Sigma_\Phi \tilde{M}$ with $\tilde{M}$ a matrix having a finite number of rows, full row rank, and an infinite number of columns; and Σ_Φ a signature matrix. This factorization leads to $\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma_\Phi M(\eta)$, where $M(\xi) := \tilde{M}\mathrm{col}(I_{\mathtt{w}}, I_{\mathtt{w}}\xi, \ldots)$ and is called a *canonical symmetric factorization of* Φ. A canonical symmetric factorization is not unique; they can all be obtained from a given one by replacing $M(\xi)$ with $UM(\xi)$, with $U \in \mathbb{R}^{\bullet\times\bullet}$ such that $U^\top \Sigma_\Phi U = \Sigma_\Phi$.

Some features of the calculus of QDFs which will be used in this paper are the following. The first one is that of *derivative of a QDF*. The functional $\frac{\mathrm{d}}{\mathrm{d}t} L_\Phi$ defined by $(\frac{\mathrm{d}}{\mathrm{d}t} Q_\Phi)(w) := \frac{\mathrm{d}}{\mathrm{d}t}(Q_\Phi(w))$ is again a QDF. It is easy to see that the two-variable polynomial matrix inducing it is $(\zeta + \eta)\Phi(\zeta, \eta)$.

Next, we introduce the notion of *integral of a QDF*. In order to make sure that the integral exists, we assume that the QDF acts on $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. The integral of Q_Φ maps $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ to $\mathbb{R}$ and is defined as $\int Q_\Phi(w) := \int_{-\infty}^{\infty} Q_\Phi(w)dt$.

Finally, we show how to associate a QDF with a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Let $\mathfrak{B} = \mathrm{im}\, M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, and let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Define $\Phi' \in \mathbb{R}_s^{\mathtt{l}\times\mathtt{l}}[\zeta, \eta]$ as

$$\Phi'(\zeta, \eta) := M^\top(\zeta)\Phi(\zeta, \eta)M(\eta);$$

if w and ℓ satisfy $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, then $Q_\Phi(w) = Q_{\Phi'}(\ell)$. The introduction of the two-variable matrix Φ allows to study the behavior Q_Φ along $\mathfrak{B}$ in terms of properties of the QDF $Q_{\Phi'}$ acting on free trajectories of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{l}})$.

A.4 Dissipative behaviors

Definition 6 Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. $\mathfrak{B}$ is called Σ-dissipative if $\int_{\mathbb{R}} Q_\Sigma(w)dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. $\mathfrak{B}$ is called strictly Σ-dissipative if there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}} Q_\Sigma(w)dt \geq \varepsilon \int_{\mathbb{R}} w^\top w dt$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. $\mathfrak{B}$ is called strictly Σ-dissipative on $\mathbb{R}_-$ if there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}_-} Q_\Sigma(w)dt \geq \varepsilon \int_{\mathbb{R}_-} w^\top w dt$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}_-, \mathbb{R}^{\mathtt{w}})$.

Note that (strict) half-line dissipativity implies (strict) dissipativity, which in turn implies dissipativity. Dissipativity is related to the concept of storage function.

Definition 7 Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Assume that $\mathfrak{B}$ is Σ-dissipative; then the QDF Q_Ψ is a *storage function* if for all $w \in \mathfrak{B}$ $\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leq Q_\Sigma(w)$. A QDF Q_Δ is a *dissipation function* if $Q_\Delta(w) \geq 0$ for all $w \in \mathfrak{B}$, and for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ it holds that $\int_{\mathbb{R}} Q_\Sigma(w) = \int_{\mathbb{R}} Q_\Delta(w)$.

The following proposition gives a characterization of dissipativity in term of storage and dissipation functions.

Proposition 8 *The following conditions are equivalent*

1. $\mathfrak{B}$ *is* Σ*-dissipative,*
2. $\mathfrak{B}$ *admits a storage function,*
3. $\mathfrak{B}$ *admits a dissipation function.*

Moreover, for every dissipation function Q_Δ there exists a unique storage function Q_Ψ, and for every storage function Q_Ψ there exists a unique dissipation function Q_Δ, such that for all $w \in \mathfrak{B}$ the dissipation equality $\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w)$ holds.

Proof See [37, Proposition 5.4]. □

Every storage function is a quadratic function of the state, in the following sense.

Proposition 9 *Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ be Σ-dissipative. Let Q_Ψ be a storage function. Then Q_Ψ is a state function, i.e. for every polynomial matrix X inducing a state map for $\mathfrak{B}$, there exists a real symmetric matrix K such that $Q_\Psi(w) = \left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w\right)^\top K\left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w\right)$.*

Proof See Theorem 5.5 of [37]. □

Now assume that $\mathfrak{B}$ is represented in image form $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ and that it is Σ-dissipative. Then it is easy to show that if Q_Ψ is a storage function, then for every $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{l}}[\xi]$ inducing a state map for $\mathfrak{B}$ acting on the latent variable, there exists a symmetric matrix K such that $Q_\Psi(w) = \left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell\right)^\top K\left(X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell\right)$ for every w and ℓ such that $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$.

In general, there exists an infinite number of storage functions; however, all of them lie between two extremal ones.

Proposition 10 *Let $\mathfrak{B}$ be Σ-dissipative; then there exist storage functions Ψ_- and Ψ_+ such that any storage function Ψ satisfies $Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$ along $\mathfrak{B}$.*

Proof See [37, Theorem 5.7]. □

The extremal storage functions Q_{Ψ_+} and Q_{Ψ_-} can be computed from anti-Hurwitz, respectively, Hurwitz spectral factorizations.

Proposition 11 *Let $\mathfrak{B} = \mathrm{im}\, M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ be Σ-dissipative, with M observable. Assume that $M(-i\omega)^\top \Sigma M(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Then the smallest and the largest storage functions Ψ_- and Ψ_+ of $\mathfrak{B}$ can be constructed as follows: let H and A be Hurwitz, respectively, anti-Hurwitz polynomial spectral factors of $M(-\xi)^\top \Sigma M(\xi)$. Then*

$$\Psi_+(\zeta,\eta) = \frac{M(\zeta)^\top \Sigma M(\eta) - A^T(\zeta)A(\eta)}{\zeta+\eta} \text{ and } \Psi_-(\zeta,\eta) = \frac{M(\zeta)^\top \Sigma M(\eta) - H^T(\zeta)H(\eta)}{\zeta+\eta}.$$

Proof See [37, Theorem 5.7]. □

If $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then the nonnegativity of all storage functions is equivalent with the half-line Σ-dissipativity of $\mathfrak{B}$, as the following result shows.

Proposition 12 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Let X be a minimal state map for $\mathfrak{B}$ acting on the external variable w. Assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then the following statements are equivalent.*

1. *$\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$;*
2. *there exists a nonnegative storage function of $\mathfrak{B}$;*
3. *all storage functions of $\mathfrak{B}$ are nonnegative;*
4. *there exists $K = K^\top > 0$ real such that $Q_K(w) := \left(X\left(\frac{d}{dt}\right)w\right)^\top K\left(X\left(\frac{d}{dt}\right)w\right)$ is a storage function of $\mathfrak{B}$;*
5. *there exists a storage function of $\mathfrak{B}$, and every real symmetric matrix $K > 0$ such that $Q_K(w) := \left(X\left(\frac{d}{dt}\right)w\right)^\top K\left(X\left(\frac{d}{dt}\right)w\right)$ is a storage function of $\mathfrak{B}$ satisfies $K > 0$.*

Proof See [37, Proposition 6.4]. □

Appendix B

It was pointed out by an anonymous reviewer that the results of Sect. 4, in particular Theorem 2, allow an alternative, somewhat shorter and more streamlined proof using results from interpolation theory and the theory of reproducing kernel Hilbert spaces. Essentially, as was indicated by the reviewer, the proof of Theorem 2 can be subdivided into a number of steps that lead to a spectral factorization of the polynomial matrix $M(-\xi)^T \Sigma M(\xi)$, even in the more general case that the transfer matrix ND^{-1} is proper (instead of strictly proper). Some of these steps can be obtained in a straightforward way from results published before in [6]. Important ingredients in the steps mentioned above are well-known results on reproducing kernel Hilbert spaces, the notion of Potapov factors (see also [26]) , and results from the theory on Schur and Nevanlinna interpolation problems.

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