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ALGORITHMS FOR TARGETING STRIKES IN A LINES-OF-COMMUNICATION (LOC) NETWORK

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PREFACE

This Memorandum presents algorithms forming the mathematical basis for lines-of-communication (LOC) models developed by the author and Captain M. J. Ondrasek, AFXOXS (Director of Operations, Systems Division, Hq USAF), while members of a special RAND-AFGOA (Operations Analysis, Hq USAF) joint Southeast Asia study group.

The Memorandum is a generalization of earlier work by the author [6]; it differs in that damage to an arc resulting from interdiction may be partial instead of total.

A model based on the material of this Memorandum is currently being applied by RAND and AFGOA; results will be reported to the Air Force as part of a follow-on effort to the joint group's activities. This model and the results of its application will be the subject of further Memoranda.

SUMMARY

This Memorandum presents two general algorithms for targeting strikes in a lines-of-communication (LOC) network. Each depicts a situation in which the LOCs are represented by a network of nodes and directed arcs. Nodes are points where two or more arcs intersect. Arcs, which can represent road, rail, or waterway segments, or points of transshipment between two modes of transport, are characterized by a beginning and ending node, upper and lower bounds on flow, and a cost that is a function of flow.

The situations depicted by the algorithms differ only in the form of the cost function on arcs. In the first case, arc costs are linear functions of flow; in the second case, arc costs are convex-piecewise-linear functions of flow with one break-point (i.e., two pieces).

It is assumed that the user of the LOCs is attempting to achieve a circulation flow at minimum cost. This is a very general goal that includes, as special cases, maximizing flow between two points, meeting a required flow between two points at minimum cost, and combinations of these two. The algorithms presented attempt to make such costs as large as possible over time.

The effect of targeting strikes is to increase the arc cost functions and to decrease arc capacities for a given period of time. Thus for the first case, where arc costs are linear, the slope of an arc's cost function increases and its capacity decreases as the number of strikes increase. In the second case, an arc's capacity and break-point (i.e., the point where the slope of the cost function changes) are both decreasing functions of the number of strikes, while the cost slopes before and after the break-point both increase with the number of strikes.

When a single strike is to be targeted against the network, the arc selected is one of maximum strike value, where an arc's strike value is defined as the cost to repair it plus the product of the time to repair it and the cost increase of a minimum-cost circulation flow resulting from it. The strike value of a particular arc may easily be found by finding a minimum-cost circulation flow and its cost, first when all arcs are assigned unstruck costs and capacities, and second when struck

values are assigned to arcs under consideration, and then performing the indicated arithmetic. Thus an arc of maximum strike value could be found by calculating the strike value of all arcs. The algorithms of this Memorandum avoid this enumeration by taking advantage of the fact that, when calculating the strike value of one arc, upper bounds for the strike values of other arcs can often be obtained that are low enough to preclude them from further consideration.

In both cases, when multiple strikes are targeted against the LOCs, optimal arc selection is approximated by repeated application of the one-strike algorithm.

CONTENTS

I. INTRODUCTION

This Memorandum presents two algorithms for targeting strikes against the arcs or links of a combat force's lines of communication (LOCs). Locations for the strikes are chosen on the basis of the degree to which they hinder the combat force or LOC user in achieving his goal. This goal has several possible formulations. Among these are maximizing flow between two points and meeting a given flow at minimum cost.

The LOCs are represented by a network of nodes and directed arcs. Arcs can represent road, rail, or waterway segments, or points of transshipment between two modes of transport. Each is characterized by a beginning node, an ending node, upper and lower bounds on flow, and a cost that is a function of flow. For the situation treated by the first algorithm, the cost function is linear; for that treated by the second algorithm, cost is convex-piecewise-linear with one break-point. Thus cost functions can be defined by unit-flow costs. In the first case the unit-flow cost is constant. In the second case it is double-valued; that is, each arc has one unit cost for flow up to a certain value, and another higher cost for flow units beyond that value. These flow costs could represent dollars, vehicle-hours, manpower units, or any other appropriate measure. Among quantities flow could represent are vehicles per hour or tons per hour.

Capacities (upper bounds on flow) and cost functions on arcs are assumed to depend on the number of strikes directed against them. For the most part, capacities decrease and costs increase with the number of strikes. Specifically, for the first case, capacities are nonincreasing and unit-flow costs are nondecreasing functions of the number of strikes. For the second case both total capacity and the point at which unit-flow cost changes are nonincreasing functions of strikes, while both unit cost functions are nondecreasing in the number of strikes.

In addition to flow bounds and costs, arcs have repair times and repair costs that are associated with targeting. Each of these is a nondecreasing function of the number of strikes. Thus the effect of striking an arc is to reduce its capacity and increase its unit-flow cost (or costs) for a period of time during which the user's cost of operation may increase and, in addition, to cause him to incur a repair cost.

It is assumed that the general goal of the user of the LOCs is to achieve a minimum-cost circulation flow. Section II will show that this includes, as special cases, maximizing flow from one node to another, meeting a required flow between two nodes at minimum cost, or combinations of these. The algorithms presented here attempt to allocate strikes against the LOC arcs in such a way that, over time, this minimum cost is as great as possible. Such allocation is done on the basis of immediate user cost, repair times, and repair cost. Specifically, if a single strike is to be targeted, it is directed against an arc of maximum strike value, where strike value is defined as the repair cost plus the resulting cost increase of a minimum-cost circulation flow multiplied by the repair time. While this arc could be selected by calculating the strike value of all arcs, this burden is avoided by taking advantage of the fact that in calculating the strike value for one arc, we can sometimes obtain upper bounds on the strike values of other arcs. We can then drop from consideration any arcs for which these upper bounds are less than the calculated strike values of other arcs since they could never be arcs of maximum strike value. For multiple strikes, no method for allocating strikes optimally, other than complete enumeration, is known. It is approximated here, however, by repeated application of the one-strike algorithm.

II. CIRCULATION FLOWS AND USER GOALS

Finding a minimum-cost circulation flow in a network is done efficiently by Fulkerson's "Out of Kilter" algorithm [3]. The formulation of such a problem is specified by a network consisting of points called nodes and of directed arcs, each of which joins two nodes. An arc originating at node i and terminating at node j is designated by the symbol (i,j). Each arc (i,j) has an upper bound, u_{ij}, and lower bound, ℓ_{ij}, on its flow, and a cost, c_{ij}, representing its cost per unit of flow. Letting x_{ij} represent the flow on arc (i,j), the problem may be formulated as follows. Find flows, $x_{ij} \geq 0$, that minimize

$$(1) \qquad \sum_{ij} c_{ij} x_{ij}$$

subject to

$$(2) \qquad \ell_{ij} \leq x_{ij} \leq u_{ij} \quad \text{all } i,j$$

$$(3) \qquad \sum_{j} x_{ji} - \sum_{j} x_{ij} = 0 \quad \text{all } i.$$

Expression (1) represents the total cost of the flow pattern, (2) states that the flow on each arc must be between its lower and upper bounds, and (3) states that the flow into any node must equal the flow out of it.

Note that an undirected arc between i and j with zero lower bound may be represented by two directed arcs, (i,j) and (j,i), and that if its cost is positive, no minimum-cost flow pattern will have strictly positive flow on both of them.

Several possible goals for a user of an LOC network can be formulated as minimum-cost circulation flows by proper data manipulation. The remainder of this section discusses some of them.

MAXIMIZING FLOW

This problem requires the maximization of flow from a node that shall be called the source, to one called the sink, subject to capacity or upper-bound constraints on the arcs. Denoting the source by S and the sink by T, the problem becomes one of finding $x_{ij} \geq 0$ that maximize the source-sink flow, v, subject to

(4) $$x_{ij} \leq u_{ij} \quad \text{all } i,j$$

(5) $$\sum_j x_{jS} - \sum_j x_{Sj} + v = 0$$

(6) $$\sum_j x_{ji} - \sum_j x_{ij} = 0 \quad \text{all } i \neq S,T$$

(7) $$\sum_j x_{jT} - \sum_j x_{Tj} - v = 0.$$

If one were to add an artificial arc (T,S) to the network defined by (4) through (7), then any source-sink flow on the original network could be represented by a circulation flow in the modified network. Specifically, (T,S) would be assigned a flow of v, and all other arcs flows identical to those in the original network. Maximizing flow would merely be maximizing x_{TS}. Thus the flow maximization problem can be converted to a minimum-cost circulation problem by constructing an artificial arc (T,S) with $c_{TS} = -1$ and $u_{TS} = M$, where M is sufficiently large.* All other arcs have zero cost, and all arcs have $\ell_{ij} = 0$.

MINIMIZING COST OF REQUIRED FLOW

This problem specifies that a required source-sink flow be met at minimum cost subject to capacity constraints on the arcs. If the constant r is the required flow, then this becomes a problem of finding $x_{ij} \geq 0$ that minimizes

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*$M > \sum_{i,j} u_{ij}$ is sufficient since (4) and (5) imply $v \leq \sum_{i,j} u_{ij}$.

(8) $$\sum_{ij} c_{ij} x_{ij}$$

subject to

(9) $$x_{ij} \leq u_{ij} \quad \text{all } i,j$$

(10) $$\sum_{j} x_{jS} - \sum_{j} x_{Sj} = -r$$

(11) $$\sum_{j} x_{ji} - \sum_{j} x_{ij} = 0 \text{ all } i \neq S,T$$

(12) $$\sum_{j} x_{jT} - \sum_{j} x_{Tj} = r$$

This can be converted to a minimum-cost circulation flow problem by adding an artificial arc (T,S) with $\ell_{TS} = u_{TS} = r$ and $c_{TS} = 0$, and, of course, setting all other $\ell_{ij} = 0$.

COMBINING REQUIRED AND MAXIMUM FLOW PROBLEMS

Several problems that are really combinations of the maximum-flow and minimum-cost-of-required-flow problems may be formulated as minimum-cost circulation flows. This section describes one in which the user actually does desire to meet a required flow, r, at minimum cost. It is not known, however, whether the network will support a flow of r; hence a more desirable formulation would be as follows. Find a minimum-cost flow pattern for a required source-sink flow, r, if such a flow is possible. If a flow of r cannot be generated, find a maximum flow.

This can be expressed as: find v, $x_{ij} \geq 0$, that minimizes

(13) $$\sum_{ij} c_{ij} x_{ij} - Mv,$$

where M is sufficiently large,* subject to $v \leq r$ and Expressions (4) through (7).

This converts to a minimum-cost circulation flow problem by assigning the artificial arc (T,S) with $u_{TS} = r$ and $c_{TS} = -M$, and then setting all lower bounds on arc flow to zero. After an appropriate pattern is found, one may wish to subtract out the cost incurred by the flow on the artificial arc, since this cost is merely an artifice to insure that either a maximum flow or a flow of r is obtained.

Note that if a flow of r cannot be obtained, and, in addition, there are several that maximize flow, this formulation will choose a least-cost one among these.

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*$M = \sum_{i,j} c_{ij} u_{ij}$ is sufficiently large.

III. ALGORITHMS FOR LINEAR COSTS

When arc costs are strictly linear functions of flow, strikes are allocated against them on the basis of upper and lower flow bounds, unit-flow costs, repair times, and repair costs. All of these except the lower flow bounds vary with the number of strikes. Symbolically, these inputs are written as follows:

(i,j)	designates the arc originating at i and terminating at j.
ℓ_{ij}	lower bound on arc flow for (i,j).
u_{ij}^{s}	upper bound on arc flow for arc (i,j) if s strikes are directed against it.
c_{ij}^{s}	cost per unit-flow for arc (i,j) if s strikes are directed against it.
t_{ij}^{s}	repair time for arc (i,j) if s strikes are directed against it.
r_{ij}^{s}	repair cost for arc (i,j) if s strikes are directed against it.

Parameters with zero superscripts are essentially "unstruck" parameters and normally all t_{ij}^{0} and r_{ij}^{0} would be equal to zero. The u_{ij}^{s} would be nonincreasing functions and the c_{ij}^{s}, t_{ij}^{s}, and r_{ij}^{s} nondecreasing functions of s. Otherwise, there are no restrictions on these quantities.

There are several possible relationships for the manner in which these parameters vary with the number of strikes. One can be derived by assuming all strikes allocated against a particular arc (i,j) are directed against the same target and each has an identical and independent probability p_{ij} of successfully destroying it. For this situation, real arc upper flow bounds, unit-flow costs, repair times, and repair costs could take on one of two values, depending on whether the arc was interdicted or, equivalently, the target successfully destroyed.

The u_{ij}^s, c_{ij}^s, t_{ij}^s, and r_{ij}^s would all be expected values depending on the probability of interdiction. Letting the superscripts INT and UNINT refer to "interdicted" and uninterdicted" values, these expected values would be:

$$u_{ij}^s = (1 - (1 - p_{ij})^s)u_{ij}^{INT} + (1 - p_{ij})^s u_{ij}^{UNINT}$$
$$c_{ij}^s = (1 - (1 - p_{ij})^s)c_{ij}^{INT} + (1 - p_{ij})^s c_{ij}^{UNINT}$$
$$t_{ij}^s = (1 - (1 - p_{ij})^s)t_{ij}^{INT}$$
$$r_{ij}^s = (1 - (1 - p_{ij})^s)r_{ij}^{INT}$$

For all relationships, however, allocation of strikes is done by repeated use of a one-strike selection algorithm. The one-strike algorithm selects for targeting the arc that maximizes repair cost plus the product of the repair time and the cost increase of a minimum-cost circulation flow. The remainder of this section is devoted to the development of the one-strike algorithm, the extension to its use for multiple strikes, and matching the inputs to conform to the user goals discussed in Sec. II.

ONE-STRIKE ALGORITHM

When only one strike is to be allocated against the LOCs, arcs can, of course, be struck only zero or one time. Thus, for simplification and for other reasons that will become clear in the next section, superscripts for parameters dependent on the number of strikes will be eliminated when referring to the one-strike algorithm and a prime will be used to denote struck values. These are now written as follows:

u_{ij} upper bound on arc flow for (i,j) if unstruck,

u'_{ij} upper bound on arc flow for (i,j) if struck.

c_{ij} cost per unit-flow for arc (i,j) if unstruck.

c'_{ij} cost per unit-flow for arc (i,j) if struck.

t_{ij} repair time for arc (i,j).

r_{ij} repair cost for arc (i,j).

For arc selection, let k be the cost of a minimum-cost circulation flow if no arcs are struck, and k_{ij} the cost of a minimum-cost circulation flow if arc (i,j) is struck but all other arcs are unstruck. Then the value of striking arc (i,j) is defined as

$$v(i,j) = (k_{ij} - k)t_{ij} + r_{ij}. \tag{14}$$

The one-strike algorithm finds the arc of maximum strike value. This arc could be found, of course, by actually calculating the strike value for each arc. However, the procedure for calculating the strike value of an arc (or, more specifically, the calculation of k_{ij}) yields a circulation flow that can sometimes be used to calculate upper bounds on strike values of other arcs. Of course, an arc with such an upper bound that is less than the calculated strike value of some other arc cannot be an arc of maximum strike value.

More specifically, suppose the calculation of a minimum-cost circulation flow when arc (a,b) is struck yields an arc flow $x_{ij} \leq u'_{ij}$ for some (i,j). Such a flow would be feasible when arc (i,j) is struck, and totaling up the arc costs for this flow pattern would yield an upper bound for k_{ij}. This upper-bound calculation would differ from the calculation of k_{ab} only in that the struck cost is used for arc (i,j) instead of for arc (a,b), which is now assigned its unstruck cost. Thus one obtains:

$$k_{ij} \leq k_{ab} + (c'_{ij} - c_{ij})x_{ij} - (c'_{ab} - c_{ab})x_{ab} \tag{15}$$

Substituting the right-hand side of Expression (15) for k_{ij} in (14) yields:

$$v(i,j) \leq \{k_{ab} + (c'_{ij} - c_{ij})x_{ij} - (c'_{ab} - c_{ab})x_{ab} - k\}t_{ij} + r_{ij}. \tag{16}$$

Expression (16) will be used in the one-strike algorithm. Before the algorithm is presented, several terms need to be defined formally, some of which have been discussed:

k = cost of minimum-cost circulation flow if no arcs are struck.

k_{ij} = cost of a minimum-cost circulation flow if arc (i,j) is struck and all others are unstruck.

$v(i,j)$ = the value of striking arc (i,j) as defined by (14).

v_{ij} = upper bound on the value of striking arc (i,j).

w = the arc of highest strike value among all those whose strike values have been calculated.

Note that if $v(w) \geq v_{ij}$ for all (i,j), then w is the arc to strike. The algorithm will start with w undefined, $v(w) = 0$, and all $v_{ij} = \infty$. Then it will find a minimum-cost circulation flow with no arcs struck, and lower the v_{ij} in accordance with a slight modification of Expression (15) wherever $x_{ij} \leq u'_{ij}$. Then k_{ab} and $v(a,b)$ are calculated for a particular arc, (a,b), which has the property that $v_{ab} \geq v_{ij}$ for all (i,j). The quantity v_{ab} is then lowered to $v(a,b)$ and, when possible, other v_{ij} values are lowered in accordance with the right-hand side of Expression (16), thus bringing them closer to the $v(i,j)$ values. If $v(a,b) \geq v(w)$, then (a,b) replaces w. Otherwise, w remains unchanged. Then (a,b) is replaced by another arc of highest v_{ij} and the procedure is repeated until a point is reached where $v(w) \geq v_{ij}$ for all (i,j). At this point, w is the arc of maximum strike value and $v(w)$ its strike value. Formally, the algorithm is as follows.

1. Set $v(w) = 0$ and all $v_{ij} = \infty$.

2. Using unstruck costs and upper bounds for arcs, find a minimum-cost circulation flow and denote its value by k. For all arcs with $x_{ij} \leq u'_{ij}$, set

$$v_{ij} = (c'_{ij} - c_{ij})x_{ij}t_{ij} + r_{ij}.$$

3. Choose an arc, (a,b), such that $v_{ab} = \max_{(i,j)} v_{ij}$. If $v_{ab} = v(w)$, terminate, since w is the arc to strike and its strike value is v(w). Otherwise continue.

4. Change (a,b)'s upper flow bound to u'_{ab} and its unit-flow cost to c'_{ab} and find a minimum-cost circulation flow. Denote its value by k_{ab}.

5. Set $v_{ab} = (k_{ab} - k)t_{ab} + r_{ab}$. If $v_{ab} \geq v(w)$, set w = (a,b) and $v(w) = v_{ab}$. Otherwise, leave w and v(w) unchanged.

6. For each arc with $x_{ij} \leq u'_{ij}$, calculate $\Delta_{ij} = \{k_{ab} + (c'_{ij} - c_{ij}) x_{ij} - (c'_{ab} - c_{ab}) x_{ab} - k\}t_{ij} + r_{ij}$ and if $\Delta_{ij} < v_{ij}$, set $v_{ij} = \Delta_{ij}$. (This step may be eliminated for arcs with $v_{ij} \leq v(w)$, since such arcs have essentially been eliminated as candidates to strike).

7. Restore the upper flow bound of (a,b) to u_{ab} and the unit flow cost of(a,b) to c_{ab} and return to step 3.

Step 1 is merely an initialization step. Step 2 calculates k and attempts to lower the v_{ij}. Step 3 tests to see if the arc of maximum strike value has been found (i.e., if $v(w) \geq v_{ij}$ for all (i,j)) and if not, selects another arc for examination. Steps 4 and 5 find the value of striking the arc being examined. Step 5 also tests to see if the arc being examined has a strike value higher than those of all other arcs that have previously been examined and, if so, designates it as such. Step 6 attempts to lower the v_{ij} of all arcs with $v_{ij} > v(w)$, and step 7 restores the parameters for the examined arc to unstruck values.

MULTIPLE STRIKES

There are no known methods, other than complete enumeration, for allocating strikes optimally, when several strikes are to be targeted. Thus optimality will be approximated by allocating strikes one at a time. Each time, the arc against which to allocate the next strike is determined by the one-strike algorithm. In applying the one-strike algorithm, arc parameters are determined by the number of strikes already allocated to them. Specifically, if the first $\bar{n}$ strikes include s to be directed against arc (i,j), then, in applying the one-strike algorithm to allocate the $\bar{n} + 1^{st}$ strike, the unstruck parameters for arc (i,j) correspond more or less to those for s strikes and the struck parameters to s + 1 strikes. More specifically, they are

$$u_{ij} = u_{ij}^{s}$$
$$u_{ij}' = u_{ij}^{s+1}$$
$$c_{ij} = c_{ij}^{s}$$
$$c_{ij}' = c_{ij}^{s+1}$$
$$t_{ij} = t_{ij}^{s+1}$$
$$r_{ij} = r_{ij}^{s+1} - r_{ij}^{s}.$$

The generalized algorithm for allocating n strikes, then, is as follows:

1. For each arc, set $u_{ij} = u_{ij}^{0}$, $u_{ij}' = u_{ij}^{1}$, $c_{ij} = c_{ij}^{0}$. $c_{ij}' = c_{ij}^{1}$, $t_{ij} = t_{ij}^{1}$, and $r_{ij} = r_{ij}^{1} - r_{ij}^{0}$. Set $\bar{n} = 1$.

2. Use the one-strike algorithm to find the location for the $\bar{n}^{th}$ strike.

3. Let (a,b) be the location for the $\bar{n}$th strike and let s be such that $u_{ab} = u_{ab}^{s}$ (i.e., among the first $\bar{n} - 1$ strikes, s is the number allocated to (a,b)). Then set $u_{ab} = u_{ab}^{s+1}$, $u_{ab}' = u_{ab}^{s+2}$, $c_{ab} = c_{ab}^{s+1}$, $c_{ab}' = c_{ab}^{s+2}$, $t_{ab} = t_{ab}^{s+2}$, $r_{ab} = r_{ab}^{s+2} - r_{ab}^{s+1}$.

4. If $\bar{n} = n$, terminate. All strikes have been allocated and the number of strikes allocated to (i,j) is s(i,j), where s(i,j) satisfies $u_{ij} = u_{ij}^{s(i,j)}$. Otherwise, increase $\bar{n}$ by one and return to Step 2.

Step 1 initializes parameters. Step 2 finds the location for the $\bar{n}$th strike. Step 3 updates parameters for the arc selected. Step 4 tests to see if all strikes have been allocated.

USER GOALS AND INPUTS

Section II showed that several possible user goals, while not appearing to consist of finding a minimum-cost circulation flow, could be formulated as such. Three such goals, each of which involved finding a source-sink flow, required the construction of an artificial arc from the sink (T) to the source (S). User goals were determined by the values of the upper and lower flow bounds and unit-flow cost for (T,S). Since some of these parameters depend on the number of strikes, their values must be generalized. Furthermore, values must also be assigned to repair times and repair costs for (T,S).

The generalization of the parameters for the artificial arc are such that its strike value is always zero. Thus repair costs and repair times are zero for all levels of strike and equal to those defined in Sec. II. Specifically, in all three special cases, $t_{TS}^{s} = 0$ and $r_{TS}^{s} = 0$ for all s. If the LOC user is attempting to maximize flow from S to T, $\ell_{TS} = 0$, $u_{TS}^{s} = M$, and $c_{TS}^{s} = -1$ for all s, where M is very large. If the goal is to meet a required flow, r, at minimum cost, $\ell_{TS} = r$, $u_{TS}^{s} = r$, and $c_{TS}^{s} = 0$ for all s. In the combined problem where the user is attempting to meet a required flow, r, at minimum cost if such a flow can be achieved, but to maximize flow if a flow of r cannot be achieved, $\ell_{TS} = 0$, $u_{TS}^{s} = r$, and $c_{TS}^{s} = -M$ for all s, where M is again very large.

IV. ALGORITHMS FOR PIECEWISE LINEAR COSTS

A RAND-AFGOA study group* found it desirable, where arc parameters are based on expected values that in turn are based on probabilities of successful targeting, to let unit-flow costs reflect the fact that arcs which are little used when unstruck may not be used at all when successfully targeted. While, ideally, one might wish to allow for arc costs to be strictly convex functions of flow, the desirability of using the programmed "out-of-kilter" algorithm led to piecewise-linear cost functions with one break-point.

This requires new definitions for the upper flow bounds and unit-flow costs for arcs. These are:

u_{ij}^s upper bound on arc flow for arc (i,j) if s strikes are targeted against it.

$\bar{u}_{ij}^s$ value at which unit-flow cost changes on arc (i,j) if s strikes are targeted against it.

$\bar{c}_{ij}^s$ unit-flow cost for units up to $\bar{u}_{ij}^s$ on arc (i,j) if s strikes are targeted against it.

c_{ij}^s unit-flow cost for units in excess of $\bar{u}_{ij}^s$ on arc (i,j) if s strikes are targeted against it.

The parameters ℓ_{ij}, t_{ij}^s, and r_{ij}^s are as defined when arc costs are linear functions of flow. The cost of a circulation flow pattern with flows x_{ij} would (with superscripts deleted) be

$$(17) \qquad \sum_{x_{ij}\le\bar{u}_{ij}} \bar{c}_{ij}x_{ij} + \sum_{x_{ij}>\bar{u}_{ij}} \bar{c}_{ij}\bar{u}_{ij} + \sum_{x_{ij}>\bar{u}_{ij}} c_{ij}(x_{ij} - \bar{u}_{ij}).$$

It is assumed that u_{ij}^s and $\bar{u}_{ij}^s$ are nonincreasing in s, and $\bar{c}_{ij}^s$, c_{ij}^s, t_{ij}^s, and r_{ij}^s are all nondecreasing in s. Of course, $\bar{u}_{ij}^s \le u_{ij}^s$

*See Preface.

and $\bar{c}_{ij}^s \leq c_{ij}^s$.*

In the RAND-AFGOA study mentioned earlier, it is assumed that all strikes directed against arc (i,j) are aimed at the same target and each has independent probability p_{ij} of interdiction or successful targeting. The upper flow bound and unit-flow cost on arc (i,j) are, respectively, u_{ij}^{INT} and c_{ij}^{INT} when interdicted, and u_{ij}^{UNINT} and c_{ij}^{UNINT} when uninterdicted. Arc parameters as a function of the number of strikes are expected values obtained when it is assumed that arcs not used to full capacity when uninterdicted will not be used at all when interdicted. Thus, they are

$$u_{ij}^s = (1 - p_{ij})^s u_{ij}^{UNINT} + (1 - (1 - p_{ij})^s) u_{ij}^{INT}$$

$$\bar{u}_{ij}^s = (1 - p_{ij})^s u_{ij}^{UNINT}$$

$$\bar{c}_{ij}^s = c_{ij}^{UNINT}$$

$$c_{ij}^s = c_{ij}^{INT}$$

$$t_{ij}^s = (1 - (1 - p_{ij})^s) t_{ij}^{INT}$$

$$r_{ij}^s = (1 - (1 - p_{ij})^s) r_{ij}^{INT}$$

where, of course, t_{ij}^{INT} and r_{ij}^{INT} are repair times and repair costs when arc (i,j) is interdicted.

As in the case of linear costs, strikes are allocated one at a time by a one-strike algorithm that chooses, as the arc to strike, the one that maximizes repair cost plus the product of repair time and the cost-increase of a minimum-cost circulation flow pattern.

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*Since $\bar{c}_{ij}^s \leq c_{ij}^s$, the problem of finding a minimum-cost circulation flow can be converted to one in which arc costs are linear functions of flow. This is done by replacing each arc, (i,j), by two arcs in parallel. The first of these has upper flow bound of $\bar{u}_{ij}^s$ and unit-flow cost of $\bar{c}_{ij}^s$, while the second has values of $u_{ij}^s - \bar{u}_{ij}^s$ and c_{ij}^s for these parameters. These inputs are now suitable for the "out-of-kilter" algorithm.

ONE-STRIKE ALGORITHM

As in the case of linear costs, an iteration of the one-strike algorithm will consist of finding the strike value of some arc, and then trying to improve upper bounds already obtained for the strike value of the arcs whose strike values have not been calculated. The algorithm terminates when the calculated strike value of some arc is at least as great as the upper bound on strike value for any other arc. The notational simplification adopted earlier of dropping superscripts and designating struck values by a prime will also be adopted here. They are now written:

u_{ij} upper bound on arc flow for (i,j) if unstruck

u'_{ij} upper bound on arc flow for (i,j) if struck

$\bar{u}_{ij}$ value at which unit-flow cost changes on arc (i,j) if unstruck

$\bar{u}'_{ij}$ value at which unit-flow cost changes on arc (i,j) if struck

$\bar{c}_{ij}$ unit-flow cost for units up to $\bar{u}_{ij}$ on (i,j) if unstruck

$\bar{c}'_{ij}$ unit-flow cost for units up to u'_{ij} on (i,j) if struck

c_{ij} unit-flow cost for units in excess of $\bar{u}_{ij}$ on (i,j) if unstruck

c'_{ij} unit-flow cost for units in excess of $\bar{u}'_{ij}$ on (i,j) if struck

t_{ij} repair time for (i,j)

r_{ij} repair cost for (i,j)

The quantities k, k_{ij}, v(i,j), v_{ij}, and w are as defined in the one-strike algorithm for linear costs.

If a minimum-cost circulation flow obtained when (a,b) is struck yields $x_{ij} \leq u'_{ij}$, then this flow is feasible when (i,j) is struck and

(18) $$k_{ij} \leq k_{ab} + \delta_{ij} - \delta_{ab} \quad \text{where}$$

$$(19)\qquad \delta_{ij} = (\bar{c}'_{ij} - \bar{c}_{ij})(\min[\bar{u}'_{ij},x_{ij}]) + (c'_{ij} - \bar{c}_{ij})(\max[0, \min(x_{ij},\bar{u}_{ij}) - \bar{u}'_{ij}]) + (c'_{ij} - c_{ij})(\max[0,x_{ij} - \bar{u}_{ij}]).$$

The quantity δ_{ij} is the increase in cost on arc (i,j) obtained by substituting struck parameters for unstruck ones. The first term in the right-hand side of (19) represents the cost-increase on (i,j) for flow units up to $\bar{u}'_{ij}$, the second for flow units between $\bar{u}'_{ij}$ and $\bar{u}_{ij}$, and the third for flow units in excess of $\bar{u}_{ij}$. Substituting the right-hand side of (18) for k_{ij} in (14), the expression for v(i,j), one obtains

$$(20)\qquad v(i,j) \leq (k_{ab} + \delta_{ij} - \delta_{ab} - k)t_{ij} + r_{ij}.$$

Thus, the one-strike algorithm for finding the arc of maximum strike value is as follows:

1. Set $v(w) = 0$ and all $v_{ij} = \infty$.

2. Using unstruck costs and upper bounds for all arcs, find a minimum-cost circulation flow and denote its value by k. For all arcs with $x_{ij} \leq u'_{ij}$, set $v_{ij} = \delta_{ij}t_{ij} + r_{ij}$ where δ_{ij} is as defined in (19).

3. Choose an arc, (a,b), such that $v_{ab} = \max_{(i,j)} v_{ij}$. If $v_{ab} = v(w)$, terminate, since w is the arc to strike and its strike value is v(w). Otherwise continue.

4. Change (a,b)'s upper flow bound to u'_{ab}, the point at which unit-flow cost changes to $\bar{u}'_{ab}$, and its low and high unit-flow costs to $\bar{c}'_{ab}$ and c'_{ab} respectively. Then find a minimum-cost circulation flow and denote its value by k_{ab}.

5. Set $v_{ab} = (k_{ab} - k)t_{ab} + r_{ab}$. If $v_{ab} \geq v(w)$ set $w = (a,b)$ and $v(w) = v_{ab}$. Otherwise leave w and v(w) unchanged.

6. For each arc with $x_{ij} \leq u'_{ij}$, calculate $\Delta_{ij} = (k_{ab} + \delta_{ij} - \delta_{ab} - k)t_{ij} + r_{ij}$, and if $\Delta_{ij} < v_{ij}$, set $v_{ij} = \Delta_{ij}$. (This step may be eliminated for arcs with $v_{ij} \leq v(w)$.)

7. Restore (a,b)'s upper flow bound to u_{ab}, the point at which unit-flow cost changes to $\bar{u}_{ab}$, and low and high unit-flow costs to $\bar{c}_{ab}$ and c_{ab} respectively, and return to step 3.

The functions performed by the individual algorithm steps are essentially the same as those in the linear-cost case. Specifically, step 1 initializes; step 2 calculates k and attempts to lower the v_{ij}; step 3 tests to see if the arc of maximum strike value has been found and, if not, selects another one for examination; steps 4 and 5 find the value of striking the arc being examined and if this value exceeds v(w), lets it replace w; step 6 attempts to lower the v_{ij}; and step 7 restores the parameters of the examined arc to unstruck values.

MULTIPLE STRIKES

The optimal allocation of several strikes against the LOCs is approximated by repeated application of the one-strike algorithm, with arc parameters updated for each application. If the one-strike algorithm is entered with s strikes already targeted to arc (i,j), then the input parameters for arc (i,j) are

$$u_{ij} = u_{ij}^{s}$$

$$u_{ij}' = u_{ij}^{s+1}$$

$$\bar{u}_{ij} = \bar{u}_{ij}^{s}$$

$$\bar{u}_{ij}' = \bar{u}_{ij}^{s+1}$$

$$\bar{c}_{ij} = \bar{c}_{ij}^{s}$$

$$\bar{c}_{ij}' = \bar{c}_{ij}^{s+1}$$

$$c_{ij} = c_{ij}^{s}$$

$$c_{ij}' = c_{ij}^{s+1}$$

$$t_{ij} = t_{ij}^{s+1}$$

$$r_{ij} = r_{ij}^{s+1} - r_{ij}^{s}.$$

Following the format for the linear-cost case, the general algorithm for targeting n strikes in the piecewise-linear-cost case is as follows:

1. For each arc, set $u_{ij} = u^0_{ij}$, $u'_{ij} = u^1_{ij}$, $\bar{u}_{ij} = \bar{u}^0_{ij}$, $\bar{u}'_{ij} = \bar{u}^1_{ij}$, $\bar{c}_{ij} = \bar{c}^0_{ij}$, $\bar{c}'_{ij} = \bar{c}^1_{ij}$, $c_{ij} = c^0_{ij}$, $c'_{ij} = c^1_{ij}$, $t_{ij} = t^1_{ij}$, and $r_{ij} = r^1_{ij} - r^0_{ij}$. Set $\bar{n} = 1$.

2. Use the one-strike algorithm to find the location for the $\bar{n}^{th}$ strike.

3. Let (a,b) be the location for the $\bar{n}^{th}$ strike and let s be such that $u_{ab} = u^s_{ab}$. Then set $u_{ab} = u^{s+1}_{ab}$, $u'_{ab} = u^{s+2}_{ab}$, $\bar{u}_{ab} = \bar{u}^{s+1}_{ab}$, $\bar{u}'_{ab} = u^{s+2}_{ab}$, $\bar{c}_{ab} = \bar{c}^{s+1}_{ab}$, $\bar{c}'_{ab} = \bar{c}^{s+2}_{ab}$, $c_{ab} = c^{s+1}_{ab}$, $c'_{ab} = c^{s+2}_{sb}$, $t_{ij} = t^{s+1}_{ij}$, and $r_{ab} = r^{s+2}_{ab} - r^{s+1}_{ab}$.

4. If $\bar{n} = n$, terminate. All strikes have been targeted and the number of strikes targeted to (i,j) is s(i,j), where s(i,j) satisfies $u_{ij} = u^{s(i,j)}_{ij}$. Otherwise, increase $\bar{n}$ by one and return to step 2.

USER GOALS AND INPUTS

With respect to the special user goals discussed in Sec. II, parameters for the artificial arc, (T,S), are quite similar to those of the linear-cost case. All $\bar{u}^s_{ij} = 0$ and $\bar{c}^s_{ij}$ may take on arbitrary values. This essentially makes cost on the artificial arc a strictly linear function of flow, with c^s_{TS} the unit-flow cost when s strikes have been targeted against it. Thus the newly defined parameters have essentially been zeroed out for this arc and, for all user goals discussed, the quantities r^s_{ij}, t^s_{ij}, u^s_{ij}, c^s_{ij}, and ℓ_{ij} take on the same values as in the linear cost case. Specifically, $t^s_{TS} = 0$ and $r^s_{TS} = 0$ for all s. When the LOC user is attempting to maximize flow from S to T, $\ell_{TS} = 0$, $u^s_{TS} = M$, and $c^s_{TS} = -1$ for all s; when attempting to meet a required flow, r, at minimum cost, $\ell_{TS} = r$, $u^s_{TS} = r$, and $c^s_{TS} = 0$ for all s; and when attempting to meet a required flow r, at minimum cost if possible, and if not possible, to maximize flow, $\ell_{TS} = 0$, $u^s_{TS} = r$, and $c^s_{TS} = -1$ for all s. In all of these cases M represents a very large number.

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