Algorithms for time-optimal control of CNC machines along curved tool paths

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the time optimal control problem

It has been an intuitive assumption for some time that if a control system is being operated from a limited source of power, and if one wishes to have the system change from one state to another in minimum time, then this can be done by at all times utilizing properly all of the power available. This hypothesis is called “the bang-bang principle.”

Bushaw accepted this hypothesis, and in 1952 showed for some simple systems with one degree of freedom that,of all bang-bang systems, there is one that is optimal. In 1953, I made the observation that the best of all bang-bang systems, if it exists, is then the best of all systems operating from the same power source.

J. P. La Salle (1960)

prior work (robotics literature)

J. E. Bobrow, S. Dubowsky, and J. S. Gibson (1985), Time-optimal control of robotic manipulators along specified paths, *International Journal of Robotics Research* **4** (3), 3–17

F. Pfeiffer and R. Johanni (1987), A concept for manipulator trajectory planning, *IEEE Journal of Robotics and Automation* **RA–3** (2), 115–123

Z. Shiller (1994), On singular time–optimal control along specified paths, *IEEE Transactions on Robotics and Automation* **10**, 561–566

Z. Shiller and H. H. Lu (1990), Robust computation of path constrained time optimal motions, Proceedings, *IEEE International Conference on Robotics and Automation*, 144–149

Z. Shiller and H. H. Lu (1992), Computation of path constrained time optimal motions with dynamic singularities, *Journal of Dynamic Systems, Measurement, and Control* **114** (March), 34–40

K. G. Shin and N. D. McKay (1985), Minimum-time control of robotic manipulators with geometric path contraints, *IEEE Transactions on Automatic Control* **AC-30** (6), 531–541

J. J. E. Slotine and H. S. Yang (1989), Improving the efficiency of time–optimal path–following algorithms, *IEEE Transaction on Robotics and Automation* **5** (1), 118–124

H. S. Yang and J. J. E. Slotine (1994), Fast algorithms for near–minimum–time control of robot manipulators, *International Journal of Robotics Research* **13**, 521–532

attractive features of Cartesian problem

- closed-form integration of $\dot{v} = a_{\min/\max}(s, v)$ ODEs
- $\Rightarrow$ $(\text{optimal feedrate})^2 =$ piecewise-rational function
- switching points $=$ roots of univariate polynomials
- analytic reduction of “interpolation integral”
  $\Rightarrow$ accurate & efficient real-time interpolator
- amenable to all types of (piecewise) polynomial
  parametric curves (Bézier, B-spline, etc.)

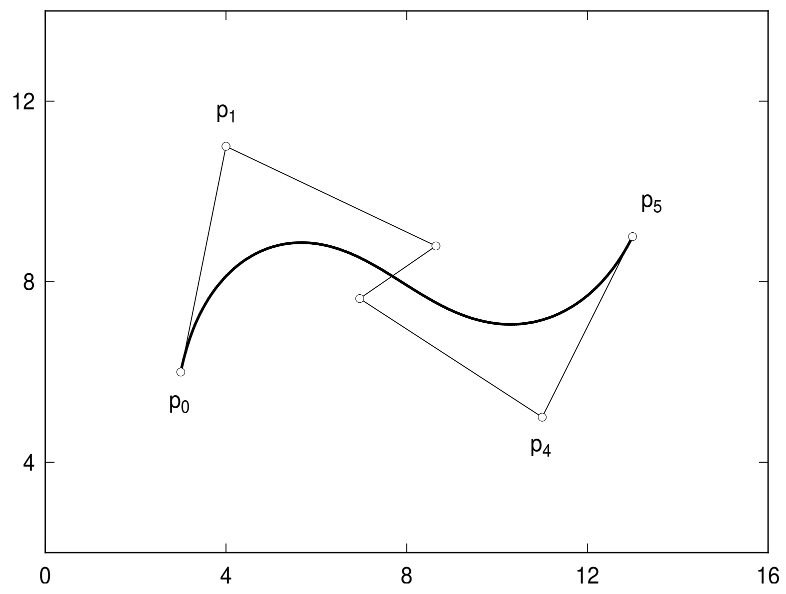
“conventional” vs. “high-speed” machining

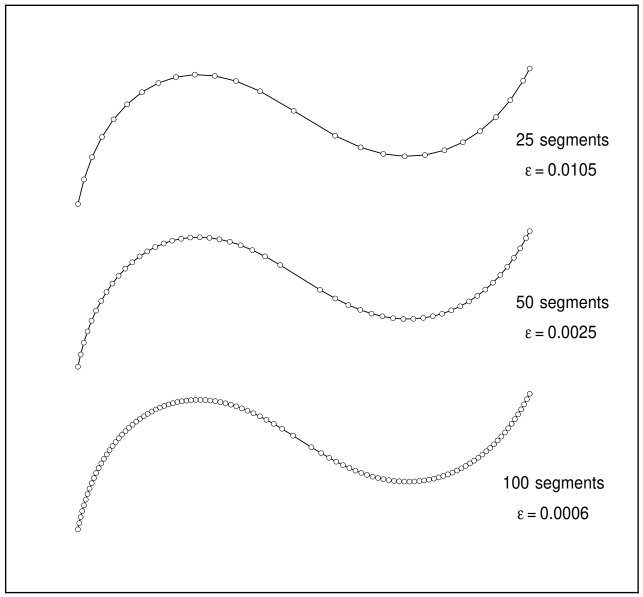
- machining times are major determinant of product cost
- usual: feedrate ≤ 100 in/min, spindle speed $\leq 6,000$ rpm
- HSM: feedrate $\leq 1,200$ in/min, spindle speed $\leq 50,000$ rpm
- in HSM inertial effects may dominate cutting forces, friction, etc. (especially for intricate curved tool paths)
- most CNC machines significantly under-perform in practice – control software, not hardware, is the limiting factor
- determine feedrate for minimal-time traversal of curved path, under given axis acceleration bounds (assuming inertial forces dominant)

3-axis “open architecture” CNC mill

- MHO Series 18 Compact Mill
- 18” $\times$ 18” $\times$ 12” work volume
- Yaskawa brushless DC motors
- zero-backlash precision ball screws
- linear encoders, ± 0.001” accuracy
- MDSI OpenCNC control software
- custom real-time interpolators

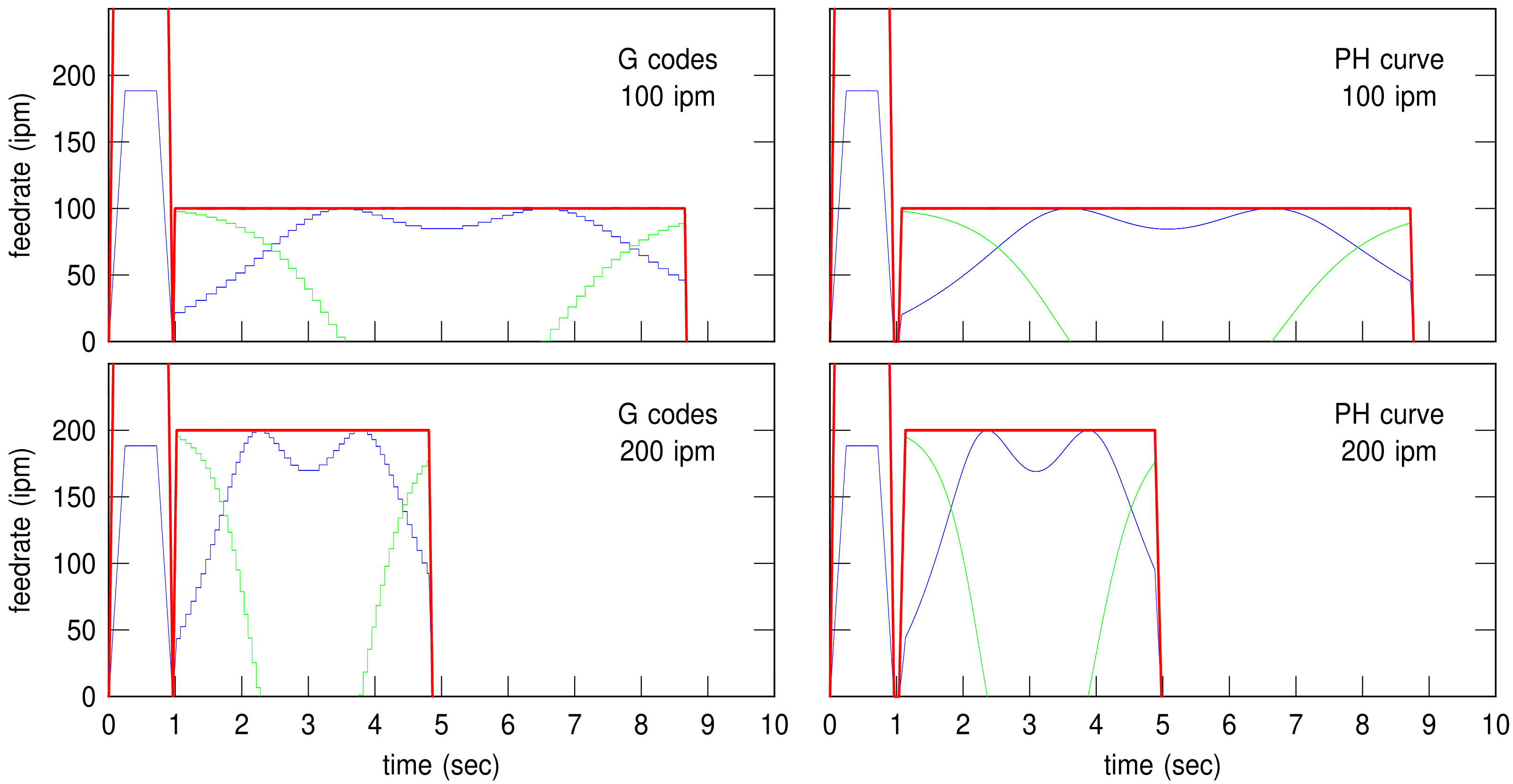
past project: real-time CNC interpolators for Pythagorean-hodograph (PH) curves





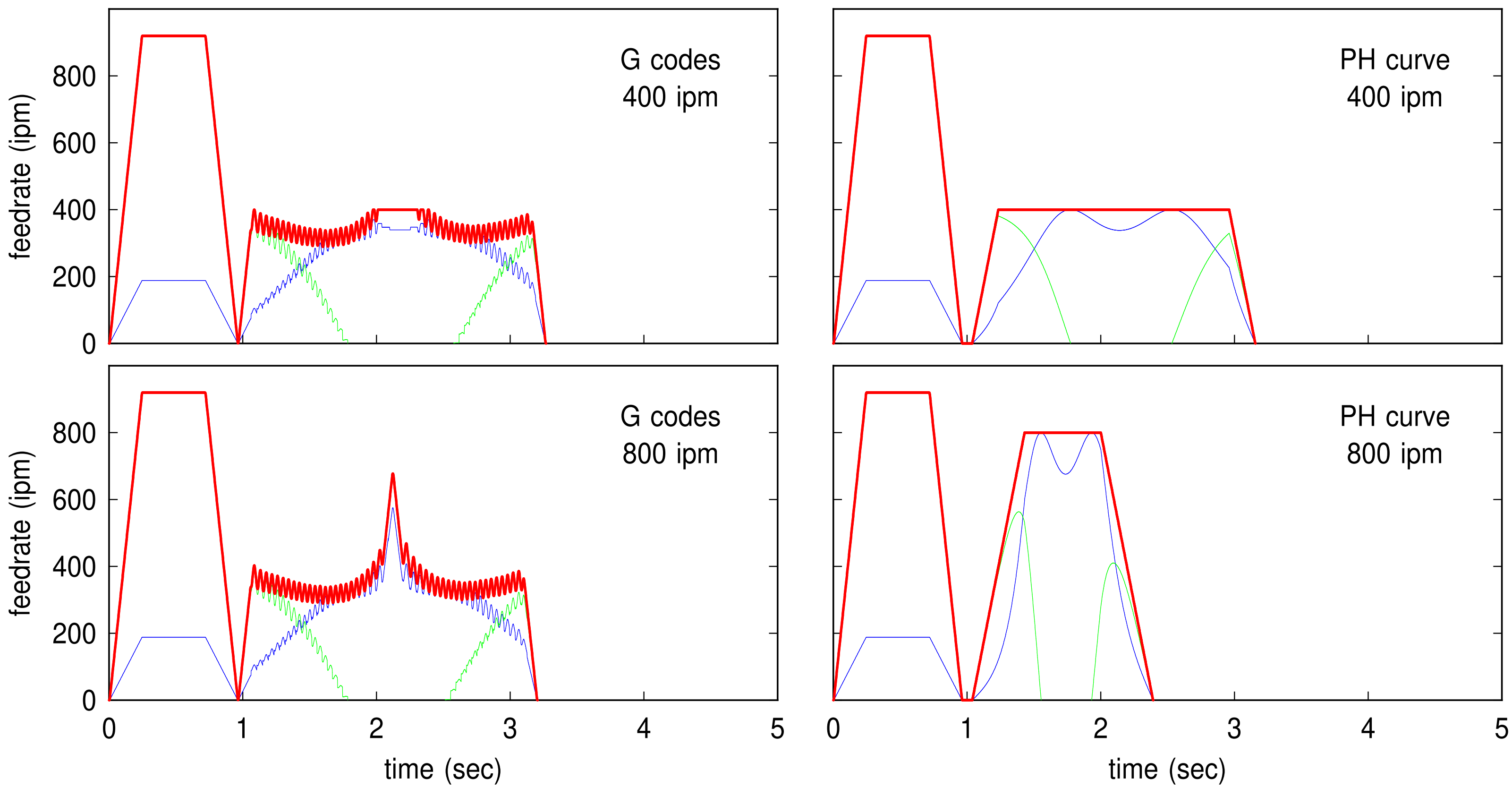
Left: analytic tool path description (quintic PH curve). Right: approximation of path to various prescribed tolerances using piecewise-linear G codes.

comparative feedrate performance: 100 & 200 ipm



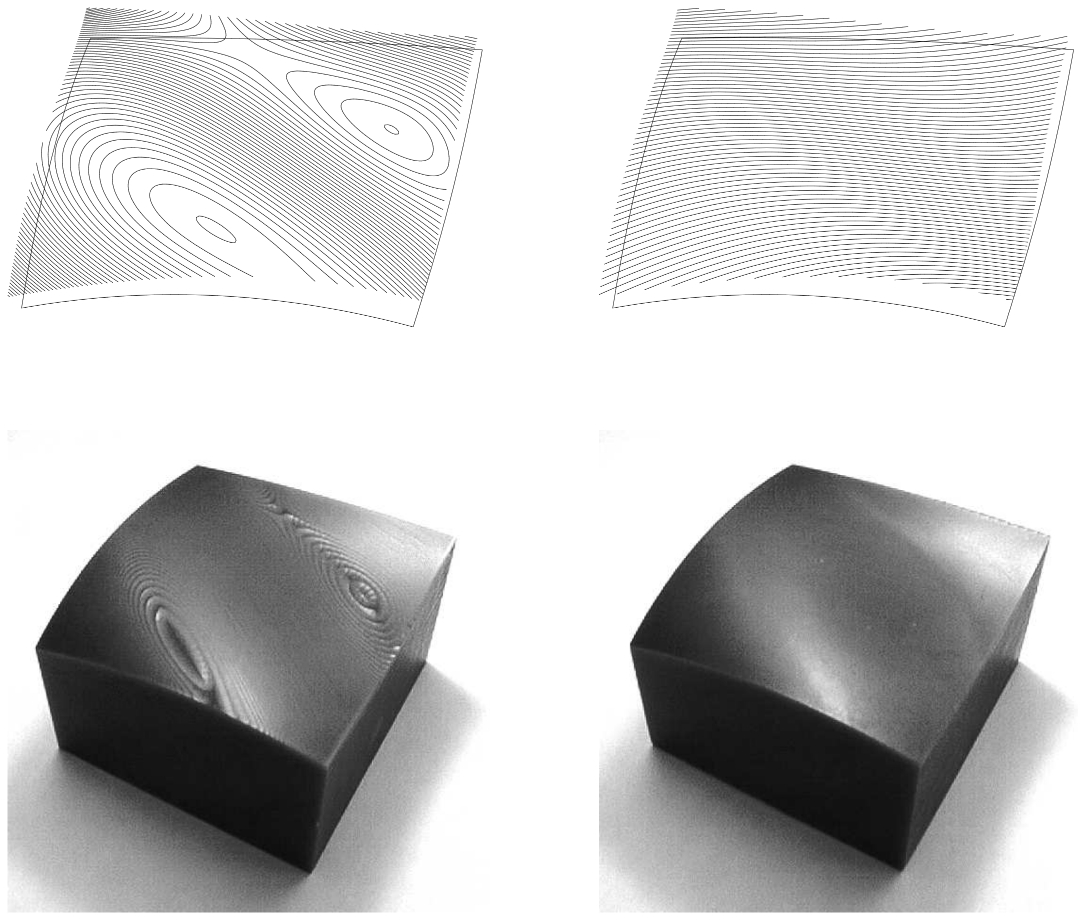
The G code and PH curve interpolators both give excellent performance (red) at 100 and 200 ipm. The "staircase" nature of the x and y feedrate components (blue and green) for the G codes indicate faithful reproduction of the piecewise-linear path, while the PH curve yields smooth variations.

comparative feedrate performance: 400 & 800 ipm

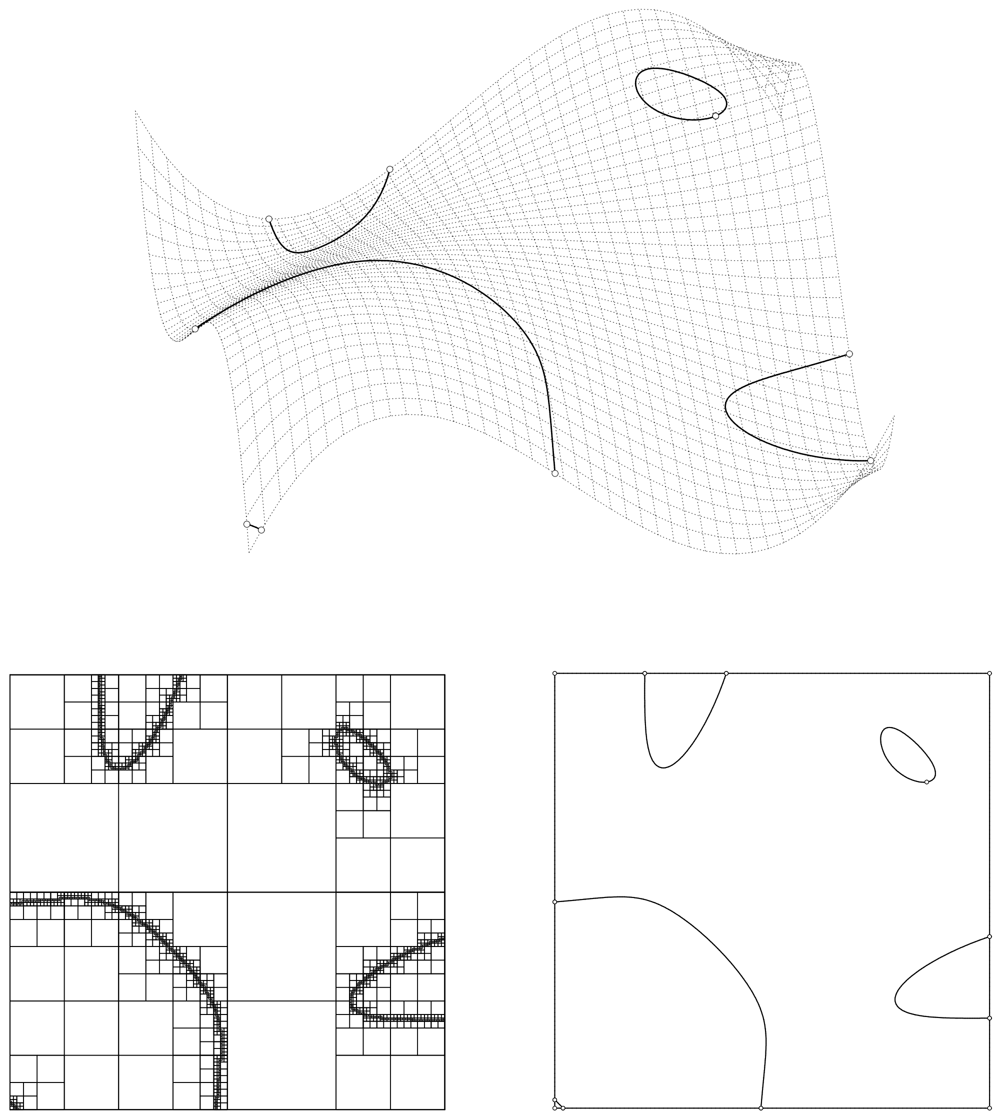


At 400 & 800 ipm, the PH curve interpolator continues to yield impeccable performance – but the performance of the G code interpolator is severely degraded by “aliasing” effects, incurred by the finite sampling frequency and discrete nature of the piecewise-linear path description.

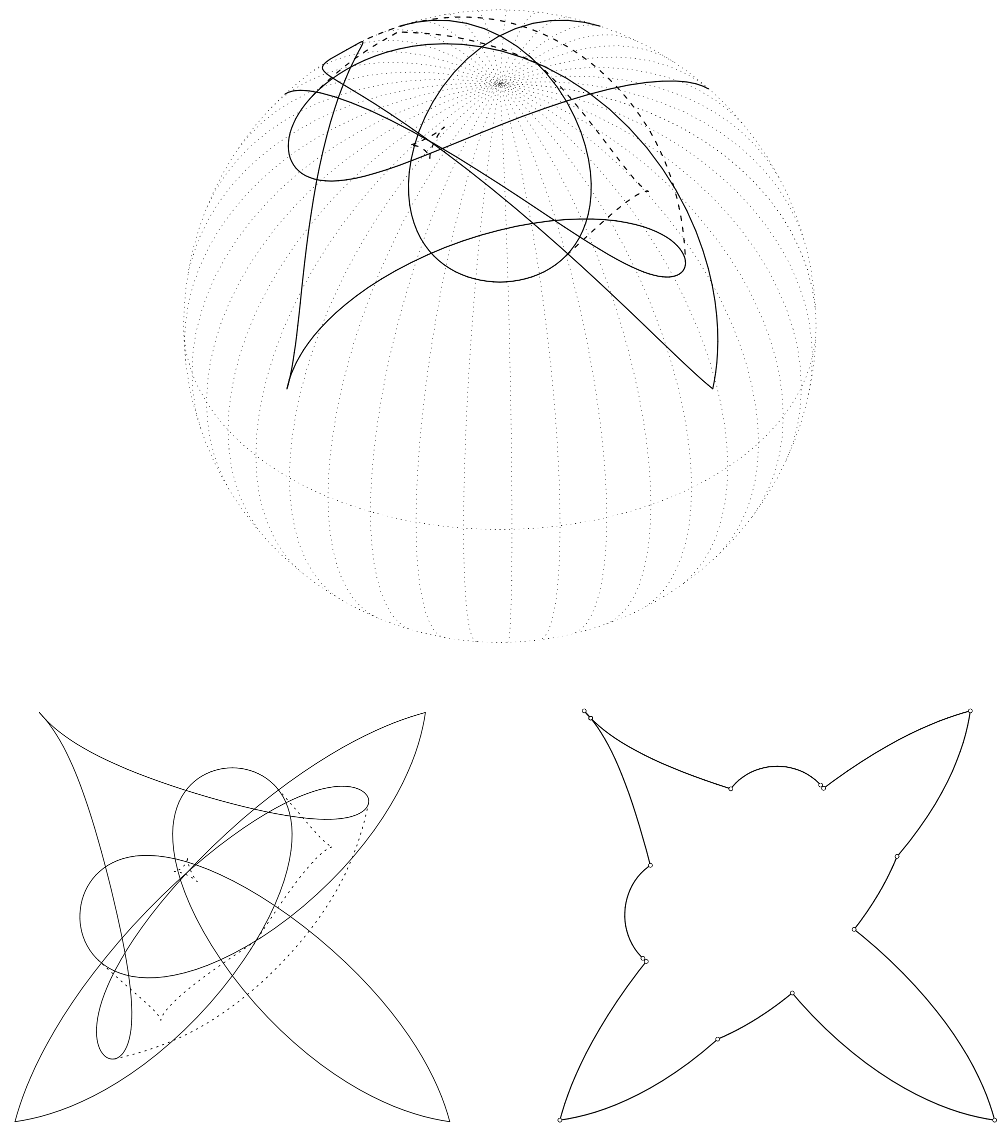
past project: optimal section-plane orientation for contour machining of free-form surfaces



parabolic lines on free-form surfaces



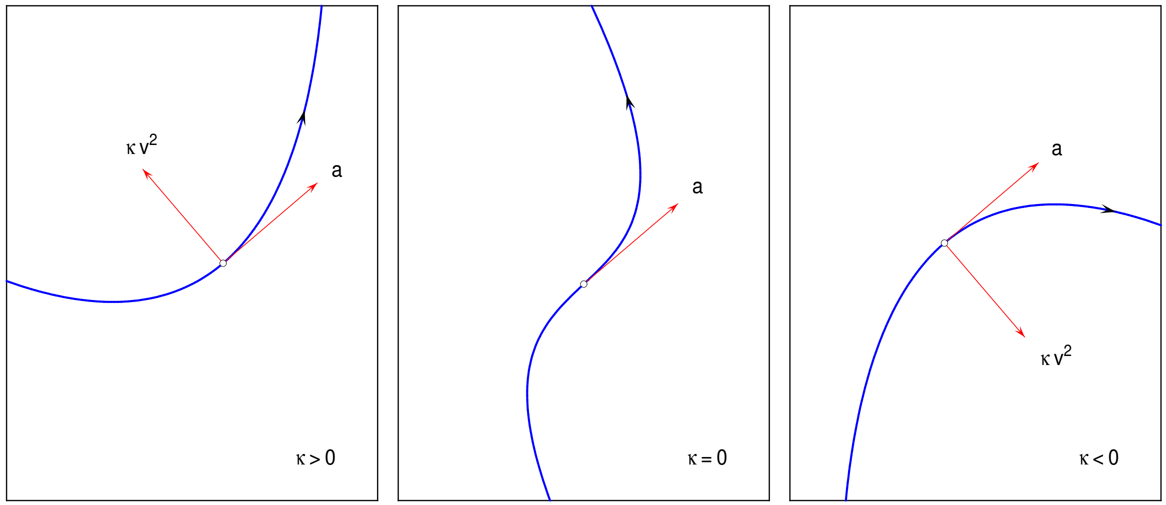
Gauss map computation for free-form surfaces



preambulatory terminology

- $s =$ arc length along curved tool path
- $v = \dfrac{\mathrm{d}s}{\mathrm{d}t} =$ feedrate (scalar speed) along path
- $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} =$ feed acceleration — note that $a = v\,\dfrac{\mathrm{d}v}{\mathrm{d}s}$
- $v_{\mathrm{lim}}(s) =$ velocity limit (maximum v consistent with axis acceleration constraints)
- $v_{\mathrm{lim}}(s)$ usually called the "velocity limit curve" or VLC
- feasible region $=$ set of states in (s, v) phase plane below the VLC, consistent with all constraints

time-optimal motion with acceleration constraints



acceleration vector $\ddot{\mathbf{r}} = a\,\mathbf{t} + \kappa v^2\,\mathbf{n}$

$\mathbf{t} = \text{tangent}, \;\; \mathbf{n} = \text{normal}, \;\; \kappa = \text{curvature}, \;\; v = \text{feedrate}, \;\; a = \text{feed acceleration}$

$$\min_{v} \; T = \int \frac{\mathrm{d}s}{v} \qquad \text{such that} \quad \left\{ \begin{array}{l} -A_x \le a\,t_x + \kappa v^2 n_x \le +A_x \\ -A_y \le a\,t_y + \kappa v^2 n_y \le +A_y \\ -A_z \le a\,t_z + \kappa v^2 n_z \le +A_z \end{array} \right.$$

for simplicity, take $A_x = A_y = A_z \; (= A, \text{ say})$ henceforth

need general (not arc-length) parameterization

"It is impossible to parameterize any curve, other than a straight line, by rational functions of its arc length"

Farouki & Sakkalis (1991)

elements of proof by contradiction:

- Pythagorean triples of polynomials
- partial fraction decomposition of rational functions
- contour integration & "calculus of residues"

$$\int \frac{g}{h}\,\mathrm{d}t \;=\; \text{rational function} \qquad \Longleftrightarrow \qquad \int_{-\infty}^{+\infty} \frac{g}{h}\,\mathrm{d}t \;=\; 0$$

degree–n Bézier curve

$$\mathbf{r}(\xi) \;=\; (x(\xi), y(\xi), z(\xi)) \;=\; \sum_{k=0}^{n} \mathbf{p}_k \binom{n}{k} (1-\xi)^{n-k} \xi^k \,, \quad \xi \in [\,0,1\,]$$

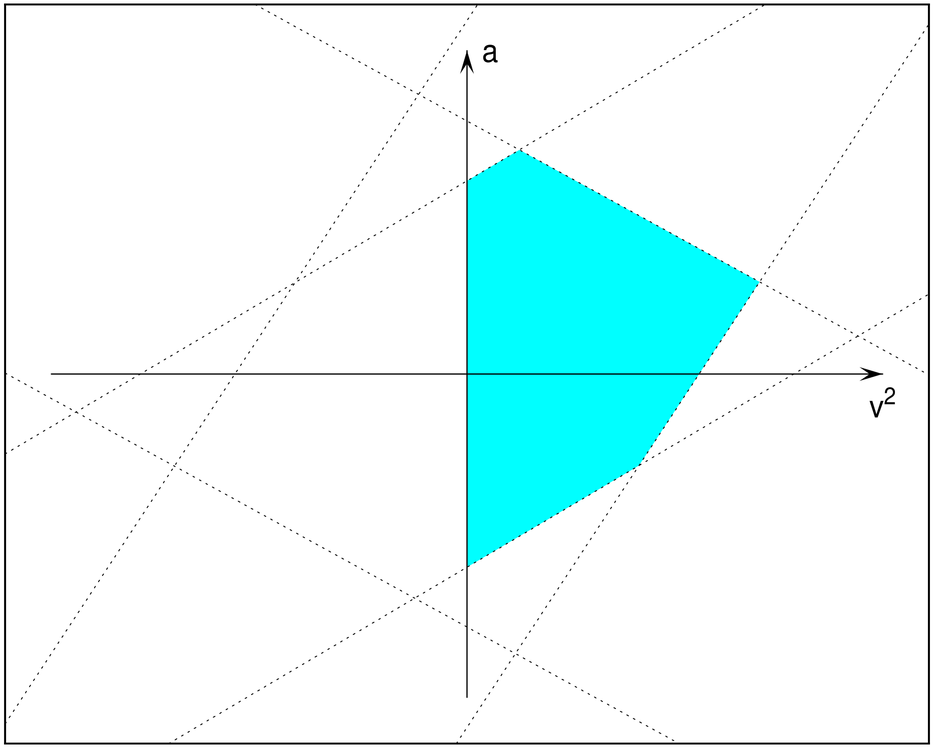
denote ξ-derivatives by primes & define parametric speed

$$\sigma(\xi) \;=\; |\mathbf{r}'(\xi)| \;=\; \frac{\mathrm{d}s}{\mathrm{d}\xi}$$

with $q = v^2$ (and hence $q' = 2\sigma a$) acceleration constraints become

$$\left.\begin{aligned} -A \;&\le\; \frac{q'}{2\sigma^2}\,x' + \frac{q}{\sigma^3}\,(\sigma x'' - \sigma' x') \;\le\; +A \\ -\,A \;&\le\; \frac{q'}{2\sigma^2}\,y' + \frac{q}{\sigma^3}\,(\sigma y'' - \sigma' y') \;\le\; +A \\ -\,A \;&\le\; \frac{q'}{2\sigma^2}\,z' + \frac{q}{\sigma^3}\,(\sigma z'' - \sigma' z') \;\le\; +A \end{aligned}\right\} \quad \text{for} \;\; \xi \in [\,0,1\,]$$

constraint parallelogram in (v^2, a) plane

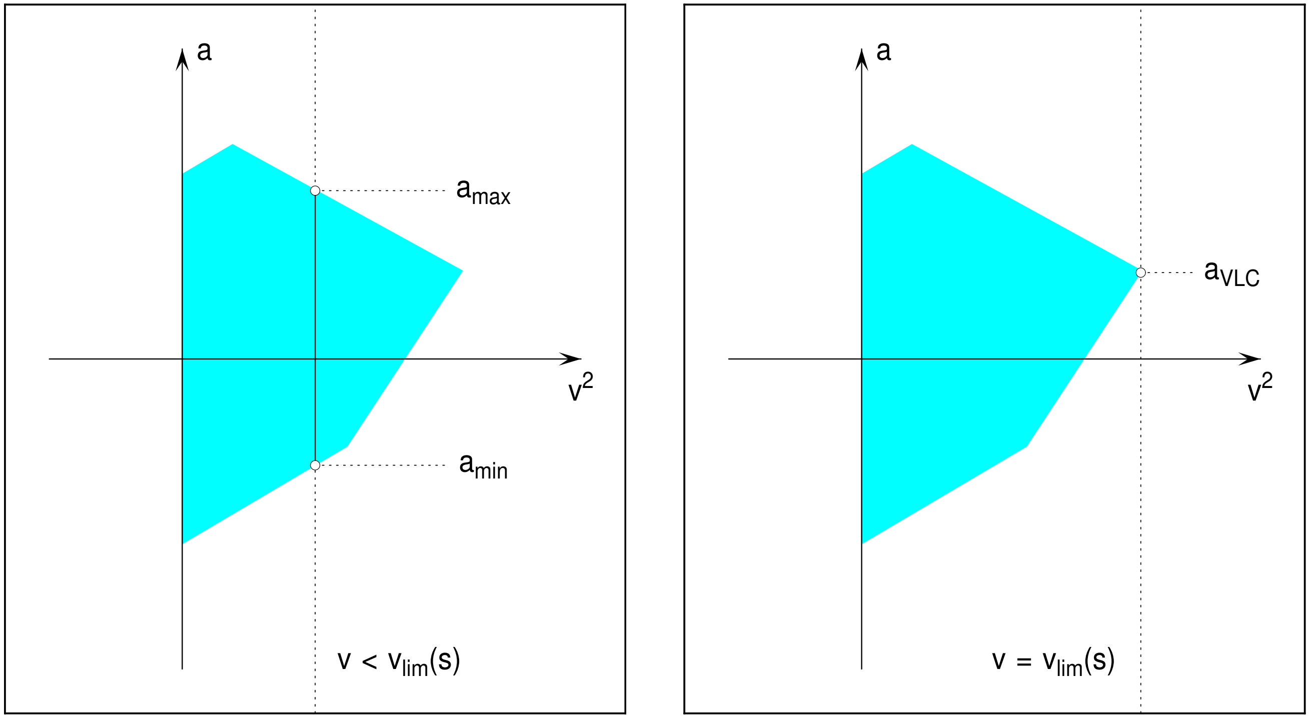


Parallelogram of possible (v^2, a) combinations, defined by pairs of linear symmetric constraints in v^2 and a (only the portion with $v^2 > 0$ is relevant).

Parallelogram varies in size, shape, and orientation with position s on curve.

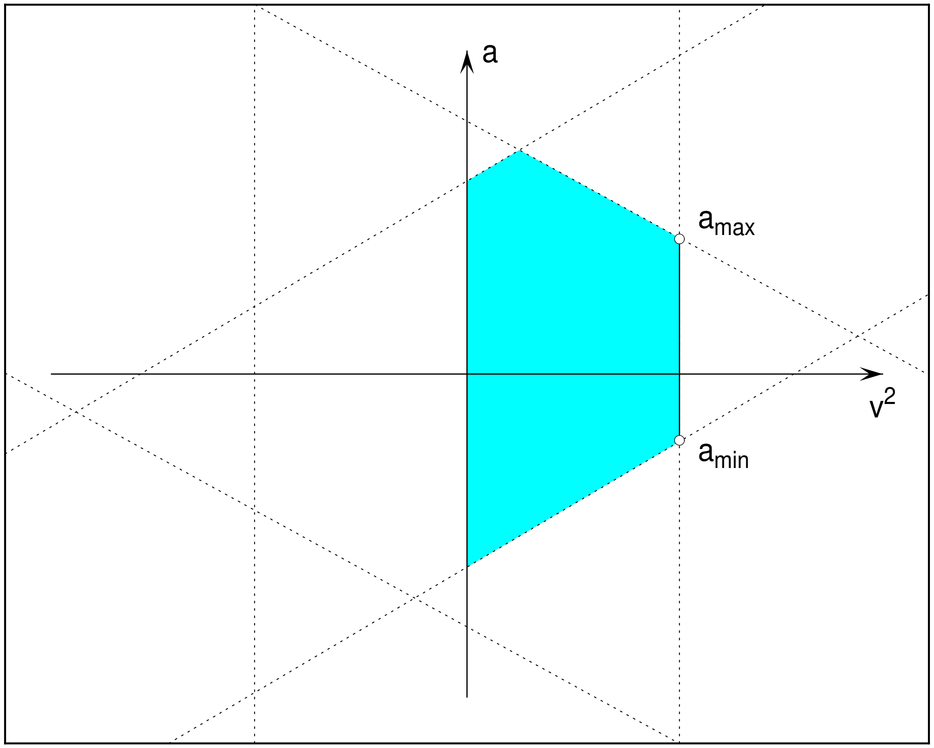
Right-most vertex of parallelogram defines velocity limit curve (VLC), $v_{\lim}(s)$.

range of possible feed accelerations



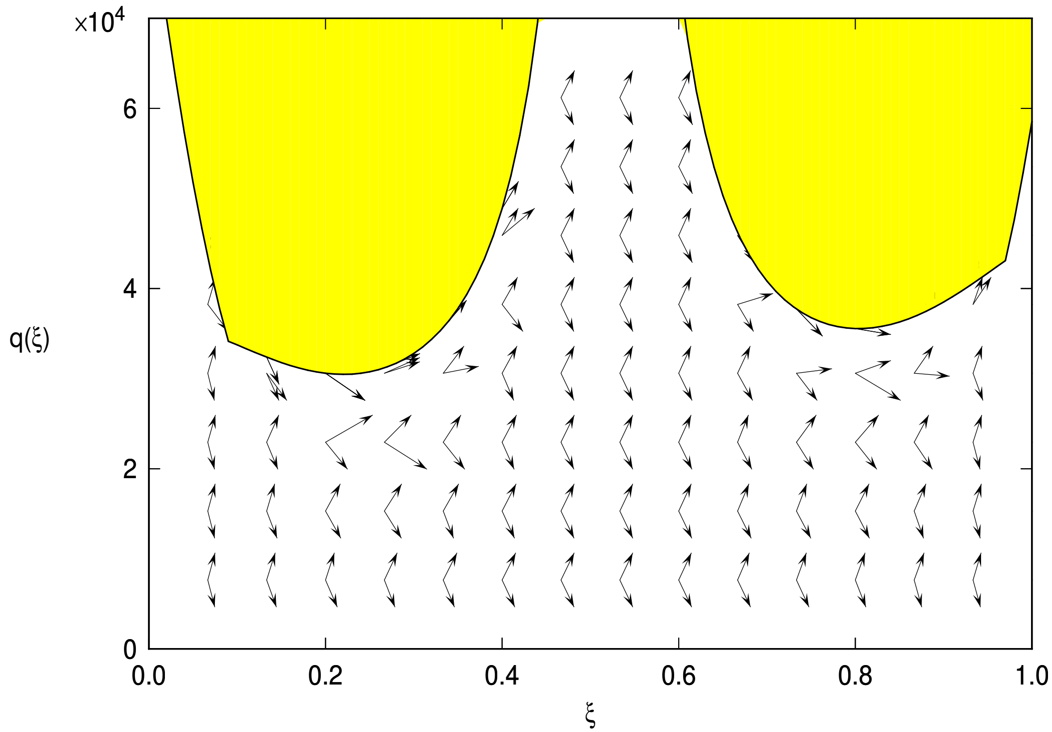
Left: for states (s, v) with $v < v_{\text{lim}}(s)$, a range of possible feed accelerations $a_{\text{min}}(s, v) \leq a \leq a_{\text{max}}(s, v)$ is defined by the intersection of a vertical line with the parallelogram in the (v^2, a) plane. Right: if $v = v_{\text{lim}}(s)$, the vertical line passes through the right–most parallelogram vertex, and the range of feed accelerations collapses to a single value, denoted by $a_{\text{VLC}}(s)$.

special behavior at critical points



At a critical point (a slope discontinuity) of the VLC, one pair of constraints is described by vertical lines — i.e., these constraints depend only on v^2, and not on a. Instead of a unique feed acceleration $a_{\mathrm{VLC}}(s)$ at such points, there is a range of possible values $a_{\min} \le a \le a_{\max}$ instead.

feasible phase-plane trajectory slopes



For a smooth parametric curve $\mathbf{r}(\xi)$, bounds $a_{\min} \le a \le a_{\max}$ on the feed acceleration $a = \dot{v}$ fix the range of possible trajectory slopes in the (ξ, q) phase plane, where $q = v^2$. On the VLC, $a_{\min} = a_{\max}$ $(= a_{\mathrm{VLC}},\ \mathrm{say})$ — except at critical points. The upper & lower bounds on trajectory slopes are continuous but not differentiable functions of position in (ξ, q) phase plane.

trajectory "sources" and "sinks" on VLC

trajectory source $=$ segment of VLC along which unique feed acceleration points in to feasible phase plane region

trajectory sink $=$ segment of VLC along which unique feed acceleration points out of feasible phase plane region

tangency point $=$ transition between source and sink segments on the VLC – serves as potential switching point between $a_{\min}$ and $a_{\max}$ phase plane trajectories

we interrupt this seminar for a

— BREAKING NEWS FLASH !! —

Hu is the new leader of China ?

(transcript of a White House conversation between George W. Bush and Condoleeza Rice upon the nomination of Hu Jintao as the new Chief of the Communist Party of China)

- **George**: Condi! Nice to see you. What's happening?
- **Condi**: Sir, I have here the report about the new leader of China.
- **George**: Great. Lay it on me.
- **Condi**: Hu is the new leader of China.
- **George**: That's what I want to know.
- **Condi**: That's what I'm telling you.
- **George**: That's what I'm asking you. Who is the new leader of China?
- **Condi**: Yes.
- **George**: I mean the fellow's name.
- **Condi**: Hu.
- **George**: The guy in China.
- **Condi**: Hu.

- **George**: The new leader of China.
- **Condi**: Hu.
- **George**: The Chinaman!
- **Condi**: Hu is leading China.
- **George**: Now whaddya' asking me for?
- **Condi**: I'm telling you Hu is leading China.
- **George**: Well, I'm asking you. Who is leading China?
- **Condi**: That's the man's name.
- **George**: That's who's name?
- **Condi**: Yes.
- **George**: Will you or will you not tell me the name of the new leader of China?
- **Condi**: Yes, sir.
- **George**: Yassir? Yassir Arafat is in China? I thought he was in the Middle East.
- **Condi**: That's correct.
- **George**: Then who is in China?
- **Condi**: Yes, sir.
- **George**: Yassir is in China?
- **Condi**: No, sir.
- **George**: Then who is?
- **Condi**: Yes, sir.
- **George**: Yassir?
- **Condi**: No, sir.

- **George**: Look, Condi. I need to know the name of the new leader of China. Get me the Secretary General of the UN on the phone.
- **Condi**: Kofi?
- **George**: No, thanks.
- **Condi**: You want Kofi?
- **George**: No.
- **Condi**: You don't want Kofi.
- **George**: No. But now that you mention it, I could use a glass of milk. And then get me the UN.
- **Condi**: Yes, sir.
- **George**: Not Yassir! The guy at the UN.
- **Condi**: Kofi?
- **George**: Milk! Will you please make the call?
- **Condi**: And call who?
- **George**: Who is the guy at the UN?
- **Condi**: Hu is the guy in China.
- **George**: Will you stay out of China?!
- **Condi**: Yes, sir.
- **George**: And stay out of the Middle East! Just get me the guy at the UN.
- **Condi**: Kofi.
- **George**: All right! With cream and two sugars. Now get on the phone. (Condi picks up the phone.)
- **Condi**: Rice, here.
- **George**: Rice? Good idea. And a couple of egg rolls, too. Maybe we should send some to the guy in China. And the Middle East. Can you get Chinese food in the Middle East?

taxonomy of break-points for optimal feedrate

- switching points on VLC: critical points & tangency points
- switching points below VLC: $a_{\min}/a_{\max}$ trajectory crossings
- break-points determined by local curve geometry: inflections, turning points, equi-orientation points
- break-points determined by phase-plane location: transition points

Can be found by polynomial root-solving. Each type incurs change in identity of acceleration constraint defining current $a_{\min}/a_{\max}$ trajectory – identity of "active" constraint between consecutive break-points must be recorded.

closed-form integration for extremal trajectories

if x is limiting axis, $a_{\min}/a_{\max}$ trajectories are defined by

$$\frac{q'}{2\sigma^2}\,x' \;+\; \frac{q}{\sigma^3}\,(\sigma x'' - \sigma' x') \;=\; \pm A$$

re-write differential equation as

$$q' \;+\; 2\left(\frac{x''}{x'} - \frac{\sigma'}{\sigma}\right) q \;=\; \pm\,\frac{2A\sigma^2}{x'}$$

and multiply by integrating factor $(x'/\sigma)^2$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{x'}{\sigma}\right)^2 q \;=\; \pm\,2Ax' \qquad \longrightarrow \qquad q \;=\; \left(\frac{\sigma}{x'}\right)^2 (C \pm 2Ax)$$

integration constant C determined by specifying known point on trajectory

“removable singularity” at VLC critical points

on $a_{\min}/a_{\max}$ trajectory $\qquad q = \left(\frac{\sigma}{x'}\right)^2 (C \pm 2Ax) = \frac{f}{g}\,, \quad \text{say}$

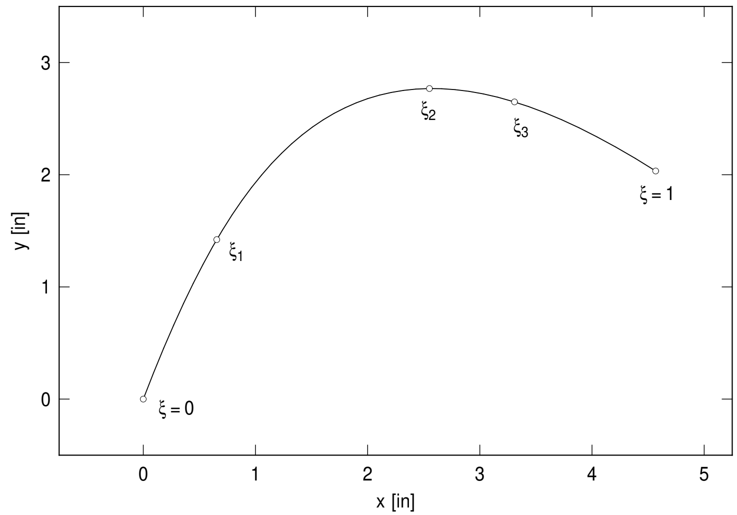
but $f(\xi_*) = g(\xi_*) = 0$ and $f'(\xi_*) = g'(\xi_*) = 0$ if ξ_* identifies a critical point !!

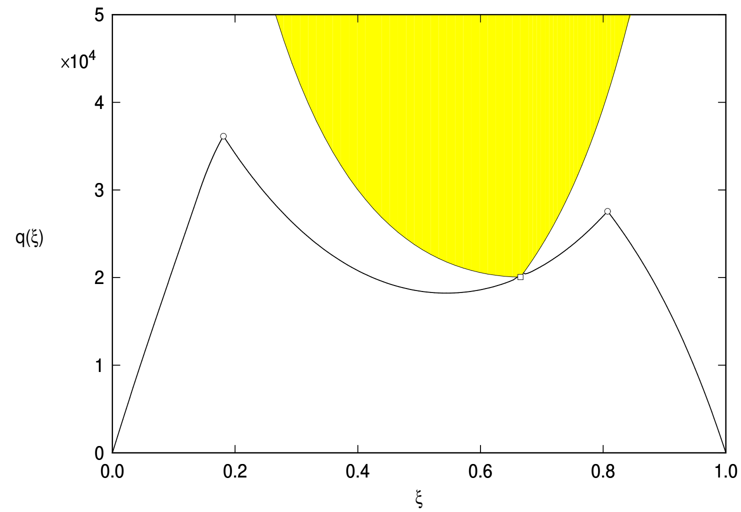
to determine C from known $q(\xi_*)$ value, we must invoke l’Hopital’s rule

$$\lim_{\xi \to \xi_*} q(\xi) = \frac{f''(\xi_*)}{g''(\xi_*)} = \pm A \, \frac{\sigma^2(\xi_*)}{x''(\xi_*)}$$

use two-fold Bernstein-form degree reduction of $f(\xi)$ and $g(\xi)$
to eliminate the common quadratic factor

simple example (planar curve)





Left: planar Bézier curve $\mathbf{r}(\xi)$, $\xi \in [\,0,1\,]$ with switching points at the three parameter values ξ_1, ξ_2, ξ_3. Right: construction of optimal feedrate function $q(\xi) = v^2(\xi)$ in the (ξ, q) phase plane. Switching point ξ_2 is a VLC critical point, and ξ_1, ξ_3 are intersections of $a_{\rm min}$ and $a_{\rm max}$ integration trajectories.

optimal feedrate algorithm (sketch)

input : curve $\mathbf{r}(\xi)$, $\xi \in [\,0,1\,]$ & axis acceleration bounds $\pm A$

0. pre-process: compute the VLC, $v_{\rm lim}(\xi)$, and the ordered set of all (potential) switching points, and their types, on it

1. integrate an $a_{\max}$ trajectory forward from $(\xi,q)=(0,0)$ and an $a_{\min}$ trajectory backward from $(\xi,q)=(1,0)$

2. if these trajectories intersect below the VLC, the optimal feedrate comprises two segments with a single switching point

3. if initial trajectories do not intersect below the VLC, they must cross the VLC at (distinct) “left” and “right” points

4. from left point, search forward along VLC for the first switching point – integrate backward & forward from this switching point to obtain an arriving $a_{\min}$ trajectory and a departing $a_{\max}$ trajectory

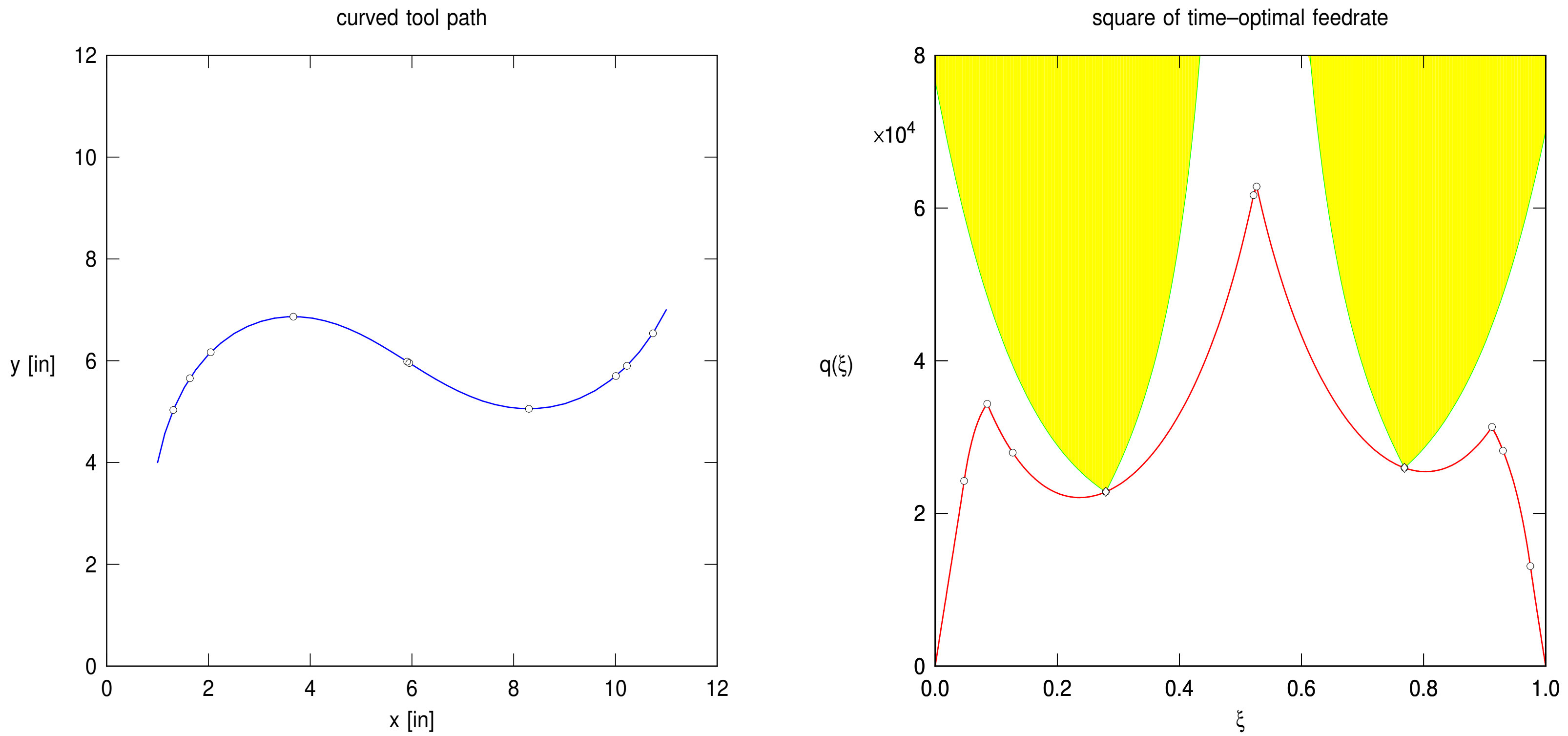
5. intersection of backward trajectory with initial $a_{\max}$ trajectory defines a new switching point below the VLC

6. if forward trajectory intersects the final $a_{\min}$ trajectory, this defines a new switching point below the VLC, and construction of the optimal feedrate is complete

7. otherwise, forward trajectory must intersect the VLC at a "left" point – using this point, repeat steps 4-6 until done

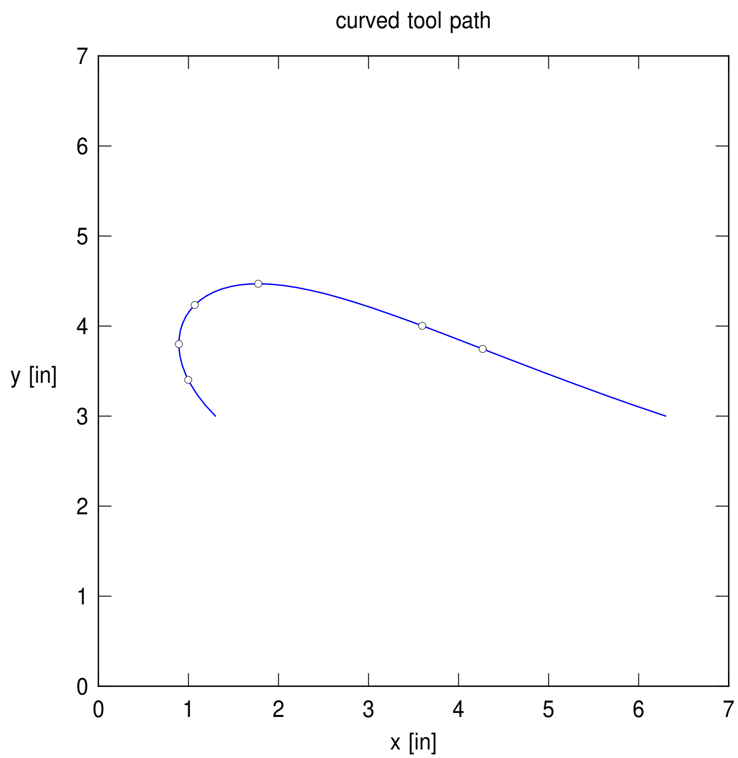
output : piecewise-rational function $q(\xi)$ for square of optimal feedrate

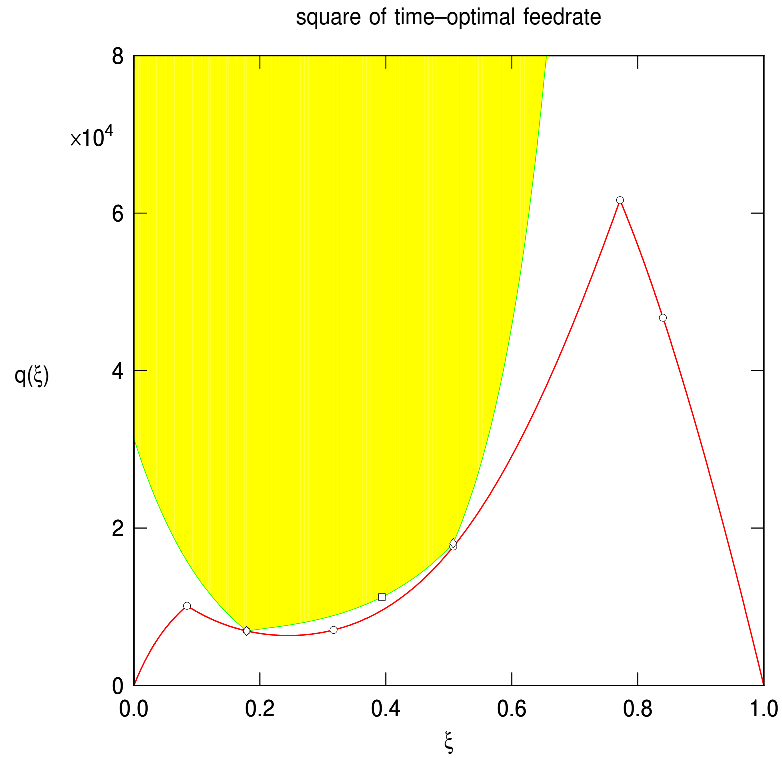
illustrative example #1



The time-optimal feedrate for the "S-shaped curve" on the left involves a total of ten switching points — two of these are critical points on the VLC, three are intersections of $a_{\min}$ and $a_{\max}$ trajectories below the VLC, and the remaining five correspond to changes in the identity of the limiting axis.

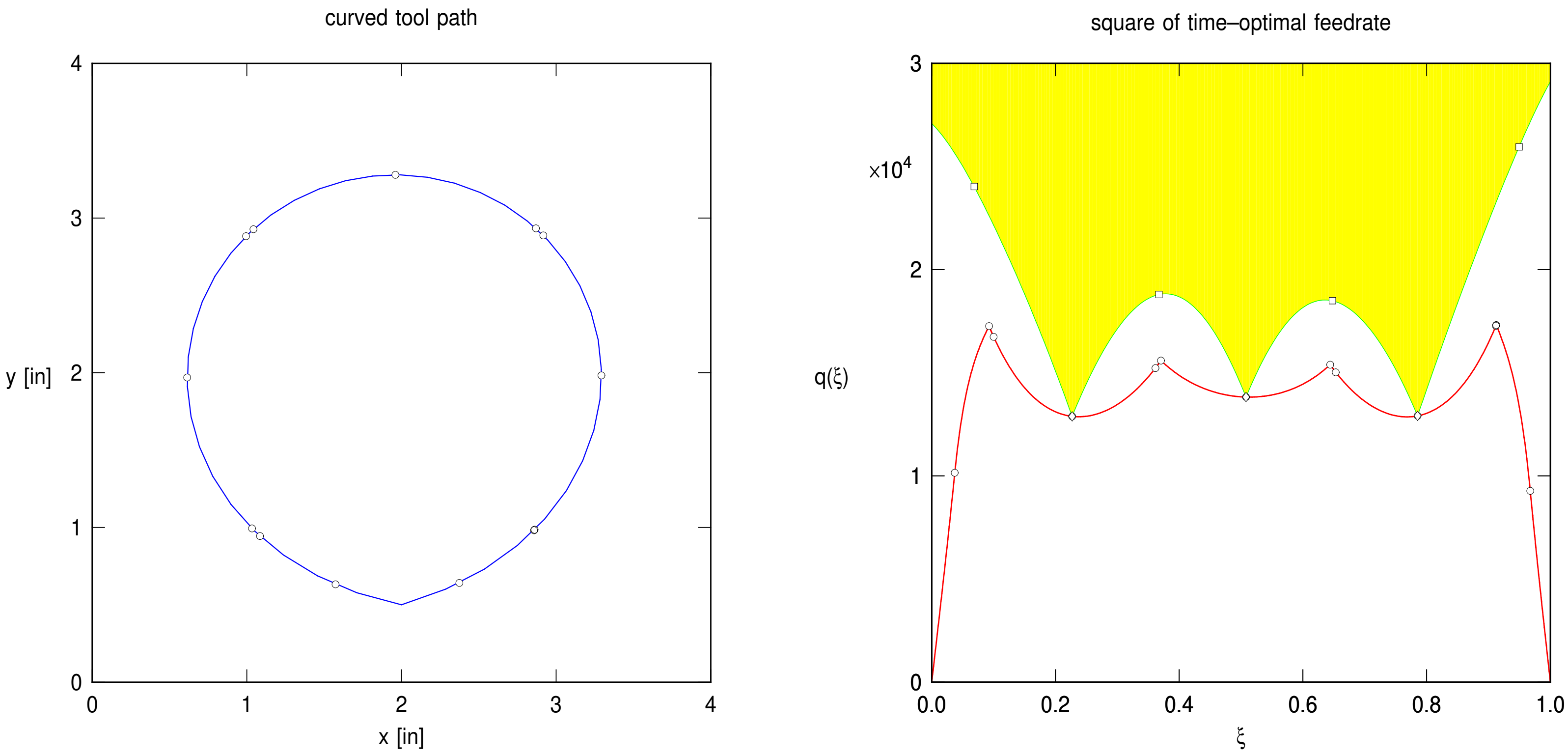
illustrative example #2





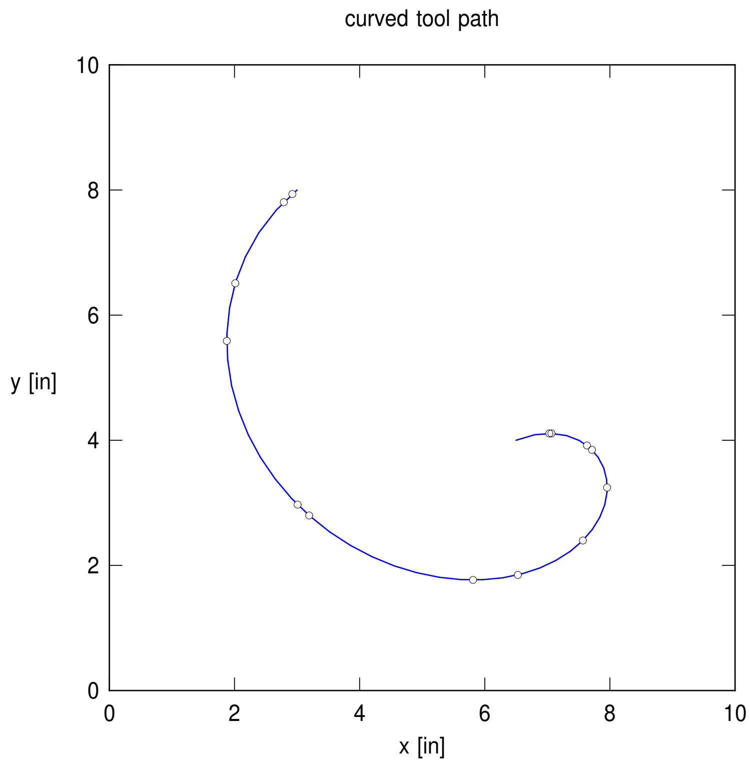
For the "J-curve" on the left, six switching points arise in constructing the time-optimal feedrate. These comprise one critical point on the VLC, two $a_{\min}/a_{\max}$ trajectory intersections, and three limiting-axis-identity changes (one of the latter is close to, but distinct from, a critical point on the VLC).

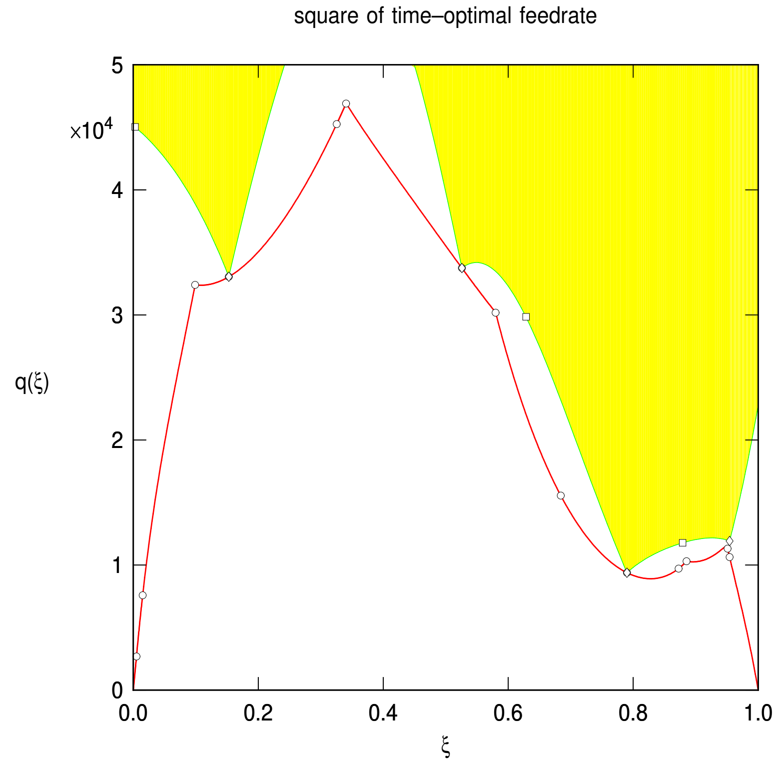
illustrative example #3



A total of twelve switching points occur in the time-optimal feedrate for the closed loop on the left: namely, three critical points on the VLC, four $a_{\min}/a_{\max}$ trajectory intersections, and five limiting-axis-identity changes.

illustrative example #4





The “spiral” on the left incurs fourteen switching points in its time-optimal feedrate function: three critical points on the VLC, five $a_{\min}/a_{\max}$ trajectory intersections, and six limiting-axis-identity changes.

real-time CNC interpolator algorithm

sampling frequency $f = 1024$ Hz, sampling interval $\Delta t = \dfrac{1}{f} \approx 0.001$ sec.

function of CNC interpolator: given curve $\mathbf{r}(\xi)$ and feedrate function $v(\xi)$, perform real-time computation of locations ξ_k at $t_k = k\Delta t,\ k = 0, 1, 2, \ldots$

locations ξ_k are solutions of (note: unknown $=$ upper limit of integration)

$$k\Delta t \, = \int_0^{\xi_k} \frac{\sigma}{v} \, \mathrm{d}\xi \, , \quad k = 0, 1, 2, \ldots$$

“interpolation integral” does not ordinarily admit a closed-form reduction however, for the time-optimal feedrate function (with x as limiting axis)

$$q \;=\; v^2 \;=\; \left(\frac{\sigma}{x'}\right)^2 (C \pm 2Ax)$$

we obtain

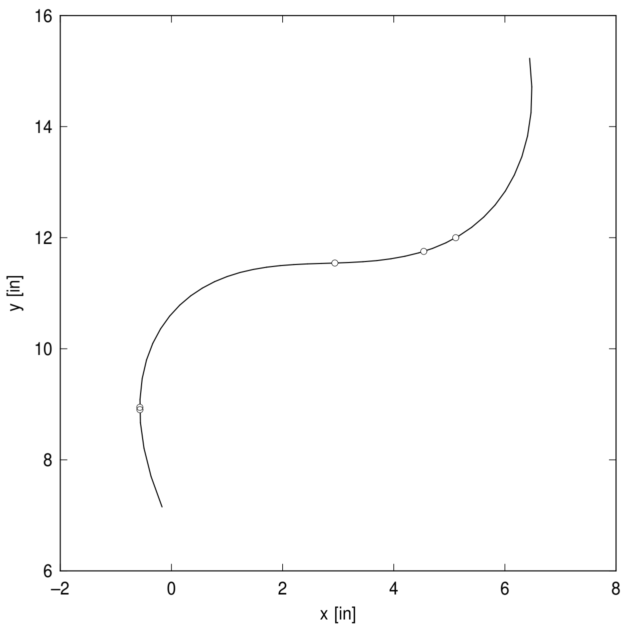
$$k\Delta t \;=\; \int_0^{\xi_k} \frac{|x'|}{\sqrt{C \pm 2Ax}}\,\mathrm{d}\xi \;=\; \frac{\sqrt{C \pm 2A\,x(\xi_k)}}{A} \,+\, K$$

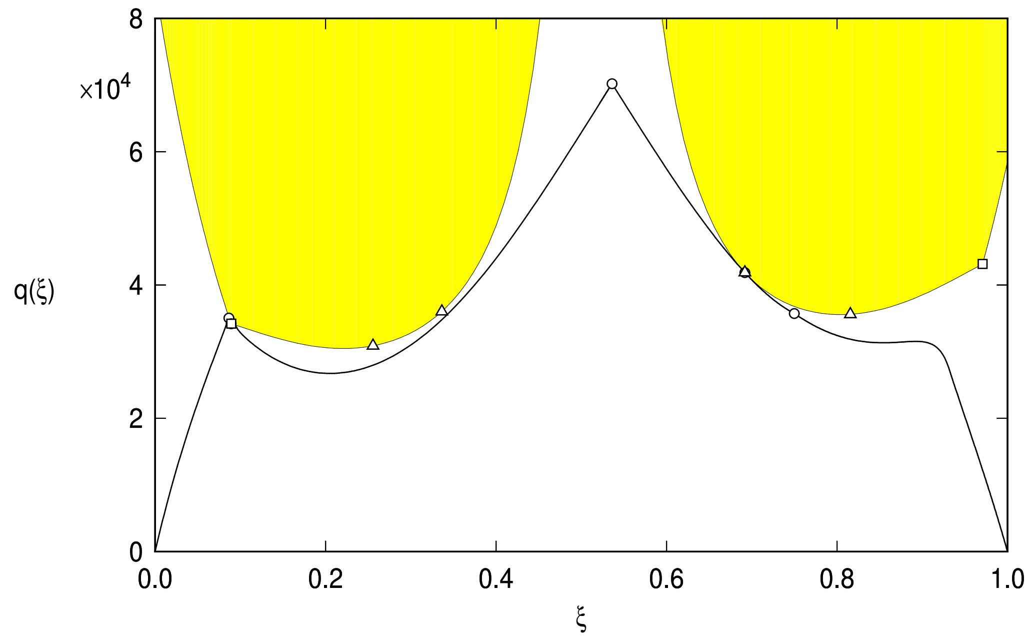
with integration constant K chosen so that $t = 0$ when $\xi = 0$

function on RHS is monotone in ξ, hence equation has unique solution
– obtained to machine precision by a few Newton-Raphson iterations

$\Longrightarrow$ highly accurate and efficient real-time CNC interpolator algorithm

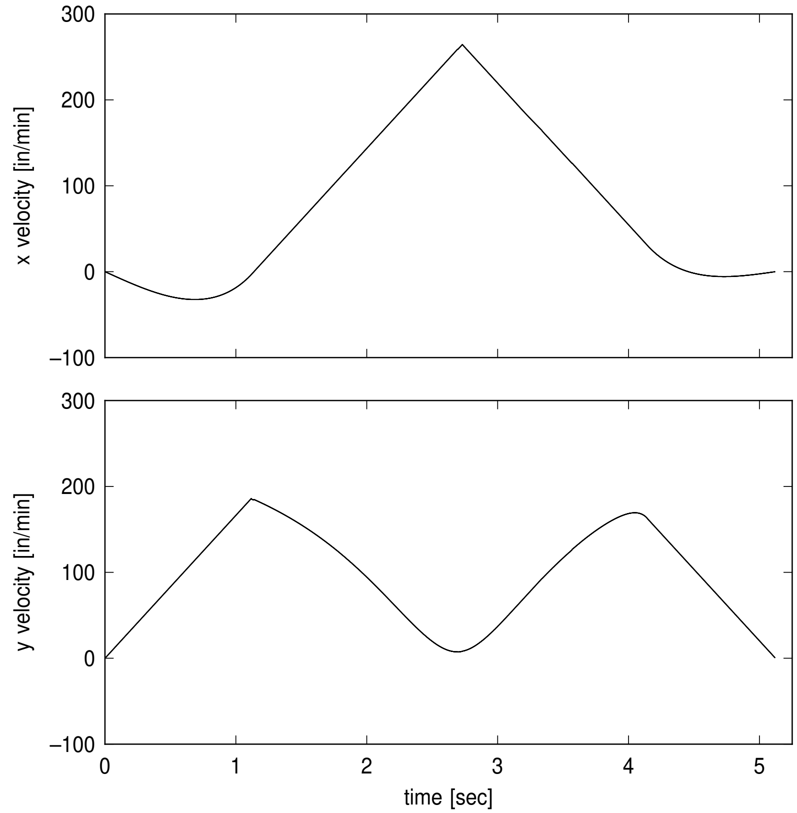
test run data from CNC machine





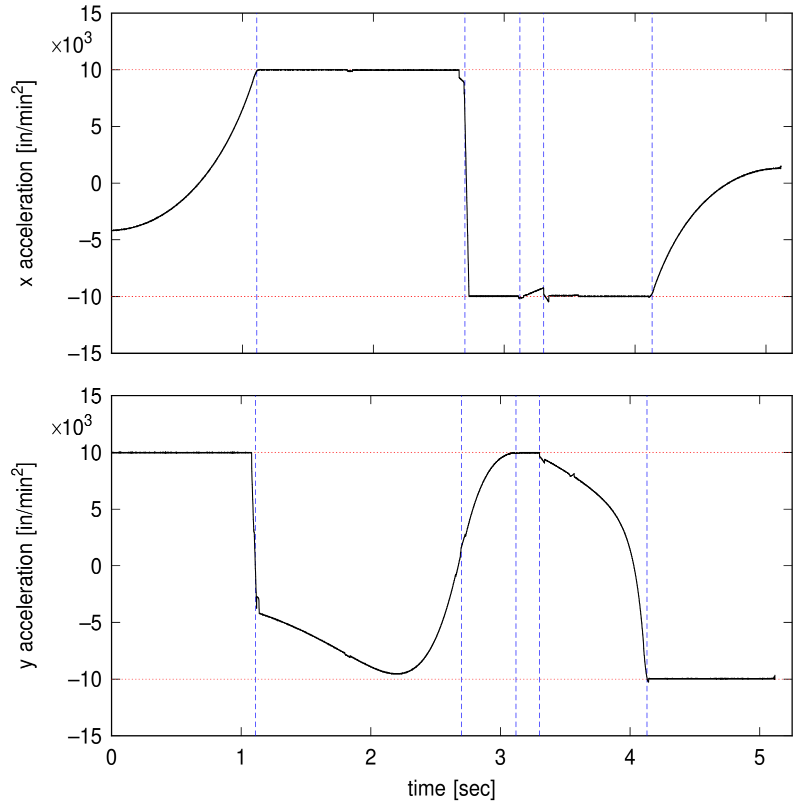
Square of time-optimal feedrate is a piecewise-rational function of ξ with five switching points (one critical and one tangency point on the VLC, and three switching points below VLC). Record real-time positional data from encoders on machine axes – obtain velocity & acceleration components by first- and second-order differencing. $A = 10^4 \text{ in/min}^2$ acceleration bound.

measured axis velocity components



Measured x and y velocity components from first differences of real-time encoder position data (NB: independent variable is time, not parameter ξ).

measured axis acceleration components



Measured x and y acceleration components from second differences of real-time encoder positions (plotted against time, not curve parameter ξ). Dashed red lines show prescribed acceleration bounds $\pm 10^4$ in/min^2, and dashed blue lines show approximate instances of switching points.

closure

- for Cartesian machines with independent axis acceleration bounds, an essentially exact piecewise-rational solution for the time-optimal "bang-bang" feedrate can be computed

- optimal feedrate admits accurate & efficient real-time interpolator, based on analytic reduction of the interpolation integral

- a number of errors & inconsistencies exist in the robotics literature on time-optimal motion along curved paths

- formulate generalizations to systems constrained by axis velocity, as well as acceleration, bounds

- explore automatic smoothing of slope discontinuities in optimal feedrate function, to obtain smooth (C^2) "near-optimal" motions

“science and poetry”

Trace science then, with modesty thy guide;
First strip off all her equipage of pride,
Deduct what is but vanity, or dress,
Or learning’s luxury, or idleness;
Or tricks to show the stretch of human brain,
Mere curious pleasure, or ingenious pain:
Expunge the whole, or lop th’excrescent parts
Of all, our vices have created arts:
Then see how little the remaining sum,
Which served the past, and must the times to come!

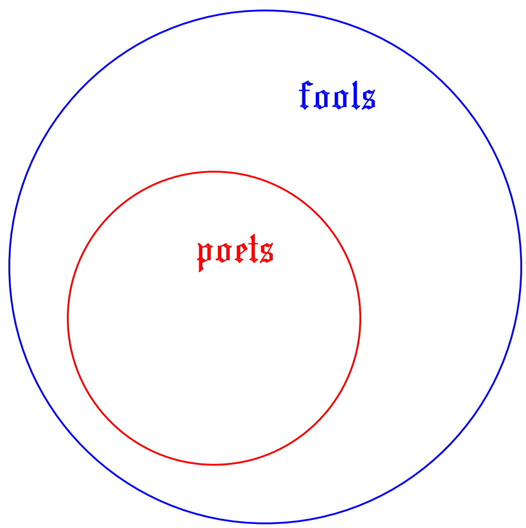
Alexander Pope (1688-1744), *Essay on Man*

"poets and fools"

Sir, I admit your general rule,
That every poet is a fool.

But you yourself may serve to show it,
That every fool is not a poet!

Alexander Pope (1688-1744)



all poets are fools, but not all fools are poets