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All 0-1 Polytopes are Traveling Salesman Polytopes

by

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L.J. Billera * and A. Sarangarajan †

Abstract

We study the facial structure of two important permutation polytopes in $\mathbf{R}^{n^2}$, the *Birkhoff* or *assignment* polytope B_n, defined as the convex hull of all $n \times n$ permutation matrices, and the *asymmetric traveling salesman* polytope T_n, defined as the convex hull of those $n \times n$ permutation matrices corresponding to n-cycles. Using an isomorphism between the face lattice of B_n and the lattice of elementary bipartite graphs, we show, for example, that every pair of vertices of B_n is contained in a cubical face, showing faces of B_n to be fairly special 0-1 polytopes. On the other hand, we show that T_n has *every* 0-1 d-polytope as a face, for $d = \log n$, by showing that every 0-1 d-polytope is the asymmetric traveling salesman polytope of some directed graph with n nodes. The latter class of polytopes is shown to have maximum diameter $\lfloor \frac{n}{3} \rfloor$.

1 *Introduction*

The (asymmetric) traveling salesman problem, to find the shortest (directed) Hamiltonian tour in a complete (directed) graph, is one of the widely studied problems in combinatorial optimization, both for its utility and for the fact that it represents, in a well-defined sense, all hard combinatorial problems. A standard approach to solving this problem is to consider it as a linear

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programming problem over the (asymmetric) traveling salesman polytope, defined as the convex hull of all (directed) Hamiltonian tours, and to use known classes of bounding inequalities (facets) to try to find an optimal tour. (See [4] for a discussion of this and other approaches to this problem.)

The difficulty with this approach is that the facets of these polytopes are not all known, and, it seems, not knowable. We give explicit evidence for this assertion for the asymmetric traveling salesman polytope (ATSP) by showing that, up to a certain dimension, *all* 0-1 polytopes are among its faces. In particular, we show that if P is a 0-1 polytope, then P appears as a face of an ATSP of sufficiently large dimension. The dimension of this ATSP is in general exponential in the dimension of P, and we show that it is not possible to get all 0-1 polytopes in $\mathbf{R}^d$ as faces of an ATSP of a dimension that is polynomial in d. Another way of viewing the main result of this paper is that every 0-1 polytope *is* the $ATSP$ of some directed graph.

In section 2 we introduce the Birkhoff polytope and associate the faces of this polytope with elementary bipartite graphs. This establishes an isomorphism between the face lattice of the Birkhoff polytope and the lattice of elementary bipartite graphs. We use this isomorphism to show that every pair of vertices of the Birkhoff polytope is contained in a cubical face. Section 3 is devoted to proving our main result on the $ATSP$. In section 4 we study the asymmetric TSP of an arbitrary directed graph and give a tight bound on its diameter.

We define some terms that will be used for the rest of this paper. We denote the set $\{1, 2, \ldots, n\}$ by $[n]$. The *symmetric group* of degree n is the set of all permutations of $[n]$. Permutations that are cycles of length k will be called *k-cycles*. If $v = (v_1, v_2, \ldots, v_n) \in \mathbf{R}^n$ then $Supp(v) := \{i : v_i \neq 0\}$ is the *support* of v. A *0-1 polytope* is a polytope whose vertices have coordinates 0 or 1. $K_{n,n}$ is the complete bipartite graph with bipartition $([n], [n])$ and edge set $\{(i,j) : 1 \leq i, j \leq n\}$. If $\mathcal{C} = \{C_1, \ldots, C_d\}$ is an *ordered collection* of permutations $C_1, \ldots, C_d$ and if $S \subseteq [d]$ then $\mathcal{C}(S) := \prod_{s \in S} C_s$; we write $\mathcal{C}[d]$ instead of $\mathcal{C}([d])$. Throughout this paper, $\mathcal{C}$ will denote an ordered collection of disjoint cycles. The graph $\tilde{G}(P)$ of a polytope P is the graph whose nodes are the vertices of P, and which has an edge joining two nodes if and only

if the corresponding vertices in P are adjacent on P. The *diameter* of P is defined as

$$diam(P) := diam(\tilde{G}(P)) = max\{d(u,v) : u,\ v \text{ are nodes of } \tilde{G}(P)\}$$

where $d(u,v)$ is the length of the shortest path between u and v in $\tilde{G}(P)$.

2 *The Birkhoff Polytope*

Let $\mathcal{S}_n$ denote the symmetric group of degree n. Given $\sigma \in \mathcal{S}_n$, we define the corresponding $n \times n$ permutation matrix $X(\sigma) \in \mathbf{R}^{n^2}$ by

$$X(\sigma)_{ij} := \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

We denote by B_n the *Birkhoff polytope* (or the *assignment polytope*) of order n, given by

$$B_n := conv\{X(\sigma) : \sigma \in \mathcal{S}_n\}.$$

It is well known that B_n is an $(n-1)^2$ dimensional polytope with the following inequality description:

$$\begin{aligned} B_n \ = \ & \{x \in \mathbf{R}^{n^2} : x_{ij} \geq 0 \ ; \ 1 \leq i \ , \ j \leq n, \ \sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \ldots, n \\ & \text{and } \sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1, \ldots, n\}. \end{aligned}$$

A detailed study of this polytope is given in [2] (see also [3]). For convenience, we shall often denote a vertex $X(\sigma)$ by σ. With each vertex $\sigma \in B_n$, we associate the bipartite graph $G(\sigma)$ which is the matching on $K_{n,n}$ with the edge set $\{(i, \sigma(i)) : i = 1, \ldots, n\}$. If F is a face of B_n, then $G(F)$ is the subgraph of $K_{n,n}$ which is the union of $G(\sigma)$ over all the vertices $\sigma \in F$. $G(F)$ has the property that every edge of $G(F)$ is in some matching. Such graphs are called *elementary graphs* (see [5]; the definition of elementary graphs given there also requires them to be connected, but we will not require that here). Also if $G \subseteq K_{n,n}$ is an elementary graph, define

$$F(G) := conv\{\sigma : G(\sigma) \subseteq G\} = B_n \cap \{x \in \mathbf{R}^{n^2} : x_{ij} = 0 \text{ if } (i,j) \notin G\}.$$

We see that $F(G)$ is an intersection of facets of B_n and therefore is a face of B_n. Clearly, if F_1, F_2 are faces of B_n, then $F_1 \subseteq F_2$ if and only if $G(F_1) \subseteq G(F_2)$. This yields the following (see [5] for details).

Theorem 2.1 *The face lattice of B_n is isomorphic to the lattice of all elementary subgraphs of $K_{n,n}$ ordered by inclusion.* □

Suppose F is a face of B_n and $G(F)$ has components $G_1, \ldots, G_k$. Suppose G_i is bipartite with bipartition $\{I_i, J_i\}$, so that the sets $I_1, \ldots, I_k$ (and $J_1, \ldots, J_k$) form a partition of $[n]$. Let B_i be the Birkhoff polytope defined on the coordinates $\{x_{kl} : k \in I_i, l \in J_i\}$. Since $G(F)$ is elementary, each G_i is elementary bipartite on $\{I_i, J_i\}$. Hence, $F_i := F(G_i)$ is a face of B_i. Then $F = F_1 \times F_2 \times \cdots \times F_k$, embedded as block diagonal matrices in $\mathbf{R}^{n^2}$ (up to permutation of rows and columns). This follows from the fact that v is a vertex of F if and only if $v = v_1 \times \cdots \times v_k$ where, for each i, v_i is a vertex of F_i for each i.

If $\sigma, \pi \in \mathcal{S}_n$, then $G(\sigma, \pi) := G(\sigma) \cup G(\pi)$ is a union of two matchings which is a set of disjoint cycles and edges. Hence, by the above remark, $F_{\sigma,\pi} := F(G(\sigma, \pi))$ is a k dimensional cube where k is the number of cycles in the graph. If $\sigma^{-1}\pi = \prod_{i=1}^{k} C_i \in \mathcal{S}_n$ where $C_1, \ldots, C_k$ are disjoint cycles in $\mathcal{S}_n$, and if $\mathcal{C} = \{C_1, \ldots, C_k\}$ then the vertices of $F_{\sigma,\pi}$ are given by $\sigma\mathcal{C}(S)$ over all subsets $S \subseteq [k]$. We note that k can be at most $\left\lfloor \frac{n}{2} \right\rfloor$. We shall associate $F_{\sigma,\pi}$ with the unit k-cube with $\sigma\mathcal{C}(S)$ corresponding to the vertex with support S (so σ corresponds to the origin and π to $(1, 1, \ldots, 1)$). For convenience, we shall often denote $F_{\sigma,\pi}$ by $F(\sigma, \mathcal{C})$. This proves the following.

Theorem 2.2 *B_n (or more generally, any face of B_n) has the property that any two of its vertices are contained in a cubical face of dimension at most $\lfloor \frac{n}{2} \rfloor$.* □

We note here that the k-cube $F_{\sigma,\pi}$ is actually a *zonotope* (Minkowski sum of line segments) generated by k mutually orthogonal segments $z_i \in \mathbf{R}^{n^2}$. Each z_i is supported on a cycle of the graph $G(\sigma, \pi)$ and has coordinates $+1$ or -1 alternately for edges of $G(\sigma)$ and $G(\pi)$.

3 *The Asymmetric TSP*

Let

$$\mathcal{T}_n := \{\sigma \in \mathcal{S}_n : \sigma \text{ is a cycle of length } n\} \subset \mathcal{S}_n$$

The *asymmetric TSP* of order n is defined by

$$T_n := conv\{X(\sigma) : \sigma \in \mathcal{T}_n\}$$

so that

$$T_n \subset B_n \subset \mathbf{R}^{n^2}$$

This means that if F is a face of B_n, then $F \cap T_n$ is a face of T_n *induced by* F. We exploit this relationship between the faces of B_n and T_n to derive some results about the faces of T_n.

We first describe the following procedure to generate some lower dimensional faces of T_n:

Let $\sigma \in \mathcal{S}_n$ and $\mathcal{C} = \{C_1, \ldots, C_k\}$ where $C_1, \ldots, C_k \in \mathcal{S}_n$ are disjoint cycles. Then $F(\sigma, \mathcal{C}) \subset B_n$ is a k-cube and if V is the set of $ATSP$ vertices of this cube then $conv(V) = F(\sigma, \mathcal{C}) \cap T_n$ is a face of T_n. Call such a face $F'(\sigma, \mathcal{C})$. We shall identify $F'(\sigma, \mathcal{C})$ with a 0-1 polytope embedded in the k-cube. We note that if $\pi = \sigma\mathcal{C}[k] = \sigma C_1 \cdots C_k$, then

$$F(\sigma, \mathcal{C}) = B_n \cap \{x \in \mathbf{R}^{n^2} : x_{ij} = 0 \text{ if } (i,j) \notin G(\sigma, \pi)\}$$

and hence

$$F'(\sigma, \mathcal{C}) = T_n \cap \{x \in \mathbf{R}^{n^2} : x_{ij} = 0 \text{ if } (i,j) \notin G(\sigma, \pi)\}.$$

Example 1: (Rao [6])
Let $n = 9$, $\sigma = (1,2,3,4,5,6,7,8,9)$ and $\mathcal{C} = \{C_1, C_2, C_3\}$ where $C_1 = (1,7,4)$, $C_2 = (2,8,5)$ and $C_3 = (3,9,6)$. Then $F(\sigma, \mathcal{C}) \subset B_9$ is a 3-cube and the other vertices of this cube are given by

$\sigma_1 = \sigma C_1 = (1,8,9)(2,3,4)(5,6,7) \sim (1,0,0)$
$\sigma_2 = \sigma C_2 = (1,2,9)(3,4,5)(6,7,8) \sim (0,1,0)$

$\sigma_3 = \sigma C_3 = (1,2,3)(4,5,6)(7,8,9) \sim (0,0,1)$
$\sigma_{12} = \sigma C_1 C_2 = (1,8,6,7,5,3,4,2,9) \sim (1,1,0)$
$\sigma_{23} = \sigma C_2 C_3 = (1,2,9,7,8,6,4,5,3) \sim (0,1,1)$
$\sigma_{13} = \sigma C_1 C_3 = (1,8,9,7,5,6,4,2,3) \sim (1,0,1)$
$\sigma_{123} = \sigma C_1 C_2 C_3 = (1,8,6,4,2,9,7,5,3) \sim (1,1,1)$

We note that 5 of the vertices of this cube are $ATSP$ vertices and thus $F'(\sigma, \mathcal{C})$ is a bipyramid over a triangle as shown in Figure 1. □

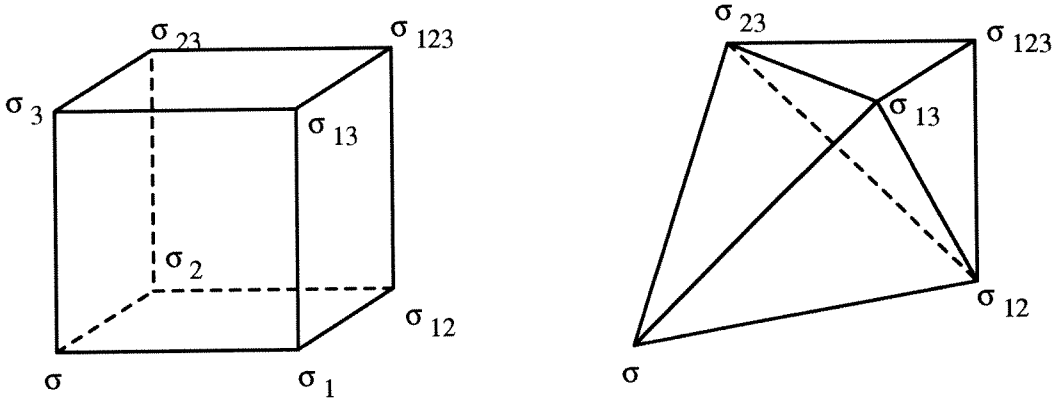

Figure 1: $F(\sigma, \mathcal{C})$ and $F'(\sigma, \mathcal{C})$ for Example 1

It is natural to ask whether any 0-1 polytope can appear as a face of the $ATSP$ in the above manner. Surprisingly, this turns out to be true; the remainder of this section is devoted to proving this assertion.

Let $\mathcal{I}_d$ denote the unit d-cube. We will usually refer to vertices of $\mathcal{I}_d$ by their corresponding supports which are subsets of $[d]$. If V is a subset of vertices of $\mathcal{I}_d$, then by $\mathcal{I}_d - V$ we mean the convex hull of the vertices of $\mathcal{I}_d$ that are not in V.

Proposition 3.1 *$\mathcal{I}_d$ is a face of T_n for $n = 3d$.*

Proof: Let $\sigma = (1, \ldots, n)$ and $\mathcal{C} = \{(1,2,3), (4,5,6), \ldots, (n-2, n-1, n)\}$. Then for any $S \subseteq [d]$, $\sigma\mathcal{C}(S) = (a_1, \ldots, a_n) \in \mathcal{T}_n$ where

$$a_{3i-2}, a_{3i-1}, a_{3i} = \begin{cases} 3i-2, 3i, 3i-1 & \text{if } i \in S \\ 3i-2, 3i-1, 3i & \text{otherwise} \end{cases}$$

This shows that $F'(\sigma,\mathcal{C}) = F(\sigma,\mathcal{C}) = \mathcal{I}_d$. □

Next we show that $\mathcal{I}_d - S$ is also a face of the T_{3d} for any vertex $S \in \mathcal{I}_d$. We begin with $S = [d]$.

Proposition 3.2 *Let d be a positive integer, $n = 3d$, $\sigma = (1,2,\ldots,n)$ and let*

$$\mathcal{C} = \{C_1, C_2, \ldots, C_d\} = \{(1,3,5),(4,6,8),\ldots,(n-2,n,2)\}$$

(i.e. $C_i = (3i-2, 3i, 3i+2)$ for $i < d$ and $C_d = (n-2,n,2)$). Then

$$F'(\sigma,\mathcal{C}) = \mathcal{I}_d - [d]$$

Proof: Since

$$\sigma\mathcal{C}[d] = (1,4,7,\ldots,n-2)(3,6,9,\ldots,n)(n-1,n-4,\ldots,2) \notin \mathcal{T}_n$$

this implies that $[d] \notin F'(\sigma,\mathcal{C})$.

Now we need to show that all the other vertices of the cube $F(\sigma,\mathcal{C})$ are also vertices of T_n. For the rest of the proof, the numbers we indicate are modulo n. Let $\mathcal{C}[i,j] = C_iC_{i+1}\cdots C_j$ (if $i > j$, then set $\mathcal{C}[i,j] = C_i\cdots C_dC_1\cdots C_j$). Let α_{ij} denote the sequence

$$3i{-}2, 3i{+}1, 3i{+}4, \ldots, 3j{+}1, 3j{+}2, 3j{-}1, 3j{-}4, \ldots, 3i{-}1, 3i, 3i{+}3, \ldots, 3j{+}3$$

that is α_{ij} increases from $3i-2$ to $3j+1$ in steps of 3 then decreases from $3j+2$ to $3i-1$ in steps of 3 and finally increases from $3i$ to $3j+3$ in steps of 3. Then

$$\sigma\mathcal{C}[i,j] = (\alpha_{ij}, 3j+4, 3j+5, \ldots, 3i-3) \in \mathcal{T}_n,$$

that is $\sigma\mathcal{C}[i,j]$ is a cyclic permutation that differs from σ by inserting α_{ij} in the interval $[3i-2, 3j+3]$.

Let $S \subset [d]$ be a proper subset of $[d]$. We write

$$\mathcal{C}(S) = \mathcal{C}[i_1,j_1]\mathcal{C}[i_2,j_2]\cdots\mathcal{C}[i_k,j_k]$$

where $i_1 < i_2 < \cdots < i_k$ and the above representation for $\mathcal{C}(S)$ is minimal (i.e., $\mathcal{C}[i_r, j_r]\mathcal{C}[i_{r+1}, j_{r+1}] \neq \mathcal{C}[i_r j_{r+1}]$ for $r = 1, \ldots, k$).

Then, $\sigma\mathcal{C}(S)$ is a cyclic permutation that differs from σ by inserting the sequences $\alpha_{i_r j_r}$ in the interval $[3i_r - 2, 3j_r + 3]$ for $r = 1, \ldots, k$, hence $\sigma\mathcal{C}(S) \in \mathcal{T}_n$ and we are done. □

Proposition 3.2 generalizes to the following result.

Proposition 3.3 *If $S \subseteq [d]$, $S \neq \emptyset$, $n = 3d$ and $\sigma = (1, \ldots, n)$ then there is an ordered collection of disjoint cycles $\bar{\mathcal{C}} = \{\bar{C}_1, \ldots, \bar{C}_d\} \subset \mathcal{S}_n$ such that $F'(\sigma, \bar{\mathcal{C}}) = \mathcal{I}_d - S$.*

Proof: Let $\mathcal{C} = \{C_1, C_2 \ldots, C_d\}$ be the ordered set as defined in the last proposition. Let $\bar{S} = [d] - S$. We show that there exists a $\pi \in \mathcal{S}_n$ such that

$$\bar{C}_i = \begin{cases} \pi^{-1} C_i \pi & i \in S \\ \pi^{-1} C_i^{-1} \pi & i \in \bar{S}. \end{cases} \tag{1}$$

Since S is non empty, $\bar{S} \neq [d]$ and hence $\hat{\sigma} := \sigma\mathcal{C}(\bar{S}) \in \mathcal{T}_n$. Let $\hat{\mathcal{C}} = \{\hat{C}_1, \ldots, \hat{C}_d\}$ where

$$\hat{C}_i = \begin{cases} C_i & i \in S \\ C_i^{-1} & i \in \bar{S}. \end{cases}$$

Then $F'(\hat{\sigma}, \hat{\mathcal{C}}) = \mathcal{I}_d - S$. Since σ and $\hat{\sigma}$ are both cycles of length n they are conjugate, that is, there is $\pi \in \mathcal{S}_n$ such that $\sigma = \pi^{-1}\hat{\sigma}\pi = \pi^{-1}\sigma\mathcal{C}(\bar{S})\pi$. Conjugating the cycles $\hat{C}_i$ by π to get $\bar{C}_i$, we see that for any subset $R \subseteq [d]$

$$\sigma\bar{\mathcal{C}}(R) = \pi^{-1}\hat{\sigma}\pi(\pi^{-1}\hat{\mathcal{C}}(R)\pi) = \pi^{-1}\hat{\sigma}\hat{\mathcal{C}}(R)\pi$$

which has the same cycle structure as $\hat{\sigma}\hat{\mathcal{C}}(R)$. Thus we must have $F'(\sigma, \bar{\mathcal{C}}) = \mathcal{I}_d - S$. □

Example 2: Let $v = (1, 0, 0, 0)$, $\sigma = (1, \ldots, 12)$. We find $\bar{\mathcal{C}} = \{\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{C}_4\}$ such that $F'(\sigma, \bar{\mathcal{C}}) = \mathcal{I}_4 - v$.

By Proposition 3.2, if

$$\mathcal{C} = \{C_1, C_2, C_3, C_4\} = \{(1, 3, 5), (4, 6, 8), (7, 9, 11), (10, 12, 2)\}$$

then $F'(\sigma,\mathcal{C}) = \mathcal{I}_4 - (1,1,1,1)$. Let $S = \{1\} = Supp(v)$, $\bar{S} = \{2,3,4\}$,
$\hat{\sigma} = \sigma\mathcal{C}(\bar{S}) = \sigma C_2C_3C_4 = (1,2,11,8,5,6,9,12,3,4,7,10) =: (a_1,a_2,\ldots,a_{12})$
with $a_1 = 1$. Setting $\pi(i) = a_i$ for $i = 1,\ldots,12$ we get
$\pi = (3,11,7,9)(4,8,12,10)$ and $\sigma = \pi^{-1}\hat{\sigma}\pi$.
Then

$$\begin{aligned}
\bar{C}_1 &= \pi^{-1}C_1\pi = (1,9,5)\\
\bar{C}_2 &= \pi^{-1}C_2^{-1}\pi = (4,6,10)\\
\bar{C}_3 &= \pi^{-1}C_3^{-1}\pi = (3,7,11)\\
\bar{C}_4 &= \pi^{-1}C_4^{-1}\pi = (2,8,12)
\end{aligned}$$

Hence

$$\bar{\mathcal{C}} = \{(1,9,5),(4,6,10),(3,7,11),(2,8,12)\}$$

Using the same method, one can show that if

$$\bar{\mathcal{C}} = \{(2,4,12),(1,5,7),(6,8,10),(3,9,11)\},$$

then $F'(\sigma,\bar{\mathcal{C}}) = \mathcal{I}_4 - (0,1,1,1)$. □

Example 2 shows how we can find $\bar{\mathcal{C}}$ of Proposition 3.3. If $\hat{\sigma} := \sigma\mathcal{C}(\bar{S}) = (a_1,\ldots,a_n)$ then setting $\pi(i) = a_i$ for $i = 1,\ldots,n$ we would get $\sigma = \pi^{-1}\hat{\sigma}\pi$. This defines $\bar{C}_i$ (by (1)) and hence $\bar{\mathcal{C}}$.

Now we tackle the general case of removing any set of vertices. Let $P \subset \mathbf{R}^d$ be a 0-1 polytope. Assume without loss of generality that $0 \in P$ and let $V := \{S_1,\ldots,S_k\}$ be the set of vertices of $\mathcal{I}_d$ that are not in P (i.e. $P = \mathcal{I}_d - V$). By Proposition 3.3, we can find $\mathcal{C}_i = \{C_{i1},\ldots,C_{id}\}$ such that $F'(\sigma,\mathcal{C}_i) = \mathcal{I}_d - S_i$ for $i = 1,\ldots,k$, where $n = 3d$, $\sigma = (1,\ldots,n)$ and each C_{ij} is a 3-cycle. The idea is to concatenate the cycles $C_{1j},\ldots,C_{kj}$ for $j = 1,\ldots,d$ to eliminate $S_1,\ldots,S_k$. For this, the cycles must be defined on distinct sets. So, assume that $\mathcal{C}_i$ is defined on $\{(i-1)n+1,\ldots,in\}$ and let $\sigma_i = ((i-1)n+1,\ldots,in)$ so that $F'(\sigma_i,\mathcal{C}_i) = \mathcal{I}_d - S_i$ for $i = 1,\ldots,k$ (i.e. $\sigma_i\mathcal{C}_i(S)$ is an n-cycle for all $S \subseteq [d]$ except when $S = S_i$).

Proposition 3.4 *There is an integer N and an ordered set of disjoint cycles $\mathcal{Y}' = \{Y_1',\ldots,Y_d'\} \subset \mathcal{S}_N$ such that if $\sigma' = (1,\ldots,N)$ then $F'(\sigma',\mathcal{Y}') = P$.*

Proof: Define the permutations $Y_j = C_{1j}C_{2j}\cdots C_{kj} \in \mathcal{S}_{kn}$ for $j = 1, \ldots, d$ and let $\mathcal{Y} = \{Y_1, \ldots, Y_d\}$, and $N = kn + (k+1)d$. The following array illustrates how Y_j is defined.

$$\begin{array}{cccccc} \mathcal{C}_1 & C_{11} & \cdots & C_{1j} & \cdots & C_{1d} \\ \vdots & \vdots & & \vdots & & \vdots \\ \mathcal{C}_k & C_{k1} & \cdots & C_{kj} & \cdots & C_{kd} \\ & & & Y_j & & \end{array} \qquad \begin{array}{c} F'(\sigma_1, \mathcal{C}_1) = \mathcal{I}_d - S_1 \\ \vdots \\ F'(\sigma_k, \mathcal{C}_k) = \mathcal{I}_d - S_k \end{array}$$

We first show that if $S \subseteq [d]$ then

$$\sigma'\mathcal{Y}(S) \in \mathcal{T}_N \Leftrightarrow \sigma_r\mathcal{C}_r(S) \text{ is an n-cycle for } r = 1, \ldots, k \tag{2}$$

To see this, note that

$$\sigma'\mathcal{Y}(S) = \sigma' \prod_{s\in S} Y_s = \sigma' \prod_{s\in S} (C_{1s}\cdots C_{ks}) = \sigma'\mathcal{C}_1(S)\cdots\mathcal{C}_k(S).$$

Suppose $\sigma_r\mathcal{C}_r(S) = (x_{(r-1)n+1}, \ldots, x_{rn})$ is an n-cycle for $r = 1, \ldots, k$ with $x_{(r-1)n+1} = (r-1)n + 1$. Then it follows that

$$\sigma'\mathcal{Y}(S) = (x_1, x_2, \ldots, x_{kn}, kn+1, kn+2, \ldots, N) \in \mathcal{T}_N$$

since $\sigma'\mathcal{Y}(S)(l) = \sigma'\mathcal{C}_r(S)(l) = \sigma_r\mathcal{C}_r(S)(l)$ if $(r-1)n < l \leq rn$, $l \neq x_{rn}$ and $\sigma'\mathcal{Y}(S)(x_{rn}) = \sigma'(rn) = rn + 1 = x_{rn+1}$ for $r = 1, \ldots, k$.

Conversely, suppose that for some r, $\sigma_r\mathcal{C}_r(S) = C'\pi$ is not an n-cycle where C' is a cycle not involving $(r-1)n+1$. Then, as above $\sigma'\mathcal{Y}(S)$ has C' as a cycle, so is not in $\mathcal{T}_N$ proving (2).

It follows from (2) that $\sigma'\mathcal{Y}(S) \in \mathcal{T}_N$ if and only if $S \notin V$, i.e., if and only if S is a vertex of P. We now modify the permutations $Y_1, \ldots, Y_d$ to define *cycles* $Y_1', \ldots, Y_d'$ such that $\sigma'\mathcal{Y}(S) \in \mathcal{T}_N \Leftrightarrow \sigma'\mathcal{Y}'(S) \in \mathcal{T}_N$ for all $S \subseteq [d]$.

Suppose the 3-cycle $C_{ij} = (a_{ij}, b_{ij}, c_{ij})$ and let δ_{ij} denote the sequence a_{ij}, b_{ij}, c_{ij}. For $j = 1, \ldots, d$ define

$$\begin{aligned} Y_j' &= (r_j + k + 1, \delta_{kj}, r_j + k, \delta_{(k-1)j}, \ldots, r_j + 2, \delta_{1j}, r_j + 1) \\ &= (r_j + k + 1, a_{kj}, b_{kj}, c_{kj}, r_j + k, \ldots, r_j + 2, a_{1j}, b_{1j}, c_{1j}, r_j + 1). \end{aligned}$$

where $r_j = kn + (j-1)(k+1)$.

Let $S \subseteq [d]$. Let $\pi = \sigma'\mathcal{Y}(S)$ and $\pi' = \sigma'\mathcal{Y}'(S)$ and write π and π' as products of disjoint cycles. Noting that $Y_j = (a_{kj}, b_{kj}, c_{kj}) \cdots (a_{1j}, b_{1j}, c_{1j})$, we see that for $i = 1, \ldots, k$ and $j \in S$, the sequence $c_{ij}, a_{ij}+1$ in π is replaced by the sequence $c_{ij}, r_j+i+1, a_{ij}+1$ in π' and the sequence $r_j+1, \ldots, r_j+k+2$ in π is replaced by r_j+1, r_j+k+2 in π'. Thus if π is an N-cycle then π' is a rearranged N-cycle and conversely. Therefore,

$$\pi' = \sigma'\mathcal{Y}(S) \in \mathcal{T}_N \Leftrightarrow \pi = \sigma'\mathcal{Y}'(S) \in \mathcal{T}_N \tag{3}$$

It follows from (2) and (3) that $F'(\sigma', \mathcal{Y}') = P$ and $N = kn + (k+1)d = 3kd + (k+1)d = (4k+1)d$. □

The following is a direct consequence of Proposition 3.4.

Theorem 3.1 *If $P \subset \mathbf{R}^d$ is a 0-1 polytope with $2^d - k$ vertices, $k > 1$, then P appears as a face of T_n for $n \geq (4k+1)d$.* □

Example 3: Let $P = \mathcal{I}_4 - \{v_1, v_2\}$ where $v_1 = (1,0,0,0)$, $v_2 = (0,1,1,1)$.

In Example 2, we found $\mathcal{C}_1$ and $\mathcal{C}_2$ such that if $\sigma = (1, \ldots, 12)$ then $F'(\sigma, \mathcal{C}_1) = \mathcal{I}_4 - v_1$ and $F'(\sigma, \mathcal{C}_2) = \mathcal{I}_4 - v_2$. Now we add 12 to every number in $\mathcal{C}_2$ to get the permutations in different sets. Let

$$\begin{aligned} \sigma_1 &= (1, \ldots, 12) \ , \ \mathcal{C}_1 = \{(1,9,5), (4,6,10), (3,7,11), (2,8,12)\} \\ \sigma_2 &= (13, \ldots, 24) \ , \ \mathcal{C}_2 = \{(14,16,24), (13,17,19), (18,20,22), (15,21,23)\} \end{aligned}$$

so that

$$F'(\sigma_1, \mathcal{C}_1) = \mathcal{I}_4 - v_1 \text{ and } F'(\sigma_2, \mathcal{C}_2) = \mathcal{I}_4 - v_2.$$

Now define

$$\begin{aligned} Y_1 &= (1,9,5)(14,16,24) \ , \ Y_1' = (\underline{27}, 14, 16, 24, \underline{26}, 1, 9, 5, \underline{25}) \\ Y_2 &= (4,6,10)(13,17,19) \ , \ Y_2' = (\underline{30}, 13, 17, 19, \underline{29}, 4, 6, 10, \underline{28}) \\ Y_3 &= (3,7,11)(18,20,22) \ , \ Y_3' = (\underline{33}, 18, 20, 22, \underline{32}, 3, 7, 11, \underline{31}) \\ Y_4 &= (2,8,12)(15,21,23) \ , \ Y_4' = (\underline{36}, 15, 21, 23, \underline{35}, 2, 8, 12, \underline{34}) \end{aligned}$$

where we have underlined the new elements in each Y_i'. Let $\sigma' = (1, \ldots, 36)$ and $\mathcal{Y}' = \{Y_1', Y_2', Y_3', Y_4'\}$. Then for example

$$\begin{aligned}\sigma' Y_3 &= (1,2,3,8,9,10,\underline{11,4},5,6,7,12,13,\ldots,18,21,\underline{22,19},20,\\ &\quad 23,24,\ldots,\underline{31,32,33,34},35,36) \in \mathcal{T}_{36}\end{aligned}$$

while

$$\begin{aligned}\sigma' Y_3' &= (1,2,3,8,9,10,\underline{11,32,4},5,6,7,12,13,\ldots,18,21,\underline{22,33,19},20,\\ &\quad 23,24,\ldots,\underline{31,34},35,36) \in \mathcal{T}_{36}\end{aligned}$$

and

$$\sigma' Y_2 Y_3 Y_4 = (14,15,\underline{22,19})\cdots \notin \mathcal{T}_{36}$$

while

$$\sigma' Y_2' Y_3' Y_4' = (14,15,\underline{22,33,19,30})\cdots \notin \mathcal{T}_{36}$$

where we have underlined the places where the permutations differ. Similarly multiplying σ' with all subsets of cycles of $\mathcal{Y}'$ we get $F'(\sigma', \mathcal{Y}') = P$. □

Remarks:

(1) The above result is not valid for B_n since B_n has the property that every pair of its vertices is contained in a cubical face. For instance, the bipyramid over a triangle (of Example 1), cannot be a face of B_n.

(2) A natural question to ask is whether the bound for n can be improved. We ask this question in two different forms. Given d, is there $n \approx d^k$ such that

(i) All 0-1 polytopes in $\mathbf{R}^d$ appear as faces of T_n?

The answer is *no* for the following reason:
The number of 0-1 polytopes in $\mathbf{R}^d$ is 2^{2^d}, since $\mathcal{I}_d$ has 2^d vertices and the convex hull of any subset of these vertices is a 0-1 polytope. If f_i is the number of i-dimensional faces of T_n, then

$$f_i \leq \binom{(n-1)!}{i+1} \leq (n-1)!^{i+1} \leq n^{(i+1)n} \leq n^{n^3} \quad \text{since } i < n^2 - 1$$

Hence, if f is the total number of faces of T_n, then

$$f \leq n^2 n^{n^3} = d^{2k} d^{kd^{3k}} \text{ if } n = d^k.$$

Then $\log(f) \leq 2k \log(d) + kd^{3k} \log(d)$ which is a polynomial in d, whereas $\log(2^{2^d}) = 2^d \log(2)$ is an exponential in d. Hence, n cannot be a polynomial in d.

(ii) All (combinatorially) distinct 0-1 polytopes in $\mathbf{R}^d$ appear as faces of T_n?

This question remains open as we do not have a good estimate of the number of distinct 0-1 polytopes in $\mathbf{R}^d$.

(3) Theorem 3.1 does not hold for all "hard" 0-1 problems:

For example, the polytope associated with the quadratic assignment problem cannot have every 0-1 polytope among its faces since it is *neighborly*, i.e., every pair of its vertices are adjacent. This is so since, choosing a hyperplane containing any pair of vertices of B_n, and no others, its normal can be used (by forming a tensor product with itself) to define a supporting hyperplane to the desired edge of the quadratic assignment polytope. See [1; 2.2.3] for some background. (This fact and its proof was pointed out to us by A.I. Barvinok.)

4 *The Asymmetric TSP of a Directed Graph*

For the rest of this section D will be a directed graph on the node set $[n]$. A Hamiltonian tour $< i_1, \ldots, i_n >$ in D (i.e. a tour comprising the edges $(i_1, i_2), (i_2, i_3), \ldots, (i_n, i_1)$) corresponds to the cyclic permutation $\sigma = (i_1, \ldots, i_n) \in \mathcal{T}_n$ and hence to the permutation matrix $X(\sigma)$.

The asymmetric TSP of the graph D is given by

$$T_D := conv\{X(\sigma) | \, \sigma \text{ corresponds to a tour in } D\}$$

If D is the complete directed graph, then $T_D = T_n$ as defined in the last section. Hence, for a general directed graph D

$$T_D = T_n \cap \{x \in \mathbf{R}^{n^2} | x_{ij} = 0 \text{ whenever } (i,j) \notin D\} \tag{4}$$

Since the nonnegativity constraints define facets of T_n, this shows that T_D is a face of T_n.

Let $\sigma, \pi \in \mathcal{S}_n$ be permutations with no fixed points (i.e. for each $i \in [n]$, $\sigma(i) \neq i$ and $\pi(i) \neq i$). Let $D(\sigma, \pi)$ be the directed graph with the edge set $\{(i,j) : \sigma(i) = j \text{ or } \pi(i) = j\}$.

Lemma 4.1 *Suppose* $\mathcal{C} = \{C_1, \ldots, C_k\}$ *where* $\sigma^{-1}\pi = C_1 \cdots C_k$. *Then* $F'(\sigma, \mathcal{C}) = T_{D(\sigma,\pi)}$.

Proof: We have seen earlier that if $G(\sigma, \pi)$ is the bipartite graph which is the union of the matchings corresponding to σ and π then the face

$$F(\sigma, \mathcal{C}) = B_n \cap \{x \in \mathbf{R}^{n^2} : x_{ij} = 0 \text{ if } (i,j) \notin G(\sigma, \pi)\}$$

is a k-cube and

$$F'(\sigma, \mathcal{C}) = F(\sigma, \mathcal{C}) \cap T_n.$$

Since the edge $(i,j) \notin G(\sigma, \pi)$ if and only if the *directed* edge $(i,j) \notin D(\sigma, \pi)$, it follows from (4) that $F'(\sigma, \mathcal{C}) = T_{D(\sigma,\pi)}$. □

We showed that any 0-1 polytope appeared as a face of the $ATSP$. In proving this result we saw that the face was of the form $F'(\sigma, \mathcal{C})$ where σ was our generic cycle $(1, \ldots, n)$. The following theorem is therefore a direct consequence of Lemma 4.1:

Theorem 4.1 *Every 0-1 polytope in* $\mathbf{R}^d$ *is the asymmetric TSP of some directed graph. In fact, if the polytope has a pair of diametrically opposite vertices, (vertices with disjoint supports such that the union of their supports is* $[d]$*) then the graph is the union of two tours. Otherwise the graph is the union of a tour and disjoint cycles that cover the graph.* □

Finally, we obtain some bounds on the diameter of T_D. We need the following preliminary results:

Lemma 4.2 *Let* $\sigma, \pi \in \mathcal{T}_n$ *correspond to tours in* D. *If* $\sigma^{-1}\pi = C_1 \cdots C_k$ *and* $\mathcal{C} = \{C_1, \ldots, C_k\}$, *then* $F'(\sigma, \mathcal{C})$ *is a face of* T_D.

Proof: Since $F'(\sigma,\mathcal{C}) = T_{D(\sigma,\pi)} \subseteq T_D \subseteq T_n$ and $F'(\sigma,\mathcal{C})$ is a face of T_n, it follows that $F'(\sigma,\mathcal{C})$ is a face of T_D, proving the lemma. □

The following is probably a well known result:

Proposition 4.1 *Let $P \subset \mathbf{R}^d$ be a 0-1 polytope and let v_1, v_2 be vertices of P such that*

(i) $Supp(v_1) \subset Supp(v_2)$ and

(ii) If v is a vertex of P such that $Supp(v_1) \subseteq Supp(v) \subseteq Supp(v_2)$, then $v = v_1$ or $v = v_2$.

Then v_1 and v_2 are adjacent on P.

Proof: Consider the face F of $\mathcal{I}_d$ spanned by the vertices v_1 and v_2. The vertices of F are all the vertices $v \in \mathcal{I}_d$ such that $Supp(v_1) \subseteq Supp(v) \subseteq Supp(v_2)$. By our assumption, the only vertices of P that lie on F are v_1 and v_2. Hence $F \cap P = conv\{v_1, v_2\}$, i.e. v_1 is adjacent to v_2 on P. □

Theorem 4.2 *The maximum diameter of T_D over all directed graphs D on n nodes is $\left\lfloor \frac{n}{3} \right\rfloor$.*

Proof: Define

$$\Theta(n) := max\{diam(T_D) : D \text{ is a directed graph on } [n]\}.$$

Let $d = \lfloor \frac{n}{3} \rfloor$ and let D be any directed graph on $[n]$. We first show that $diam(T_D) \leq d$. Let σ, $\pi \in \mathcal{T}_n$ be vertices of T_D. Let $\sigma^{-1}\pi = C_1C_2\cdots C_k$ and $\mathcal{C} = \{C_1, \ldots, C_k\}$. Let $\sigma_i = \sigma C_1 \cdots C_i$ and $\sigma_0 = \sigma$.

Choose $0 = i_0 < i_1 < \cdots < i_m = k$ so that

$$\{i_1, \ldots, i_m\} = \{j \in [k] : \sigma_j \in \mathcal{T}_n\}.$$

We assume that the cycles $C_1, \ldots, C_k$ are arranged in such a way that if $S \subseteq [k]$ such that $[i_j] \subseteq S \subseteq [i_{j+1}]$ and $\sigma\mathcal{C}(S) \in \mathcal{T}_n$ then $S = [i_j]$ or $S = [i_{j+1}]$. Then by Proposition 4.1, the vertices σ_{i_j} and $\sigma_{i_{j+1}}$ are adjacent on $F'(\sigma,\mathcal{C})$ (and hence on T_D since $F'(\sigma,\mathcal{C})$ is a face of T_D by Lemma 4.2) for $j = 0, 1, \ldots, m-1$.

Since $\sigma_{i_j}^{-1}\sigma_{i_{j+1}} = C_{i_j+1}\cdots C_{i_{j+1}}$ is an even permutation, it follows that for $j = 0, \ldots, m-1$

$$l_{i_j+1} + \cdots + l_{i_{j+1}} \geq 3$$

where l_i is the length of the cycle C_i. Adding these m inequalities, we get

$$3m \leq l_1 + l_2 + \cdots + l_k \leq n$$

i.e. $m \leq d$. We have exhibited a path from σ to π of length at most d which shows that $diam(T_D) \leq d$. This implies that $\Theta(n) \leq d$.

To show that $\Theta(n) = d$, we have to find a find a directed graph D such that $diam(T_D) = d$. Let $\sigma = (1, \ldots, n)$, $\mathcal{C} = \{C_1, \ldots, C_d\}$ where $C_i = (3i-2, 3i-1, 3i)$ for $i = 1, \ldots, d$. Let $\pi = \sigma C_1 \cdots C_d$. Then, $T_{D(\sigma,\pi)} = F'(\sigma, \mathcal{C})$ is a d-cube by Proposition 3.1, and hence has diameter d. □

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