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All Normal Extensions of S5-squared Are Finitely Axiomatizable

Abstract. We prove that every normal extension of the bi-modal system $\mathbf{S5}^2$ is finitely axiomatizable and that every proper normal extension has NP-complete satisfiability problem.

Keywords: modal logic, finite axiomatization, NP-complete, better-quasi-ordering

1. Introduction

Recall that the language of $\mathbf{S5}^2$ is the propositional language based on a fixed countably infinite set of propositional variables and equipped with the two modal operators \Box_1 and \Box_2 . For a formula φ we let $\Diamond_i\varphi$ abbreviate $\neg\Box_i\neg\varphi$ for $i = 1, 2$. We recall that $\mathbf{S5}^2$ is the smallest set of formulas containing all substitution instances of the following axiom schemas, for $i = 1, 2$:

- 1) All tautologies of the classical propositional calculus;
- 2) $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$;
- 3) $\Box_i p \rightarrow p$;
- 4) $\Box_i p \rightarrow \Box_i \Box_i p$;
- 5) $\Diamond_i \Box_i p \rightarrow p$;
- 6) $\Box_1 \Box_2 p \leftrightarrow \Box_2 \Box_1 p$;

and closed under the following rules of inference:

Modus Ponens (MP): from φ and $\varphi \rightarrow \psi$ infer ψ ;
Necessitation (N)_i: from φ infer $\Box_i \varphi$.

Recall also that a set of formulas L is called a *logic* if it contains all tautologies of the classical propositional calculus and is closed under the rule of modus ponens. A modal logic is called *normal* if it contains axiom schema 2) (see above) and is closed under the rule of necessitation. A logic L_1 is an *extension* of L_2 if $L_2 \subseteq L_1$.

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It is well known that $\mathbf{S5}^2$ has the exponential size model property, and that its satisfiability problem is NEXPTIME-complete [6]. *In this paper, by the complexity of a logic we will mean the complexity of its satisfiability problem.* It is shown in [3] that in contrast to $\mathbf{S5}^2$, every proper normal extension L of $\mathbf{S5}^2$ has the poly-size model property. That means that there is a polynomial $P(n)$ such that any L -consistent formula φ (that is, $\neg\varphi \notin L$) has a model over a frame validating L and with at most $P(|\varphi|)$ points, where $|\varphi|$ is the length of φ .

It was conjectured in [3] that every proper normal extension of $\mathbf{S5}^2$ is finitely axiomatizable and NP-complete. In this paper we prove this conjecture. In fact, we show that for every proper normal extension L of $\mathbf{S5}^2$, there is a finite set \mathbf{M}_L of finite $\mathbf{S5}^2$ -frames such that an arbitrary finite $\mathbf{S5}^2$ -frame is a frame for L iff it does not have any frame in \mathbf{M}_L as a p -morphic image. This condition yields a finite axiomatization of L . We also show that the condition is decidable in deterministic polynomial time. This, together with the poly-size model property, implies NP-completeness of (satisfiability for) L .

Finally, we note that general complexity results for (uni)modal logics were investigated before. Bull and Fine proved that every normal extension of $\mathbf{S4.3}$ has the finite model property, is finitely axiomatizable and therefore is decidable (see [4, Theorems 4.96, 4.101]). Hemaspaandra strengthened the second result by showing that every normal extension of $\mathbf{S4.3}$ is NP-complete [4, Theorem 6.41]. The proof of finite axiomatizability uses Kruskal's theorem on well-quasi-orderings [4, Theorem 4.99]. Kracht uses the same technique for showing that every extension of the intermediate logic of leptonic strings is finitely axiomatizable [8, Theorem 14, Proposition 15]. This paper takes the same line of research beyond unimodal logics. However, as we will see below, the theory of well-quasi-orderings does not suffice for our purposes; instead, we will use better-quasi-orderings.

2. Preliminaries

Recall that a triple $\mathcal{F} = (W, E_1, E_2)$ is an $\mathbf{S5}^2$ -frame (i.e., it validates the axioms of $\mathbf{S5}^2$: see, e.g., [5, Corollary 5.10]) iff W is a non-empty set and E_1 and E_2 are equivalence relations on W such that

$$\mathcal{F} \models (\forall w, v, u)(wE_1v \wedge vE_2u \rightarrow (\exists z)(wE_2z \wedge zE_1u)).$$

For $i = 1, 2$ we call the E_i -equivalence classes E_i -clusters. The E_i -cluster containing $w \in W$ is denoted by $E_i(w)$, and for $X \subseteq W$ we let $E_i(X)$ denote $\bigcup_{x \in X} E_i(x)$.

We identify non-negative integers with ordinals, so that for $n \geq 0$ we have $n = \{0, 1, \dots, n - 1\}$. For positive integers n and m , let $\mathbf{n} \times \mathbf{m}$ denote the $\mathbf{S5}^2$ -frame with domain $n \times m$ and with $(x_1, x_2)E_1(y_1, y_2)$ iff $x_2 = y_2$ and $(x_1, x_2)E_2(y_1, y_2)$ iff $x_1 = y_1$. Then it is well known that $\mathbf{S5}^2$ is complete with respect to $\{\mathbf{n} \times \mathbf{n} : n \geq 1\}$ [11].

Given two $\mathbf{S5}^2$ -frames $\mathcal{F} = (W, E_1, E_2)$ and $\mathcal{G} = (U, S_1, S_2)$, a mapping $f : U \rightarrow W$ is called a *p-morphism from \mathcal{G} to \mathcal{F}* if for each $i = 1, 2$,

$$(\forall t \in U)(\forall w \in W)(f(t)E_i w \leftrightarrow (\exists u \in U)(tS_i u \wedge f(u) = w)).$$

It is easy to check that a map $f : U \rightarrow W$ is a *p-morphism* iff the f -image of every S_i -cluster of \mathcal{G} is an E_i -cluster of \mathcal{F} , for $i = 1, 2$. We say that \mathcal{F} is *isomorphic* to \mathcal{G} if there exists a bijection $g : W \rightarrow U$ such that $wE_i w' \iff g(w)S_i g(w')$ for each $w, w' \in W$ and each $i = 1, 2$. It is easy to see that \mathcal{F} is isomorphic to \mathcal{G} iff there is a one-one *p-morphism* from \mathcal{G} onto \mathcal{F} . We call \mathcal{F} a *p-morphic image* of \mathcal{G} if there is a *p-morphism* from \mathcal{G} onto \mathcal{F} . It is well known that in this case, any formula valid in \mathcal{G} is valid in \mathcal{F} .

We call $\mathcal{F} = (W, E_1, E_2)$ *rooted* if there is a point $w \in W$ that is related to every point $v \in W$ by the reflexive transitive closure of $E_1 \cup E_2$. It is easy to check that an $\mathbf{S5}^2$ -frame \mathcal{F} is rooted iff

$$\mathcal{F} \models (\forall w, v)(\exists u)(wE_1 u \wedge uE_2 v).$$

Choose a set $\mathbf{F}_{\mathbf{S5}^2}$ of representatives of the isomorphism types of finite rooted $\mathbf{S5}^2$ -frames. That is, for each finite rooted $\mathbf{S5}^2$ -frame, there is exactly one frame in $\mathbf{F}_{\mathbf{S5}^2}$ that is isomorphic to it.

Let L be a normal extension of $\mathbf{S5}^2$. An $\mathbf{S5}^2$ -frame \mathcal{F} is called an *L-frame* if \mathcal{F} validates all formulas in L . Let \mathbf{F}_L be the set of all *L-frames* in $\mathbf{F}_{\mathbf{S5}^2}$. Then L is complete with respect to \mathbf{F}_L [1]. Thus, for our purposes it suffices to consider only finite rooted $\mathbf{S5}^2$ -frames. *From now on, we will use the term “frame” to mean this.*

For $\mathcal{F}, \mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$ we put

$$\mathcal{F} \leq \mathcal{G} \text{ iff } \mathcal{F} \text{ is a } p\text{-morphic image of } \mathcal{G}.$$

Then it is routine to check that \leq is a partial order on $\mathbf{F}_{\mathbf{S5}^2}$. We write $\mathcal{F} < \mathcal{G}$ if $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \not\leq \mathcal{F}$. Then $\mathcal{F} < \mathcal{G}$ implies $|\mathcal{F}| < |\mathcal{G}|$ and we see that there are no infinite descending chains in $(\mathbf{F}_{\mathbf{S5}^2}, <)$. Thus, for any non-empty $A \subseteq \mathbf{F}_{\mathbf{S5}^2}$, the set $\min(A)$ of $<$ -minimal elements of A is non-empty, and indeed for any $\mathcal{G} \in A$ there is $\mathcal{F} \in \min(A)$ such that $\mathcal{F} \leq \mathcal{G}$.

3. Finite axiomatizability

In this section we will prove the first main result of the paper — that every normal extension of $\mathbf{S5}^2$ is finitely axiomatizable.

First we recall the Jankov-Fine formulas for $\mathbf{S5}^2$ (see [4, §3.4] and [5, §8.4 p.392]). Consider a frame $\mathcal{F} = (W, E_1, E_2)$. For each point $p \in W$ we introduce a propositional variable, denoted also by p , and consider the formulas

$$\begin{aligned} \alpha(\mathcal{F}) &= \Box_1 \Box_2 \left(\bigvee_{p \in W} (p \wedge \neg \bigvee_{p' \in W \setminus \{p\}} p') \right. \\ &\quad \wedge \bigwedge_{\substack{i=1,2 \\ p, p' \in W, p E_i p'}} (p \rightarrow \Diamond_i p') \quad \wedge \quad \bigwedge_{\substack{i=1,2 \\ p, p' \in W, \neg(p E_i p')}} (p \rightarrow \neg \Diamond_i p') \left. \right), \\ \chi(\mathcal{F}) &= \neg \alpha(\mathcal{F}). \end{aligned}$$

LEMMA 3.1. *For any frames $\mathcal{F} = (W, E_1, E_2)$ and $\mathcal{G} = (U, S_1, S_2)$ we have that \mathcal{F} is a p -morphic image of \mathcal{G} iff $\mathcal{G} \not\models \chi(\mathcal{F})$.*

PROOF. (Sketch) Suppose \mathcal{F} is a p -morphic image of \mathcal{G} . Define a valuation V on \mathcal{F} by putting $V(p) = p$ for any $p \in W$. Then $\mathcal{F} \not\models_V \chi(\mathcal{F})$ by the definition of $\chi(\mathcal{F})$. Now if $\mathcal{G} \models \chi(\mathcal{F})$, then since p -morphic images preserve validity of formulas, we would also have $\mathcal{F} \models \chi(\mathcal{F})$, a contradiction. Therefore, $\mathcal{G} \not\models \chi(\mathcal{F})$.

For the converse, we use the argument of [5, Claim 8.36]. Suppose that $\mathcal{G} \not\models \chi(\mathcal{F})$. Then there is a valuation V' on \mathcal{G} and a point $u \in U$ such that $\mathcal{G}, u \not\models_{V'} \chi(\mathcal{F})$. Therefore, $\mathcal{G}, u \models_{V'} \alpha(\mathcal{F})$. Define a map $f : U \rightarrow W$ by putting $f(t) = p \iff \mathcal{G}, t \models_{V'} p$, for every $t \in U$ and $p \in W$. From \mathcal{G} being rooted and the truth of the first conjunct of $\alpha(\mathcal{F})$ it follows that f is well defined. The truth of the first two conjuncts of $\alpha(\mathcal{F})$ together with \mathcal{F} being rooted implies that f is surjective. Finally, the truth of the second and third conjuncts of $\alpha(\mathcal{F})$ guarantees that f is a p -morphism. Therefore, \mathcal{F} is a p -morphic image of \mathcal{G} . ■

Let L be a proper normal extension of $\mathbf{S5}^2$. By completeness of $\mathbf{S5}^2$ with respect to $\mathbf{F}_{\mathbf{S5}^2}$, the set $\mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L$ is non-empty. Let $\mathbf{M}_L = \min(\mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L)$.

THEOREM 3.2. *For any proper normal extension L of $\mathbf{S5}^2$ and $\mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$, $\mathcal{G} \in \mathbf{F}_L$ iff no $\mathcal{F} \in \mathbf{M}_L$ is a p -morphic image of \mathcal{G} .*

PROOF. Let $\mathcal{G} \in \mathbf{F}_L$; then since p -morphisms preserve validity of formulas, every p -morphic image of \mathcal{G} belongs to \mathbf{F}_L and hence can not be in \mathbf{M}_L .

Conversely, if $\mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L$ then there is $\mathcal{F} \in \mathbf{M}_L$ such that $\mathcal{F} \leq \mathcal{G}$ — that is, \mathcal{F} is a p -morphic image of \mathcal{G} . ■

THEOREM 3.3. *Every proper normal extension L of $\mathbf{S5}^2$ is axiomatizable by the axioms of $\mathbf{S5}^2$ plus $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$.*

PROOF. Let $\mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$. Then by Theorem 3.2, $\mathcal{G} \in \mathbf{F}_L$ iff there is no $\mathcal{F} \in \mathbf{M}_L$ with $\mathcal{F} \leq \mathcal{G}$, iff (by Lemma 3.1) there is no $\mathcal{F} \in \mathbf{M}_L$ with $\mathcal{G} \not\models \chi(\mathcal{F})$, iff $\mathcal{G} \models \chi(\mathcal{F})$ for all $\mathcal{F} \in \mathbf{M}_L$. Thus, $\mathcal{G} \models \{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$ iff $\mathcal{G} \in \mathbf{F}_L$.

Let L' be the logic axiomatized by the axioms of $\mathbf{S5}^2$ plus $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$. From the above it is clear that $\mathbf{F}_{L'} = \mathbf{F}_L$. But L (L') is sound and complete with respect to \mathbf{F}_L ($\mathbf{F}_{L'}$, respectively). So, $L' = L$. ■

It follows that $L \supset \mathbf{S5}^2$ is finitely axiomatizable whenever \mathbf{M}_L is finite. We now proceed to show that \mathbf{M}_L is indeed finite for every proper normal extension L of $\mathbf{S5}^2$.

Suppose $\mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}$. For $i = 1, 2$, we say that the E_i -depth of \mathcal{G} is n , and write $d_i(\mathcal{G}) = n$, if the number of E_i -clusters of \mathcal{G} is n .

Fix a proper normal extension L of $\mathbf{S5}^2$. Since $\mathbf{S5}^2$ is complete with respect to $\{\mathbf{n} \times \mathbf{n} : n \geq 1\}$, there is $n \geq 1$ such that $\mathbf{n} \times \mathbf{n} \notin \mathbf{F}_L$. Let $n(L)$ be the least such.

LEMMA 3.4. *Let L be as above, and write n for $n(L)$.*

1. If $\mathcal{G} \in \mathbf{F}_L$, then $d_1(\mathcal{G}) < n$ or $d_2(\mathcal{G}) < n$.
2. If $\mathcal{G} \in \mathbf{M}_L$, then $d_1(\mathcal{G}) \leq n$ or $d_2(\mathcal{G}) \leq n$.

PROOF. 1. If $\mathcal{G} \in \mathbf{F}_L$ and $d_1(\mathcal{G}) \geq n$ and $d_2(\mathcal{G}) \geq n$, then by [3, Lemma 5], $\mathbf{n} \times \mathbf{n}$ is a p -morphic image of \mathcal{G} . So, $\mathbf{n} \times \mathbf{n} \in \mathbf{F}_L$, a contradiction.

2. If $\mathcal{G} \in \mathbf{M}_L$ and both depths of \mathcal{G} are greater than n , then again $\mathbf{n} \times \mathbf{n}$ is a p -morphic image of \mathcal{G} . Therefore, $\mathbf{n} \times \mathbf{n} < \mathcal{G}$. However, \mathcal{G} is a minimal element of $\mathbf{F}_{\mathbf{S5}^2} \setminus \mathbf{F}_L$, implying that $\mathbf{n} \times \mathbf{n}$ belongs to \mathbf{F}_L , which is false. ■

COROLLARY 3.5. *\mathbf{M}_L is finite iff $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$.*

PROOF. By Lemma 3.4, $\mathbf{M}_L = \bigcup_{k \leq n(L)} \{\mathcal{F} \in \mathbf{M}_L : d_1(\mathcal{F}) = k\} \cup \bigcup_{k \leq n(L)} \{\mathcal{F} \in \mathbf{M}_L : d_2(\mathcal{F}) = k\}$. Thus, \mathbf{M}_L is finite if and only if $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$. ■

Since \mathbf{M}_L is a \leq -antichain in $\mathbf{F}_{\mathbf{S5}^2}$, to show that $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$, it is enough to prove that for any k , the set $\{\mathcal{F} \in \mathbf{F}_{\mathbf{S5}^2} : d_i(\mathcal{F}) = k\}$ does not contain an infinite \leq -antichain. Without loss of generality we can consider the case when $i = 2$.

Fix $k \in \omega$. For every $n \in \omega$ let \mathcal{M}_n denote the set of all $n \times k$ matrices¹ (m_{ij}) with coefficients in ω ($i < n, j < k$). Let $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$. Define \preceq on \mathcal{M} by putting $(m_{ij}) \preceq (m'_{ij})$ if we have $(m_{ij}) \in \mathcal{M}_n, (m'_{ij}) \in \mathcal{M}_{n'}, n \leq n'$, and there is a surjection $f : n' \rightarrow n$ such that $m_{f(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. It is easy to see that (\mathcal{M}, \preceq) is a quasi-ordered set (i.e., \preceq is reflexive and transitive).

Let $\mathbf{F}_{\mathbf{S5}^2}^k = \{\mathcal{F} \in \mathbf{F}_{\mathbf{S5}^2} : d_2(\mathcal{F}) = k\}$. For each $\mathcal{F} \in \mathbf{F}_{\mathbf{S5}^2}^k$ we fix enumerations F_0, \dots, F_{n-1} of the E_1 -clusters of \mathcal{F} (where $n = d_1(\mathcal{F})$) and F^0, \dots, F^{k-1} of the E_2 -clusters of \mathcal{F} . Define a map $H : \mathbf{F}_{\mathbf{S5}^2}^k \rightarrow \mathcal{M}$ by putting $H(\mathcal{F}) = (m_{ij})$ if $|F_i \cap F^j| = m_{ij}$ for $i < d_1(\mathcal{F})$ and $j < k$. As $\mathcal{F} \in \mathbf{F}_{\mathbf{S5}^2}$, it follows that $m_{ij} > 0$ for each such i, j . Recall that a map $f : P \rightarrow P'$ between ordered sets (P, \leq) and (P', \leq') is called *order reflecting* if $f(w) \leq' f(v)$ implies $w \leq v$ for any $w, v \in P$.

LEMMA 3.6. $H : (\mathbf{F}_{\mathbf{S5}^2}^k, \leq) \rightarrow (\mathcal{M}, \preceq)$ is an order-reflecting injection.

PROOF. Since $\mathbf{F}_{\mathbf{S5}^2}$ consists of non-isomorphic frames, H is one-one. Now let $\mathcal{F} = (W, E_1, E_2), \mathcal{G} = (U, S_1, S_2), \mathcal{F}, \mathcal{G} \in \mathbf{F}_{\mathbf{S5}^2}^k$, and $(m_{ij}), (m'_{ij}) \in \mathcal{M}$ be such that $H(\mathcal{F}) = (m_{ij}), H(\mathcal{G}) = (m'_{ij})$, and $(m_{ij}) \preceq (m'_{ij})$. We need to show that $\mathcal{F} \leq \mathcal{G}$. Suppose $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. Then there is surjective $f : n' \rightarrow n$ such that $m_{f(i)j} \leq m'_{ij}$ for $i < n'$ and $j < k$. Then $|G_i \cap G^j| \geq |F_{f(i)} \cap F^j| > 0$ for any $i < n'$ and $j < k$. Hence there exists a surjection $h_i^j : G_i \cap G^j \rightarrow F_{f(i)} \cap F^j$. Define $h : U \rightarrow W$ by putting $h(u) = h_i^j(u)$, where $i < n', j < k$, and $u \in G_i \cap G^j$. It is obvious that h is well defined and onto.

Now we show that h is a p -morphism. If uS_1v , then $u, v \in G_i$ for some $i < n'$. Therefore, $h(u), h(v) \in F_{f(i)}$, and so $h(u)E_1h(v)$. Analogously, if uS_2v , then $u, v \in G^j$ for some $j < k$, $h(u), h(v) \in F^j$, and so $h(u)E_2h(v)$. Now suppose $u \in G_i \cap G^j$ for some $i < n'$ and $j < k$. If $h(u)E_2h(v)$, then $h(u), h(v) \in F^j$ and $v \in G^j$. As both u and v belong to G^j it follows that uS_2v . Finally, if $h(u)E_1h(v)$, then $h(u) \in F_{f(i)} \cap F^j$ and $h(v) \in F_{f(i)} \cap F^{j'}$, for some $j' < k$. Therefore, there exists $z \in G_i \cap G^{j'}$ (since $z \in G_i$ we have uS_1z) such that $h(z) = h(v)$. Thus, h is an onto p -morphism, implying that $\mathcal{F} \leq \mathcal{G}$. Thus, H is order reflecting. ■

¹ By an $n \times k$ matrix we mean a matrix with n rows and k columns.

COROLLARY 3.7. *If $\Delta \subseteq \mathbf{F}_{S5^2}^k$ is a \leq -antichain, then $H(\Delta) \subseteq \mathcal{M}$ is a \preceq -antichain.*

PROOF. Immediate. ■

Now we will show that there are no infinite \preceq -antichains in \mathcal{M} . For this we define a quasi-order \sqsubseteq on \mathcal{M} included in \preceq and show that there are no infinite \sqsubseteq -antichains in \mathcal{M} . To do so we first introduce two quasi-orders \sqsubseteq_1 and \sqsubseteq_2 on \mathcal{M} and then define \sqsubseteq as the intersection of these quasi-orders. For $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$, we say that:

- $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ if there is a one-one order-preserving map $\varphi : n \rightarrow n'$ (i.e., $i < i' < n$ implies $\varphi(i) < \varphi(i')$) such that $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$;
- $(m_{ij}) \sqsubseteq_2 (m'_{ij})$ if there is a map $\psi : n' \rightarrow n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$.

Let \sqsubseteq be the intersection of \sqsubseteq_1 and \sqsubseteq_2 .

LEMMA 3.8. *For any $(m_{ij}), (m'_{ij}) \in \mathcal{M}$, if $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \preceq (m'_{ij})$.*

PROOF. Suppose $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. If $(m_{ij}) \sqsubseteq (m'_{ij})$, then $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ and $(m_{ij}) \sqsubseteq_2 (m'_{ij})$. By $(m_{ij}) \sqsubseteq_1 (m'_{ij})$ there is a one-one order-preserving map $\varphi : n \rightarrow n'$ with $m_{ij} \leq m'_{\varphi(i)j}$ for all $i < n$ and $j < k$; and by $(m_{ij}) \sqsubseteq_2 (m'_{ij})$ there is a map $\psi : n' \rightarrow n$ such that $m_{\psi(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. Let $\text{rng}(\varphi) = \{\varphi(i) : i < n\}$. Define $f : n' \rightarrow n$ by putting

$$f(i) = \begin{cases} \varphi^{-1}(i), & \text{if } i \in \text{rng}(\varphi), \\ \psi(i), & \text{otherwise.} \end{cases}$$

Then f is a surjection. Moreover, for $i < n'$ and $j < k$, if $i \in \text{rng}(\varphi)$, then $m_{f(i)j} = m_{\varphi^{-1}(i)j} \leq m'_{ij}$ by the definition of \sqsubseteq_1 ; and if $i \notin \text{rng}(\varphi)$, then $m_{f(i)j} = m_{\psi(i)j} \leq m'_{ij}$ by the definition of \sqsubseteq_2 . Therefore, $m_{f(i)j} \leq m'_{ij}$ for all $i < n'$ and $j < k$. Thus, $(m_{ij}) \preceq (m'_{ij})$. ■

Thus, it is left to show that there are no infinite \sqsubseteq -antichains in \mathcal{M} . For this we use the theory of *better-quasi-orderings (bqos)*. Our main source of reference is Laver [9].

For any set $X \subseteq \omega$ let $[X]^{<\omega} = \{Y \subseteq X : |Y| < \omega\}$, and for $n < \omega$ let $[X]^n = \{Y \subseteq X : |Y| = n\}$. We say that Y is an initial segment of X if there is $n \in \omega$ such that $Y = \{x \in X : x \leq n\}$.

DEFINITION 3.9. Let X be an infinite subset of ω . We say that $\mathcal{B} \subseteq [X]^{<\omega}$ is a barrier on X if $\emptyset \notin \mathcal{B}$ and:

- for every infinite $Y \subseteq X$, there is an initial segment of Y in \mathcal{B} ;
- \mathcal{B} is an antichain with respect to \subseteq .

A barrier is a barrier on some infinite $X \subseteq \omega$.

Note that for any $n \geq 1$, $[\omega]^n$ is a barrier on ω .

DEFINITION 3.10.

1. If s, t are finite subsets of ω , we write $s \triangleleft t$ to mean that there are $i_1 < \dots < i_k$ and j ($1 \leq j < k$) such that $s = \{i_1, \dots, i_j\}$ and $t = \{i_2, \dots, i_k\}$.
2. Given a barrier \mathcal{B} and a quasi-ordered set (Q, \leq) , we say that a map $f : \mathcal{B} \rightarrow Q$ is good if there are $s, t \in \mathcal{B}$ such that $s \triangleleft t$ and $f(s) \leq f(t)$.
3. Let (Q, \leq) be a quasi-order. We call \leq a better-quasi-ordering (bqo) if for every barrier \mathcal{B} , every map $f : \mathcal{B} \rightarrow Q$ is good.

Now we recall basic constructions and properties of bqos.

PROPOSITION 3.11. If (Q, \leq) is a bqo, there are no infinite \leq -antichains in Q .

PROOF. Let $(\xi_n)_{n \in \omega}$ be an infinite sequence of distinct elements of Q . As we pointed out, $\mathcal{B} = [\omega]^1 = \{\{n\} : n < \omega\}$ is a barrier. Define a map $\theta : \mathcal{B} \rightarrow Q$ by putting $\theta(\{n\}) = \xi_n$. Since (Q, \leq) is a bqo, θ is good. Therefore, there are $\{n\}, \{m\} \in \mathcal{B}$ such that $\{n\} \triangleleft \{m\}$ (i.e., $n < m$) and $\xi_n \leq \xi_m$. So, no infinite subset of Q forms a \leq -antichain. ■

We write On for the class of all ordinals. Let (Q, \leq) be a quasi-order. Define \leq^* on the class $\bigcup_{\alpha \in On} Q^\alpha$, and on any set contained in it, by putting $(x_i)_{i < \alpha} \leq^* (y_i)_{i < \beta}$ if there is a one-one order-preserving map $\varphi : \alpha \rightarrow \beta$ such that $x_i \leq y_{\varphi(i)}$ for all $i < \alpha$.

Let $\wp(Q)$ be the power set of Q . The order \leq can be extended to $\wp(Q)$ as follows: For $\Gamma, \Delta \in \wp(Q)$, we say that $\Gamma \leq \Delta$ if for all $\delta \in \Delta$ there is $\gamma \in \Gamma$ with $\gamma \leq \delta$.

Recall that (P, \leq') is called a suborder of (Q, \leq) if $P \subseteq Q$ and $\leq' = \leq \cap P^2$.

THEOREM 3.12.

1. (ω, \leq) is a bqo.

2. Any suborder of a bqo is a bqo.
3. If \leq and \leq' are bqos on Q , then $\leq \cap \leq'$ is also a bqo on Q .
4. If (Q, \leq) is a bqo, then $(\bigcup_{\alpha \in \mathcal{O}_n} Q^\alpha, \leq^*)$ is also a (proper class) bqo. Hence, by (2), its suborders (Q^k, \leq^*) and $(\bigcup_{n < \omega} Q^n, \leq^*)$ are bqos.
5. If (Q, \leq) is a bqo, then $(\wp(Q), \leq)$ is a bqo.

PROOF. (1) follows from Lemma 1.2 of [9]. (2) is trivial.

(3): By [9, Lemma 1.8], $(Q \times Q, \leq \otimes \leq')$ is a bqo, where we define $(x, x') \leq \otimes \leq' (y, y')$ iff $x \leq y$ and $x' \leq' y'$. By (2), its suborder $(\{(q, q) : q \in Q\}, \leq \otimes \leq')$ is also a bqo, and this is isomorphic to $(Q, \leq \cap \leq')$.

(4) — see [9, Theorem 1.10].

(5) Finally to show $(\wp(Q), \leq)$ is a bqo we adapt the proof of Lemma 1.3 of [9]. Let \mathcal{B} be a barrier and consider $f : \mathcal{B} \rightarrow \wp(Q)$. Suppose f is not good. Then for each $s, t \in \mathcal{B}$ with $s \triangleleft t$ we have $f(s) \not\leq f(t)$. Let $\mathcal{B}(2) = \{s \cup t : s, t \in \mathcal{B} \text{ and } s \triangleleft t\}$. Thus for every element $s \cup t \in \mathcal{B}(2)$ there is an element $\delta_{st} \in f(t)$ such that for every $\gamma \in f(s)$ we have $\gamma \not\leq \delta_{st}$.

Define a map $h : \mathcal{B}(2) \rightarrow Q$ by putting $h(s \cup t) = \delta_{st}$ for every $s \cup t \in \mathcal{B}(2)$. It can be checked that h is well defined. It is known (see, e.g., [9, p. 35]) that $\mathcal{B}(2)$ is a barrier. Since (Q, \leq) is a bqo, h is good, so there exist $s \cup t, s' \cup t' \in \mathcal{B}(2)$ with $s \cup t \triangleleft s' \cup t'$ and $h(s \cup t) \leq h(s' \cup t')$. It is easy to check (see [9, p. 35]) that $t = s'$. But now $\delta_{s't'} = h(s' \cup t') \geq h(s \cup t) \in f(t) = f(s')$. This contradicts the definition of $\delta_{s't'}$, hence f is good and therefore $(\wp(Q), \leq)$ is a bqo. ■

REMARK 3.13. A quasi-order \leq on a set Q is called a *well-quasi-ordering* (wqo) if for any sequence $(x_i)_{i < \omega}$ in Q there exist $i < j < \omega$ with $x_i \leq x_j$. As we said in the introduction, wqos have been used to prove finite axiomatizability results in modal logic on many previous occasions. The following facts are known about them (cf. Theorem 3.12):

1. Any bqo is a wqo.
2. If \leq and \leq' are wqos on Q , then $\leq \cap \leq'$ is also a wqo on Q .
3. (Higman's Lemma, proved in [7]) If (Q, \leq) is a wqo then $(\bigcup_{n \in \omega} Q^n, \leq^*)$ is also a wqo.

An example of a wqo (Q, \leq) with $(\bigcup_{\alpha \in \mathcal{O}_n} Q^\alpha, \leq^*)$ not a wqo was constructed by Rado [10]: let $Q = \{(i, j) : i < j < \omega\}$, ordered by $(i, j) \leq (k, l)$ iff either $i = k$ and $j \leq l$, or else $i, j < k$. This is a wqo on Q . Now for $i < \omega$ let ξ_i be the sequence $((i, i + 1), (i, i + 2), \dots)$. Then $\xi_i \not\leq^* \xi_j$ for all $i < j < \omega$. This example can be used to show that for a wqo (Q, \leq) , in general $(\wp(Q), \leq)$ fails

to be a wqo, even if we restrict to finite subsets of Q (see also the discussion on p. 33 of [9]). This failure is why we use bqos and not wqos here.

By Proposition 3.11, to show that there are no \sqsubseteq -antichains in \mathcal{M} it suffices to show that $(\mathcal{M}, \sqsubseteq)$ is a bqo. It follows from Theorem 3.12(3) that the intersection of two bqos is again a bqo. Hence, it is enough to prove that $(\mathcal{M}, \sqsubseteq_1)$ and $(\mathcal{M}, \sqsubseteq_2)$ are bqos.

LEMMA 3.14. *$(\mathcal{M}, \sqsubseteq_1)$ is a bqo.*

PROOF. By Theorem 3.12(1), (ω, \leq) is a bqo. By Theorem 3.12(4), (ω^k, \leq^*) is also a bqo. By Theorem 3.12(4) again, $(\mathcal{M}, \sqsubseteq_1) \cong (\bigcup_{n < \omega} (\omega^k)^n, \leq^{**})$ is a bqo as well.² ■

It remains to show that $(\mathcal{M}, \sqsubseteq_2)$ is a bqo.

LEMMA 3.15. *$(\mathcal{M}, \sqsubseteq_2)$ is a bqo.*

PROOF. For a matrix $(m_{ij}) \in \mathcal{M}_n$ let $m_i = (m_{i0}, \dots, m_{ik-1})$ denote the i -th row of (m_{ij}) . Note that each row of (m_{ij}) is a $1 \times k$ matrix, and so $m_i \in \mathcal{M}_1$ for any $i < n$. We write $\text{row}(m_{ij})$ for the set $\{m_i : i < n\}$. Obviously, $\text{row}(m_{ij}) \in \wp(\mathcal{M}_1) \subseteq \wp(\mathcal{M})$. Consider an arbitrary barrier \mathcal{B} and a map $f : \mathcal{B} \rightarrow \mathcal{M}$. We need to show that f is good with respect to \sqsubseteq_2 . Define $g : \mathcal{B} \rightarrow \wp(\mathcal{M})$ by $g(s) = \text{row}(f(s))$. Since $(\mathcal{M}, \sqsubseteq_1)$ is a bqo, by Theorem 3.12(5), $(\wp(\mathcal{M}), \sqsubseteq_1)$ is also a bqo. Hence, there are $s, t \in \mathcal{B}$ such that $s \triangleleft t$ and $g(s) \sqsubseteq_1 g(t)$. Therefore, for each $\delta \in g(t)$ there is $\gamma \in g(s)$ with $\gamma \sqsubseteq_1 \delta$.

Now we show that $f(s) \sqsubseteq_2 f(t)$. Write (m_{ij}) for $f(s)$ and (m'_{ij}) for $f(t)$. Suppose that $(m_{ij}) \in \mathcal{M}_n$ and $(m'_{ij}) \in \mathcal{M}_{n'}$. We define $\psi : n' \rightarrow n$ as follows. Let $i < n'$. Then $m'_i \in g(t)$. By the above, we may choose $\psi(i) < n$ such that $m_{\psi(i)} \sqsubseteq_1 m'_i$. This defines ψ , and we have $m_{\psi(i)j} \leq m'_{ij}$ for any $i < n'$ and $j < k$. Thus, $f(s) \sqsubseteq_2 f(t)$, f is a good map, and so $(\mathcal{M}, \sqsubseteq_2)$ is a bqo. ■

It follows that $(\mathcal{M}, \sqsubseteq)$ is a bqo. Therefore, there are no infinite \sqsubseteq -antichains in \mathcal{M} . Thus, by Lemma 3.8 there are no infinite \preceq -antichains in \mathcal{M} .

Now we are in a position to prove the first main theorem of this paper.

THEOREM 3.16. *Every normal extension of $\mathbf{S5}^2$ is finitely axiomatizable.*

² To apply this theorem, we needed to require in the definition of \sqsubseteq_1 on \mathcal{M} that φ is order preserving. This is the only time this assumption is used.

PROOF. Clearly, $\mathbf{S5}^2$ is finitely axiomatizable. Suppose L is a proper normal extension of $\mathbf{S5}^2$. Then by Theorem 3.3 L is axiomatizable by the $\mathbf{S5}^2$ axioms plus $\{\chi(\mathcal{F}) : \mathcal{F} \in \mathbf{M}_L\}$. Since there are no infinite \preceq -antichains in \mathcal{M} , by Corollary 3.7 there are no infinite antichains in $\mathbf{F}_{\mathbf{S5}^2}^k$, for each $k \in \omega$. Therefore, $\{\mathcal{F} \in \mathbf{M}_L : d_i(\mathcal{F}) = k\}$ is finite for every $k \leq n(L)$ and $i = 1, 2$. Thus, \mathbf{M}_L is finite by Corollary 3.5. It follows that L is finitely axiomatizable. ■

COROLLARY 3.17. *The lattice of normal extensions of $\mathbf{S5}^2$ is countable.*

PROOF. Immediately follows from Theorem 3.16 since there are only countably many finitely axiomatizable normal extensions of $\mathbf{S5}^2$. ■

REMARK 3.18. In algebraic terminology, Corollary 3.17 says that the lattice of subvarieties of the variety \mathbf{Df}_2 of two-dimensional diagonal-free cylindric algebras is countable. This is in contrast with the variety \mathbf{CA}_2 of two-dimensional cylindric algebras (with diagonals), since, as was shown in [2], the cardinality of the lattice of subvarieties of \mathbf{CA}_2 is that of continuum.

4. Complexity

Note that Theorem 3.16, and the fact that every normal extension L of $\mathbf{S5}^2$ is complete with respect to a class of finite frames (\mathbf{F}_L) for which (up to isomorphism) membership is decidable, imply that L is decidable. This section will be devoted to showing that if L is a proper normal extension, then its satisfiability problem is NP-complete. Fix such an L . We will see in Corollary 4.3 below that NP-completeness follows from the poly-size model property if we can decide in time polynomial in $|W|$ whether a finite structure $\mathcal{A} = (W, R_1, R_2)$ is in \mathbf{F}_L (up to isomorphism). It suffices to decide in polynomial time (1) whether \mathcal{A} is a (rooted $\mathbf{S5}^2$ -) frame; (2) whether a given frame is in \mathbf{F}_L . The first is easy. We concentrate on the second.

By Lemma 3.4(1), there is $n(L) \in \omega$ such that for each frame $\mathcal{G} = (U, S_1, S_2)$ in \mathbf{F}_L we have $d_1(\mathcal{G}) < n(L)$ or $d_2(\mathcal{G}) < n(L)$. So, if both depths of a given frame \mathcal{G} are greater than or equal to $n(L)$ (which obviously can be checked in polynomial time in the size of \mathcal{G}), then $\mathcal{G} \notin \mathbf{F}_L$. So, without loss of generality we can assume that $d_1(\mathcal{G}) < n(L)$.

By Theorem 3.2, \mathcal{G} is in \mathbf{F}_L iff it has no p -morphic image in \mathbf{M}_L . Because \mathbf{M}_L is a fixed finite set, it suffices to provide, for an arbitrary fixed frame $\mathcal{F} = (W, E_1, E_2)$, an algorithm that decides in time polynomial in the size of \mathcal{G} whether there is a p -morphism from \mathcal{G} onto \mathcal{F} . If we considered every map $f : U \rightarrow W$ and checked whether it is a p -morphism, it would take

exponential time in the size of \mathcal{G} (since there are $|W|^{|U|}$ different maps from U to W). Now we will give a different algorithm to check in polynomial time in $|U|$ whether the fixed frame \mathcal{F} is a p -morphic image of a given frame $\mathcal{G} = (U, S_1, S_2)$ with $d_1(\mathcal{G}) < n(L)$.

LEMMA 4.1. *\mathcal{F} is a p -morphic image of \mathcal{G} iff there is a partial surjective map $g : U \rightarrow W$ with the following properties:*

1. For each $u \in U$, there is $v \in \text{dom}(g)$ such that uS_1v .
2. For each $v \in \text{dom}(g)$, the restriction $g \upharpoonright (\text{dom}(g) \cap S_1(v))$ is one-one and has range $E_1(g(v))$.
3. For each $u \in U$ there is $w \in W$ such that
 - (a) $g(v)E_2w$ for all $v \in \text{dom}(g) \cap S_2(u)$,
 - (b) for each $w' \in W$, writing

$$\begin{aligned} X_{w'} &= S_1(g^{-1}(E_1(w'))) \cap S_2(u), \\ Y_{w'} &= E_1(w') \cap E_2(w), \end{aligned}$$

we have $|Y_{w'} \setminus \text{rng}(g \upharpoonright [\text{dom}(g) \cap X_{w'}])| \leq |X_{w'} \setminus \text{dom}(g)|$.

PROOF. Recall that a map $f : U \rightarrow W$ is a p -morphism iff the f -image of every S_i -cluster of \mathcal{G} is an E_i -cluster of \mathcal{F} , for $i = 1, 2$.

Suppose there is a surjective p -morphism $f : U \rightarrow W$. Then for each S_1 -cluster $C \subseteq U$, the map $f \upharpoonright C$ is a surjection from C onto $E_1(f(u))$ for any $u \in C$, so we may choose $C' \subseteq C$ such that $f \upharpoonright C'$ is a bijection from C' onto $E_1(f(u))$. Let $U' = \bigcup \{C' : C \text{ is an } S_1\text{-cluster of } \mathcal{G}\}$. Then it is easy to check that $g = f \upharpoonright U'$ satisfies conditions 1-2 of the lemma. To check condition 3, take any $u \in U$, and put $w = f(u)$. Condition 3a is clearly true. For 3b, fix any $w' \in W$. Pick any $x \in S_2(u)$. Note that $f(x) \in E_2(w)$. Define $X_{w'}, Y_{w'}$ as in the lemma. Then $x \in X_{w'}$ iff $x \in S_1(g^{-1}(E_1(w')))$, iff there is $y \in U'$ such that xS_1y and $g(y)E_1w'$, iff $f(x)E_1w'$, iff $f(x) \in Y_{w'}$. Now f maps $S_2(u)$ onto $E_2(w)$, so $f(S_2(u)) \supseteq Y_{w'}$. It now follows that f maps $X_{w'}$ onto $Y_{w'}$. Plainly, f must therefore map a subset of $X_{w'} \setminus U'$ onto $Y_{w'} \setminus g(X_{w'} \cap U')$, so we must have $|X_{w'} \setminus U'| \geq |Y_{w'} \setminus g(X_{w'} \cap U')|$ as required.

Conversely, let g be as stated. We will extend g to a surjective p -morphism $f : U \rightarrow W$. Since U is a disjoint union of S_2 -clusters, it is enough to define f on an arbitrary S_2 -cluster of \mathcal{G} . Pick $u \in U$. We will extend $g \upharpoonright S_2(u)$ to the whole of $S_2(u)$. Pick $w \in W$ according to condition 3 of the lemma. By condition 3a, $\text{rng}(g \upharpoonright S_2(u)) \subseteq E_2(w)$. Now we extend g to

f such that $\text{rng}(f \upharpoonright S_2(u)) = E_2(w)$ and $f(x)E_1g(v)$ whenever $v \in \text{dom}(g)$ and $x \in S_2(u) \cap S_1(v)$.

For each $w' \in W$, define $X_{w'}, Y_{w'}$ as in the lemma. By conditions 1 and 2, $S_2(u) = \bigcup \{X_{w'} : w' \in W\}$, and $X_{w'} \cap X_{w''} = \emptyset$ whenever $\neg(w'E_1w'')$. For each $w' \in W$, we take the restriction of g to $X_{w'}$ (this restriction may be empty), observe that its range is a subset of $Y_{w'}$, and extend it to a surjection from $X_{w'}$ onto $Y_{w'}$. By condition 3, $|X_{w'} \setminus \text{dom}(g)| \geq |Y_{w'} \setminus \text{rng}(g \upharpoonright X_{w'})|$. So, there exists a surjection $f_{X_{w'}} : X_{w'} \rightarrow Y_{w'}$ extending g . Repeating this for a representative w' of each E_1 -cluster in turn yields an extension of g to $S_2(u)$. Repeating for a representative u of each S_2 -cluster in turn yields an extension of g to U as required.

It is left to show that f is a p -morphism. But it follows immediately from the construction of f that $f \upharpoonright S_i(u) : S_i(u) \rightarrow E_i(f(u))$ is surjective for each $u \in U$ and each $i = 1, 2$. As we pointed out above this implies that f is a p -morphism. ■

COROLLARY 4.2. *It is decidable in polynomial time in the size of \mathcal{G} , whether \mathcal{F} is a p -morphic image of \mathcal{G} .*

PROOF. By Lemma 4.1 it is enough to check whether there exists a partial map $g : U \rightarrow W$ satisfying conditions 1–3 of the lemma. There are at most $n(L)$ S_1 -clusters in \mathcal{G} , and the restriction of g to each S_1 -cluster is one-one; hence, $d = |\text{dom}(g)| \leq n(L) \cdot |W|$, and this is independent of \mathcal{G} . There are at most $d^{|W|}$ maps from a set of size at most d into W . Obviously, there are $\binom{|U|}{d} \leq |U|^d$ subsets of U of size d . Hence there are at most $d^{|W|}|U|^d$ partial maps which may satisfy conditions 1 and 2 of the lemma. Our algorithm enumerates all partial maps from U to W with domain of size at most d , and for each one, checks whether it satisfies conditions 1–3 or not. It is not hard to see that this check can be done in p -time; indeed, it is clear that conditions 1 and 2 can be checked in time polynomial in $|U|$ and there is a first-order sentence $\sigma_{\mathcal{F}}$ such that $\mathcal{G} \models \sigma_{\mathcal{F}}$ iff \mathcal{G} satisfies condition 3. The algorithm states that \mathcal{F} is a p -morphic image of \mathcal{G} if and only if it finds a map satisfying the conditions. Therefore, this is a p -time algorithm checking whether \mathcal{F} is a p -morphic image of \mathcal{G} . ■

COROLLARY 4.3. *Let L be a proper normal extension of $\mathbf{S5}^2$.*

1. *It can be checked in polynomial time in $|U|$ whether a finite $\mathbf{S5}^2$ -frame $\mathcal{G} = (U, S_1, S_2)$ is an L -frame.*
2. *The satisfiability problem for L is NP-complete.*
3. *The validity problem for L is co-NP-complete.*

- PROOF. 1. Follows directly from Theorem 3.2, Corollary 4.2, and the fact (shown in the proof of Theorem 3.16) that \mathbf{M}_L is finite.
2. It is a well known result of modal logic (see, e.g., [4, Lemma 6.35]) that if L is a consistent normal modal logic having the poly-size model property, and the problem of whether a finite structure \mathcal{A} is an L -frame is decidable in time polynomial in the size of \mathcal{A} , then the satisfiability problem of L is NP-complete. The poly-size model property of every $L \supset \mathbf{S5}^2$ is proven in [3, Corollary 9]. (1) implies that the problem $\mathcal{G} \in \mathbf{F}_L$ can be decided in polynomial time in the size of \mathcal{G} . The result follows.
3. Follows directly from (2). ■

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