All-optical generation of states for "Encoding a qubit in an oscillator"

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Most quantum computation schemes propose encoding qubits in two-level systems. Others exploit the use of an infinite-dimensional system. In "Encoding a qubit in an oscillator" [Phys. Rev. A **64**, 012310 (2001)], Gottesman, Kitaev, and Preskill (GKP) combined these approaches when they proposed a fault-tolerant quantum computation scheme in which a qubit is encoded in the continuous position and momentum degrees of freedom of an oscillator. One advantage of this scheme is that it can be performed by use of relatively simple linear optical devices, squeezing, and homodyne detection. However, we lack a practical method to prepare the initial GKP states. Here we propose the generation of an approximate GKP state by using superpositions of optical coherent states (sometimes called "Schrödinger cat states"), squeezing, linear optical devices, and homodyne detection.

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The Gottesman, Kitaev, and Preskill (GKP) scheme [1], constitutes a type of linear optical quantum computer, as do other schemes based on the proposals of Knill *et al.* [2] and schemes based on the proposal of Ralph *et al.* [3]. In the GKP scheme, the qubit is encoded in the continuous Hilbert space of an oscillator's position and momentum variables. This scheme is applicable to any type of quantum harmonic oscillator, but we focus on an optical implementation in traveling modes. The GKP encoding provides a natural error-correction scheme to correct errors due to small shifts (applications of the displacement operator) on the conjugate quadrature variables x and p [1].

The ideal GKP logical 0 qubit state, $|\bar{0}\rangle$, is defined as a state whose x-quadrature wave function is an infinite series of delta-function peaks, while the ideal $|\bar{1}\rangle$ x-quadrature wave function is displaced a distance $\sqrt{\pi}$ from the $|\bar{0}\rangle$ state. Since these states are unphysical, GKP described approximate states whose x-quadrature wave function is a series of Gaussian peaks with width Δ, contained in a larger Gaussian envelope of width $1/k$. The approximation of $|\bar{0}\rangle$ has the wave function

$$\psi_{\mathrm{GKP}}(x) = N \sum_{s=-\infty}^{\infty} e^{-\frac{1}{2}\left(2sk\sqrt{\pi}\right)^2} e^{-\frac{1}{2}\left(\frac{x-2s\sqrt{\pi}}{\Delta}\right)^2}, \tag{1}$$

where N is a normalization factor. If Δ and k are small, then ψ_{GKP} will better approximate an ideal GKP state, and the wave function will have many sharp Gaussian peaks contained in a wide envelope. We can think of the deviation from an ideal GKP state as corresponding to nonzero probability that the state has suffered from errors causing displacement in the x or p variables. If all displacements are smaller than $\sqrt{\pi}/6$, then the errors will not increase during the error-correction protocol. GKP states with $\Delta < 0.15$ and $k < 0.15$ will have a probability greater than 0.99 of being free of shift errors larger than $\sqrt{\pi}/6$ [4]. Figure 1 shows an example of an approximate GKP state's x-quadrature wave function.

Preparing GKP states is a difficult task, and to our knowledge no experiment has yet demonstrated preparation of such states. GKP proposed preparing these states by coupling an optical mode to an oscillating mirror in [1]. Another proposal was made by Travaglione and Milburn in [5], where the qubit states are prepared in the oscillatory motion of a trapped ion rather than the photons in an optical mode. Pirandola *et al.* [6] discusses the preparation of optical GKP states by use of a two-mode Kerr interaction followed by a homodyne measurement of one of the modes. The same authors describe two proposals for generating GKP states in the position and momentum of an atom by using a cavity mediated interaction with light [7,8].

Here we propose the generation of an approximate GKP state by using superpositions of optical coherent states ("cat states"), linear optical devices, squeezing, and homodyne detection. The basic idea is, first, prepare two cat states (each of which contains two Gaussian peaks in its x-quadrature wave function), squeeze both cats (to reduce the width of the Gaussian peaks), interfere them at a beam splitter, then perform homodyne detection on one of the beam splitter's output ports. Depending on the measurement result, we will find an approximate GKP state (with three Gaussian peaks) in

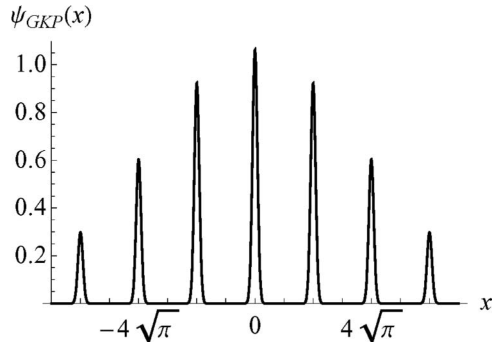

Fig. 1. Approximate GKP state's x-quadrature wave function. This shows the logical 0 state $\psi_{\mathrm{GKP}}(x)$ with $\Delta = k = 0.15$.

the beam splitter's other output port. This procedure can be repeated to produce states with larger numbers of Gaussian peaks.

A cat state is a superposition of coherent states, such as

$$|\psi_{\mathrm{cat}}(\alpha)\rangle = \frac{|-\alpha\rangle + |\alpha\rangle}{\sqrt{2(1+e^{-2\alpha^2})}}, \tag{2}$$

where α is the amplitude of the coherent state, which may be complex, but we assume it is real below. Several experimental proposals to create cat states are reviewed in [9]. Cat states of this form have been created in several experiments with $|\alpha|$ up to 1.75 and fidelities of 0.6 to 0.7 [10–15].

Cat states' x-quadrature wave functions are superpositions of Gaussian peaks. To simplify notation, we denote a Gaussian with $G(x,V,\mu) = \exp[-(x-\mu)^2/(2V)]$. These Gaussians represent wave functions, so the unnormalized vacuum state is $G(x,1,0)$. Suppose two modes (1 and 2) contain states with unnormalized wave functions $G(x_1,V,\mu_1)$ and $G(x_2,V,\mu_2)$. These modes meet at a beam splitter with transmissivity 1/2, which performs the transformation $x_1 \to (x_1+x_2)/\sqrt{2}$ and $x_2 \to (x_1-x_2)/\sqrt{2}$.

After the beam splitter, we use a homodyne detector to measure mode 2's p quadrature. In the case that the measurement result is $p_2 = 0$, this entire procedure produces the transformation

$$G(x_1,V,\mu_1)G(x_2,V,\mu_2) \to \sqrt{V}G\left(x_1,V,\frac{\mu_1+\mu_2}{\sqrt{2}}\right). \tag{3}$$

We can write Eq. (2) in the x-quadrature basis as $\tilde{\psi}_{\mathrm{cat}}(x,\alpha) = G(x,1,-\sqrt{2}\alpha) + G(x,1,\sqrt{2}\alpha)$, where the tilde signals that the state is not normalized. We now squeeze this state by an amount ζ, obtaining $\tilde{\psi}_{\mathrm{sqcat}}(x,\alpha,\zeta) = G(x,e^{-2\zeta},-\sqrt{2}\alpha) + G(x,e^{-2\zeta}\sqrt{2}\alpha e^{-\zeta})$. We will choose the cat state amplitude to be $\alpha = \sqrt{2}^{m-1}\sqrt{\pi}e^{\zeta}$, where m is an integer greater than or equal to 1, which we will later use to count iterations of our scheme. Suppose we have two copies of this squeezed cat in modes 1 and 2. They meet at a beam splitter with transmissivity 1/2, and the p quadrature of mode 2 is measured to be $p_2 = 0$. If we choose $m = 1$, the resulting unnormalized state of mode 1 is

$$\tilde{\beta}(x_1,\zeta,m=1) = G\left(x_1,e^{-2\zeta},-2\sqrt{\pi}\right) + 2G(x_1,e^{-2\zeta},0) + G\left(x_1,e^{-2\zeta},2\sqrt{\pi}\right), \tag{4}$$

which we call the first binomial state; it is similar to an approximate GKP logical qubit 0, except only the central three peaks are present.

Consider the order m binomial state given by

$$\tilde{\beta}(x,\zeta,m) = \sum_{n=0}^{2^m}\binom{2^m}{n}G\left[x,e^{-2\zeta},2\sqrt{\pi}(n-2^{m-1})\right]. \tag{5}$$

This state is a series of Gaussian peaks separated by $2\sqrt{\pi}$ along the x-quadrature axis, and the amplitudes of the peaks are given by the (2^m)th row of Pascal's triangle (where row 0 contains only 1). We will show that, given two copies of the order m binomial state, one can make the $m+1$ order binomial state. We begin with one copy of the state given by Eq. (5) in each of the modes 1 and 2. These two modes meet in a beam splitter of transmissivity 1/2, and we measure the p quadrature, obtaining the result $p_2 = 0$. The new state is given by applying Eq. (3) to $\tilde{\beta}(x_1,\zeta,m)\tilde{\beta}(x_2,\zeta,m)$. The result is

$$\sum_{n_1=0}^{2^m}\sum_{n_2=0}^{2^m}\binom{2^m}{n_1}\binom{2^m}{n_2}\times G\left(x_1,e^{-2\zeta},\sqrt{2}\sqrt{\pi}(n_1+n_2-2^m)\right). \tag{6}$$

After a little algebra and application of Vandermont's identity, we obtain

$$\sum_{q=0}^{2^{m+1}}\binom{2^{m+1}}{q}G\left(x_1,e^{-2\zeta},\sqrt{2}\sqrt{\pi}(q-2^m)\right). \tag{7}$$

This state is equivalent to $\beta(x_1,\zeta,m+1)$, except that the Gaussian peaks are separated by only $\sqrt{2}\sqrt{\pi}$ rather than $2\sqrt{\pi}$. We can compensate for this shrinking of the spacing between the Gaussian peaks if we begin the procedure with two binomial states with spacing of $2\sqrt{2}\sqrt{\pi}$. As m increases, these binomial states will approach the shape of a series of Gaussian peaks in a Gaussian envelope, like $\psi_{\mathrm{GKP}}(x)$.

In Fig. 2, we plot the probability for measuring a certain value $p_2 = R$ as a function of R when making the $m = 1$ binomial state from two cat states. If we measure $R = 0$, we obtain the wave function as shown in Fig. 2, whose Gaussian peaks' widths are determined by the degree of squeezing applied to the initial cat states and whose heights are proportional to the second row of Pascal's triangle (1,2,1), as given by Eq. (5). In Fig. 3, we show the $m = 3$ binomial state.

Creating the order m binomial state requires a minimum of 2^m cats, but the true number may be much larger, because we require that $p_2 = 0$ at each measurement event. However, initial investigations indicate that some cases in which $p_2 \neq 0$ can be recovered by applying a p-quadrature displacement whose size depends on the measurement result. Creating high-quality states will require larger squeezing ζ and higher-order m binomial states, but values for ζ and m sufficient to achieve scalable fault-tolerant quantum computation are not

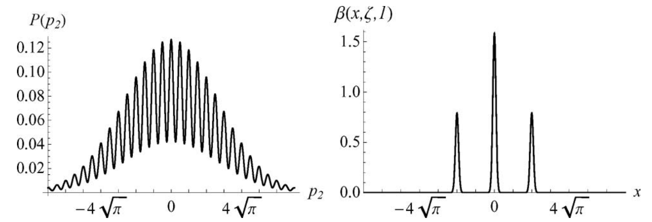

Fig. 2. Left, probability for measuring $p_2 = R$ as a function of R. Right, x-quadrature wave function for the state $\beta(x,\zeta,1)$. In both cases $\alpha = \sqrt{\pi}e^{\zeta}$, and $\zeta = 1.9$ is chosen to produce Gaussian peaks with the same width as shown in Fig. 1. This ζ is equivalent to −16 dB of quadrature noise power reduction in conventional squeezing experiments.

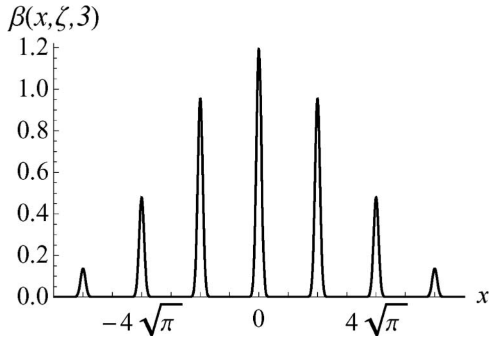

Fig. 3. Wave function of binomial state $\beta(x, \zeta, 3)$, a closer approximation of the logical 0 GKP state. Again, here $\zeta = 1.9$. Creating this state would require at least eight cat states, each with $\alpha = 2\sqrt{\pi}e^{\zeta}$. The $\beta(x, \zeta, 3)$ state also has small peaks at $\pm 8\sqrt{\pi}$, which are not visible here.

known. [Nor are sufficient values for Δ and k in Eq. (1) known.]

In this Letter, we began construction of approximate GKP states with a source of cat states. The most popular method to make cat states is by subtracting photon(s) from a squeezed vacuum state. It may be possible to alter the photon subtraction scheme to benefit our method to make GKP states. Note also that the experiment described in [13] naturally produces squeezed cats, so it may be a good candidate for an initial demonstration of our technique.

Although our scheme is built of apparently simple, well-understood optical operations, it will be difficult to achieve in an experiment. Matching the transverse and longitudinal shapes of all of the optical modes, especially during the squeezing stage [16,17], may be very difficult. A second concern is controlling photon loss during squeezing and storage of photons waiting for the next iteration. A third concern is the need for relative phase coherence of all-optical modes in this scheme. However, the level of phase coherence required for an initial demonstration is achieved in many modern quantum optics experiments. Take, for example, the continuous variable quantum teleportation experiments [18,19], which are able to maintain phase coherence during homodyne measurement, feed-forward, and displacement of optical modes.

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