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ALL SELF-DUAL MULTIMONOPOLES FOR ARBITRARY GAUGE GROUPS

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A B S T R A C T

The ADHM formalism is adapted to self-dual multimonopoles for arbitrary charge and arbitrary gauge group. Each configuration is characterized by a solution of a certain ordinary non-linear differential equation, which has chances to be completely integrable. For axially symmetric configurations it reduces to the integrable Toda lattice equations. The construction of the potential requires the solution of a further ordinary linear differential equation.

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ABSTRACT

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1. INTRODUCTION

The SU(4) supersymmetry (with arbitrary gauge group G) is the only known non-trivial quantum field theory in 3+1 dimensions which has good chances to be explicitly solvable[1]. In particular, there are indications that the conformal invariance of this theory is not broken by any anomalies. However, it may be broken spontaneously by a non-zero expectation value of the scalar field, as supersymmetry guarantees in certain directions of the field space a flat effective potential. Simultaneously, the gauge group G is broken down to a subgroup K, which always contains an Abelian factor. With respect to the latter, the theory contains magnetic monopoles.

These monopoles may be considered as excitations of the field of a dual quantum field theory. Dynamically, they are expected to behave exactly as a supersymmetry multiplet containing massive gauge bosons[2]. More precisely, the dual theory should have the same form as the original one, but with the dual gauge group G^{V} and a corresponding subgroup K^{V} [3].

To check these ideas, monopole solutions of the classical theory should be helpful. For simplicity let us consider the case where only one component ϕ of the six SU(4) components of the scalar field has a non-zero value at infinity, which corresponds to a non-zero vacuum expectation value in the quantum field theory. To obtain static solutions we want to find the most general finite energy solution of the equation

$$B_i = D_i \phi \ , \tag{1}$$

where the D_i are covariant derivatives, and B_i is the corresponding magnetic field. Of course, ϕ has to be in the adjoint representation of G.

If one introduces a dummy space direction with coordinate x^0, the scalar field may be considered as the corresponding component of the gauge potential. Then Eq. (1) takes the same form as the self-duality equation for instantons

$$B_i = E_i \ . \tag{2}$$

The ADHM construction yields the most general finite action solution of this equation[4]. In a suitable gauge, the potentials turn out to be algebraic.

2. THE ADHM CONSTRUCTION FOR MULTIMONOPOLES

The condition of finite energy and x^0-independence is somewhat more difficult to handle than the condition of finite action. Nevertheless we shall see that the ADHM construction can be adapted to our case, though its algebraic character becomes somewhat less obvious.

If one complexifies the R^4 to C^4, self-dual fields yield an analytic vector bundle over the space CP^3-CP^1 of anti-self-dual planes in C^4. The points in a fibre are functions χ from the corresponding plane to the fundamental representation of G which are constant with respect to covariant differentiations in the plane. CP^3-CP^1 can be covered by two charts, and the corresponding transition function of the bundle essentially yields back the potential.

Instead of using a transition function, the ADHM construction embeds the bundle in the direct product of the base with a larger linear space. In this larger space, χ is now defined globally and fulfils the equation[5]

$$\left(D_\mu e^\mu\right)^{A'A} \chi = \tilde{\Omega}^{A'A} + \Omega^{A'}_{B'} x^{B'A} \tag{3}$$

where for fixed B',B both Ω and $\tilde{\Omega}$ have to fulfil the Dirac equation

$$\left(D_\mu e^{+\mu}\right)_{AA'}\tilde{\Omega}^{A'B} = \left(D_\mu e^{+\mu}\right)_{AA'}\Omega^{A'}_{B'} = 0 \ . \tag{4}$$

Here the e^μ form the usual basis of the quaternions, represented by 2×2 matrices, with $e^0 = 1_2$. Moreover,

$$x = x_\mu e^\mu \tag{5}$$

represents the coordinates in C^4.

In order to avoid confusion, we now shall denote base space coordinates by X^μ. The points in the fibre over an anti-self-dual plane

$$X^{A'A}\pi_A = \omega^{A'} \tag{6}$$

fulfil the additional condition

$$\tilde{\Omega}^{A'A}\pi_A + \Omega^{A'}_{B'}\omega^{B'} = 0 \ . \tag{7}$$

Still this leaves too much freedom, as there are solutions χ which vanish all over some planes. One may either divide them out or replace the basis directly by R^4, with points in the fibre given by

$$\tilde{\Omega}^{A'A} + \Omega^{A'}_{B'}X^{B'A} = 0 \ . \tag{8}$$

For $G = SU(N)$, Eq. (8) yields for each X an N-dimensional complex hyperplane in the vector space of normalizable solutions of Eqs. (3) and (4). This vector space has a scalar product, which induces a natural connection on the C^N-bundle over R^4 described by Eq. (8) and in this way gives back the potential on which the construction was based.

Equation (8) may be written in the form

$$\Delta^+(X)v_i(X) = 0 \quad \text{for} \quad i = 1, \ldots, N \ , \tag{9}$$

where the $v_i(X)$ are orthonormal solutions of Eqs. (3) and (4) and Δ is of the form

$$\Delta = a + bX \tag{10}$$

with X-independent linear operators a,b. The potential is given by

$$A_\mu^{ij} = v_i^+ \partial_\mu v_j \ . \tag{11}$$

Different choices of the v_i yield gauge equivalent potentials.

All this applies equally for instantons and self-dual monopoles, with the single change that the condition of normalizability requires a four-dimensional integration in the former case and a three-dimensional one in the latter.

Now Eqs. (9) to (11) can be applied without knowing the details of the domain of Δ^+ [given by Eqs. (3) and (4)] or of its range (given by the normalizable solutions of the Dirac equation). One just needs some general information on the linear operators a,b and can show *a posteriori* that all potentials constructed by Eqs. (9) and (10) using such operators yield self-dual fields with the required properties. The main information taken over from the ADHM construction is that $\Delta^+\Delta$ is invertible and commutes with the quaternions. This already guarantees that the resulting field will be regular and self-dual[6].

In the instanton case the only further information used is the dimension of domain and range of Δ^+. The Dirac equation has k linearly independent normalizable solutions, where k is the instanton number, and the number of linearly independent normalizable solutions of Eqs. (3) and (4) is 2k + N.

In the monopole case we have to take account of the dummy coordinate x^0. We introduce its conjugate momentum z and take for the Dirac equation a basis of solutions of the form

$$\psi(\vec{x},x^0) = \exp\,(ix^0 z)\psi(\vec{x}) \ . \tag{12}$$

The number k(z) of normalizable solutions of this form is known[7,8]. It is zero, if z lies outside the interval spanned by the extremal eigenvalues of $\phi(\infty)$.

For Eqs. (3) and (4), χ and Ω can also be written in the form of Eq. (12). Moreover one obtains

$$\tilde{\Omega}^{A'A} = -\Omega^{A'}_{B'}x^{B'A} + 2\left(D_\mu e^\mu\right)^{A'A}(D^2)^{-1}\Omega^{B'}_{B'} \ , \tag{13}$$

except for those values of z for which D^2 is not invertible. These values are the eigenvalues of $\phi(\infty)$ and yield the jumping points of k(z).

Let us label the basis of normalizable solutions of Eqs. (3) and (4) by Ω of the form (12). Then in Eq. (13) the multiplication by x^0 may be expressed as a derivation with respect to z. Everything else commutes with ∂_0. Thus our Δ is of the form

$$\Delta = (i\partial_z + X)1_{k(z)} + iA(z) \ . \tag{14}$$

Here $1_{k(z)}$ is the k(z) dimensional unit matrix and A(z) also is k(z) dimensional. The matrix elements are quaternions.

Let us write

$$A(z) = e^{\mu}T_{\mu}(z) \ . \tag{15}$$

To obtain a $\Delta^+\Delta$ which commutes with the quaternions, the $T^{\mu}(z)$ must be anti-hermitian. $T^0(z)$ can always be absorbed by an equivalence transformation

$$\Delta \to U(z)^+\Delta U(z) \ , \quad U(z) \in U(k(z)) \ , \tag{16}$$

which represents a different choice of the basis of solutions of Eqs. (3) and (4) for each z. For the $T_i(z)$, i = 1, 2, 3, one obtains the differential equations

$$T_i'(z) = \frac{1}{2}\,\varepsilon_{ijk}\left[T_j(z)T_k(z)\right] \ . \tag{17}$$

Together with suitable boundary conditions at the jumping points of k(z) these equations yield the most general self-dual multimonopole configuration.

The $T_i(z)$ are meromorphic and can have at most simple poles. For physical values of z those can only occur at the jumping points of k(z). In a mathematically precise formulation, domain and range of Δ are Sobolev spaces, such that Δ and Δ^+ are bounded[9]. Moreover, $\Delta^+\Delta$ can easily be seen to be bounded below by a positive constant, as

$$\Delta^+\Delta = -\partial_z^2 + (x_i + iT_i)^+(x_i + iT_i) \ . \tag{18}$$

Thus $\Delta^+\Delta$ is invertible and all configurations constructed this way are regular.

According to Eq. (17) the trace part of the $T_i(z)$ is constant and can be absorbed by a translation of X. This defines a unique centre for arbitrary multimonopole configurations. Under rotations, the $T_i(z)$ transform as a vector. The traceless part of the symmetric tensor $tr(T_i(z)T_j(z))$ is independent of z and may be diagonalized by a rotation. This defines a system of natural axes for the multimonopole.

Apart from the translations, the general solution of Eq. (17) depends on $3(k(z)^2-1)$ parameters, of which $k(z)^2-1$ can be absorbed by an equivalence transformation of type (16), but with a z-independent U. For an axially symmetric multimonopole configuration, rotations around the symmetry axis can be represented as such equivalence transformations. If one decomposes SU(k(z)) with respect to a Cartan subalgebra which contains the generator of these equivalence transformations, Eq. (17) yields the completely integrable Toda lattice equations[10]. Note that these equations occurred in a configuration space analysis of spherically symmetric monopoles, while in z space the same equations describe the more general axially symmetric case. In general, the boundary conditions are different.

Let $t_i e^i$ be the residue of A(z), as z approaches some pole z_s. According to Eq. (17) the t_i must form an SU(2) subalgebra of SU(k(z)). The maximal embedding can only occur at the extremal eigenvalues of $\phi(\infty)$, as otherwise the equations above and below z_s decouple completely and yield independent monopoles in two direct factors of G.

3. THE CASE G = SU(2)

For G = SU(N) the number N is according to Eq. (9) the dimension of the kernel of Δ^+. According to Eq. (18) the dimension of the cokernel vanishes, such that

$$N = \text{index}(\Delta^+) \ . \tag{19}$$

To evaluate this index we have to find all local non-normalizable solutions of Eq. (9). Let us consider at some pole of A(z) an irreducible representation of the t_i of dimension d. With

$$t = t_i e^i \tag{20}$$

one finds

$$t^2 + t = -\sum_i t_i^2 = (d^2-1)/4 \ . \tag{21}$$

As the trace of t vanishes, all eigenvalues are known. One obtains d + 1 solutions of type

$$v(z) \sim (z-z_s)^{(d-1)/2} \tag{22}$$

and d − 1 non-normalizable solutions of type

$$v(z) \sim (z-z_s)^{(-d-1)/2} \ . \tag{23}$$

Let $\phi(\infty)$ only have two distinct eigenvalues, between which $k(z)$ is a constant k. To obtain $N = 2$, Eq. (9) must have $2k-2$ non-normalizable local solutions. This is only possible if at both jumping points of $k(z)$ the t_i form SU(2) algebras which are maximally embedded into SU(k). In this case, k is the magnetic charge[7,8]. One can check the behaviour of the scalar field by solving Eq. (9) in the limit $r \to \infty$. Up to normalizations the solutions behave like

$$v_{\pm}(z) \sim (z \mp z_s)^{(k-1)/2} \exp(\pm rz) \ . \tag{24}$$

Equation (11) yields for the eigenvalues of $\phi(\infty)$

$$\phi_{\pm} = \pm\left(z_s - \frac{k}{2r}\right) , \tag{25}$$

as it should be. The case $k = 1$ yields the simple BPS monopole, to which the ADHM formalism has been applied before[11].

Now let us count the number of multimonopole parameters. Let $T_i(z) + \delta_i(z)$ be an infinitesimal perturbation of a solution of Eq. (17) which still is a solution. We have to find zero modes of the operator

$$(P\delta)_i = \delta_i' - \varepsilon_{ijk}\left[T_j \delta_k\right] , \tag{26}$$

which are non-singular at the boundaries. On δ_i of the form

$$\delta_i = \left[T_i u\right] \tag{27}$$

this operator acts simply as a differentiation of u. Constant u yields an equivalence transformation and may be neglected. Thus we may consider P to be an operator defined on δ_i modulo δ_i of form (27). Then P^+P is bounded away from zero, such that the dimension of the kernel of P is equal to its index. Thus we only have to consider local zero modes of P close to the jumping points. With

$$(p\delta)_i = \varepsilon_{ijk}\left[t_j \delta_k\right] \tag{28}$$

one finds

$$p^2 + p = -\sum t_i^2 \ . \tag{29}$$

If one includes δ_i of type (27), the trace of p vanishes, such that one can easily calculate all eigenvalues. Under the adjoint action of the t_i, the algebra of SU(k) decomposes into $k-1$ irreducible representations of dimensions d_r, $r = 1, \ldots, k-1$. For each one there are $d_r + 2$ solutions of type

$$\delta \sim (z - z_s)^{(d_r-1)/2} \tag{30}$$

and $d_r - 2$ solutions of type

$$\delta \sim (z - z_s)^{-(d_r+1)/2} \; . \tag{31}$$

Thus

$$\text{index}(P) = 2(k^2 - 1) - 2\sum_r (d_r - 2) = 4(k - 1) \; . \tag{32}$$

If one adds the translations, one has the correct number of degrees of freedom.

Finally let us solve Eq. (17) for k = 2. If one rotates the configuration to the natural system of axes, the $T_i(z)$ form up to normalizations a standard set of SU(2) generators, and according to Eq. (17) this set does not depend on z. Thus we may write

$$T_i(z) = -if_i(z)\sigma_i/2 \; . \tag{33}$$

One obtains the first integrals

$$f_i(z)^2 - f_j(z)^2 = C_{ij} \tag{34}$$

and

$$f_i'(z) = \prod_{j\neq 1} \left(f_i^2(z) + C_{ij}\right)^{\frac{1}{2}} \; . \tag{35}$$

Thus we obtain Jacobi elliptic functions with a pure real and a pure imaginary period.

According to the choice of the z interval, one obtains SU(2), SU(3), and SU(4) configurations. In the latter case, the $f_i(z)$ must stay regular at the boundary of the interval, in the SU(2) case they must have poles at both ends.

The solutions of Eq. (9) can only be multiplied by a constant matrix, when z is shifted by a period. Thus they are essentially elliptic sigma functions.

Axial symmetry yields $C_{12} = 0$ and up to rescaling

$$f_3 = -\cot z \; , \tag{36}$$

$$f_1 = f_2 = -\frac{1}{\sin z} \tag{37}$$

or the corresponding hyperbolic functions. For this case the solutions of Eq. (9) can be read off from Ref. 9.

The axially symmetric configurations of higher charge can also be obtained from Eqs. (36) and (37), if one replaces the matrices σ_i in Eq. (33) by a maximal embedding of SU(2) in SU(k), i.e. by a k-dimensional representation of SU(2). This procedure also has more general applications.

Spherical symmetry requires $c_{12} = c_{13} = 0$, and

$$f_i(z) = -\frac{1}{z} . \tag{38}$$

In this case, only SU(3) and SU(4) configurations are possible. The SU(3) solution is known[12], but the SU(4) solution, which contains a free parameter, appears to be new.

4. OPEN PROBLEMS

Equations (17) and (9) yield all self-dual multimonopole configurations. If $\phi(\infty)$ has more than two different eigenvalues the boundary conditions for A(z) across the corresponding jumping points of k(z) still have to be studied in detail.

Equation (17) is non-linear, but it might be possible to characterize all solutions in a relatively simple way. More precisely, one might hope to represent the solutions as automorphic functions on some compact Riemann surface. The linear Eq. (9) may not allow such a treatment and in the general case may have to be treated numerically.

It will be interesting to compare the ADHM representation of the multimonopoles with other approaches[13,14].

REFERENCES

1. M. Sohnius and P. West., Phys. Lett. 100B (1981) 245.
2. H. Osborn, Phys. Lett. 83B (1979) 321.
3. P. Goddard and D. Olive, DAMTP 81/11 (1981).
4. M. Atiyah, N. Hitchin, V. Drinfeld and Yu. Manin, Phys. Lett. 65A (1978) 185.
5. H. Osborn, CERN-TH 3210 (1981).
6. E. Corrigan, D. Fairlie, P. Goddard and S. Templeton, Nucl. Phys. B140 (1978) 31.
7. C. Callias, Comm. Math. Phys. 62 (1978) 213.
8. R. Bott and R. Seeley, Comm. Math. Phys. 62 (1978) 235.
9. W. Nahm, Phys. Lett. 93B (1980) 42.
10. D. Olive, ICTP 80-81-1 and 80-81-41.
11. W. Nahm, Phys. Lett. 90B (1980) 412.
12. A. Bais and H. Weldon, Phys. Rev. Lett. 41 (1978) 601.
13. E. Corrigan and P. Goddard, DAMTP 81/9 (1981).
14. P. Forgacs, Z. Horvath and L. Palla, KFKI-1981-21, KFKI-1981-23 with further references.