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All separable Banach spaces admit for every $\varepsilon > 0$ fundamental total and bounded by $1 + \varepsilon$ biorthogonal sequences

by

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Abstract. It is proved that in every infinite dimensional Banach space X for every $\varepsilon > 0$ there exists a biorthogonal sequence (x_n, x_n^*) such that (i) linear combinations of the x_n 's are dense in X , (ii) if $x \in X$ and $x_n^*(x) = 0$ for all n , then $x = 0$, (iii) $\|x_n\| \|x_n^*\| < 1 + \varepsilon$ for all n .

The proof bases upon the following result due to Milman:

(M) *If E is a finite dimensional subspace of an infinite dimensional Banach space X , then, for every $\varepsilon > 0$, there exists a subspace F of X with $\dim F > \varepsilon^{-1}$ such that*

$$\|e + f\| \geq (1 - \varepsilon) \max(\|e\|, \|f\|) \quad \text{for every } e \text{ in } E \text{ and } f \text{ in } F.$$

A proof of (M) is included.

Introduction. It is known (cf., e.g. [1], p. 238 or [15]) that if X is a finite dimensional Banach space (say, $\dim X = m$), then X admits a biorthogonal sequence $(x_n, x_n^*)_{n=1}^m$ with $\|x_n\| \|x_n^*\| = 1$ for $n = 1, \dots, m$. In the present paper we improve results of [3] and [11]. We establish the following.

THEOREM 1. *Let X be an infinite dimensional separable Banach space. Then, for every $\varepsilon > 0$, there exists in X a fundamental total and bounded by $1 + \varepsilon$ biorthogonal sequence.*

Recall that a sequence (e_n, e_n^*) , where e_n are elements of X and e_n^* are elements of X^* — the dual of X , is biorthogonal if $e_n^*(e_m) = \delta_n^m$ for $n, m = 1, 2, \dots$, is total if, for every $x \in X$, the condition $e_n^*(x) = 0$ for every $n = 1, 2, \dots$ implies $x = 0$, is fundamental if, for every $x^* \in X^*$, the condition $x^*(e_n) = 0$ for every $n = 1, 2, \dots$ implies $x^* = 0$, is bounded by a $c \geq 1$ if $\|e_n\| \|e_n^*\| \leq c$ for every $n = 1, 2, \dots$.

The paper consists of two sections. Theorem 1 is proved in the first section. The proof bases on the following result due to Dvoretzky [5] and Milman [10], Theorem 5.8:

(D-M). *Given $\delta > 0$ and positive integers n, m, N . There exists a positive integer $K = K(n, m, N, \delta)$ such that if Y is a Banach space of dimension greater than K , then, for every n -dimensional linear subspace E of Y ,*

and for every linear subspace Y_1 of Y of codimension m , there exists a linear subspace F of Y such that

- (i) $\dim F = N$, $F \subset Y_1$, $F \cap E = \{0\}$,
- (ii) if $G = E + F$, the direct sum of subspaces E and F , then

$$\max(\|P\|, \|I_G - P\|) < 1 + \delta,$$

where $P: G \xrightarrow{\text{onto}} F$ is the projection with $\ker P = E$,

- (iii) there exists an isomorphism $T: \ell_N^2 \xrightarrow{\text{onto}} F$ such that

$$\max(\|T\|, \|T^{-1}\|) < 1 + \delta.$$

Here by I_G we denote the identity operator on G and by ℓ_N^2 the Hilbert space of dimension N .

The second section of the paper contains a proof of (D-M). We concentrate ourselves on the case of complex Banach spaces which is not treated by Milman [8], [10].

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1. Proof of Theorem 1. Since X is separable, there exist sequences (x_j) of elements of X and (x_j^*) of elements of X^* such that

- (1) if $x \in X$ and $x_j^*(x) = 0$ for all j , then $x = 0$,
- if $x^* \in X^*$ and $x^*(x_j) = 0$ for all j , then $x^* = 0$.

After fixing such sequences (x_j) and (x_j^*) and ε with $0 < \varepsilon < 1$ we define inductively a sequence $(e_m)_{m \geq 0}$ of elements of X , a sequence $(e_m^*)_{m \geq 0}$ of elements of X^* and an increasing sequence $(n_m)_{m \geq 0}$ of the indices so that

- (2) the sequence $(e_1, e_2, \dots, e_{n_s}; e_1^*, e_2^*, \dots, e_{n_s}^*)$ is biorthogonal for $s = 1, 2, \dots$,
- (3) $\|e_j\| \|e_j^*\| < 1 + \varepsilon$ for $1 \leq j \leq n_s$ and for $s = 1, 2, \dots$,
- (4) $\text{span}(e_j)_{j \leq n_{2q-1}} \supset \text{span}(x_p)_{p \leq q}$ for $q = 1, 2, \dots$,
- (5) $\text{span}(e_j^*)_{j \leq n_{2q}} \supset \text{span}(x_p^*)_{p \leq q}$ for $q = 1, 2, \dots$.

(By $\text{span}(a_j)_{j \leq k}$ we denote the smallest linear subspace spanned by the vectors a_1, a_2, \dots, a_k .)

We put $e_0 = 0$, $e_0^* = 0$ and $n_0 = 0$. Assume that, for some $t \geq 0$, elements e_0, e_1, \dots, e_{n_t} , functionals $e_0^*, e_1^*, \dots, e_{n_t}^*$ and indices n_0, n_1, \dots, n_t have been defined to satisfy conditions (2)–(5) for all $s \leq t$. We shall define the next index n_{t+1} , the elements e_j and the functionals e_j^* for $n_t < j \leq n_{t+1}$. We consider separately two cases.

Let $n_{t+1} = 2q - 1$ for some $q = 1, 2, \dots$. First using the standard Schmidt biorthogonalization procedure and the inductive hypothesis, we pick a $y \in X$ and $y^* \in X^*$ so that

$$(6) \quad \text{span}(e_1, e_2, \dots, e_{n_t}, y) \supset \text{span}(x_p)_{1 \leq p \leq q}$$

and

$$(7) \quad \text{the sequence } (e_1, e_2, \dots, e_{n_t}, y; e_1^*, e_2^*, \dots, e_{n_t}^*, y^*) \text{ is biorthogonal.}$$

Next we pick an integer $r \geq 1$ so that

$$(8) \quad 2^{-r/2} (\|y\| + \|y^*\|) < \varepsilon/4$$

and we put $n_{t+1} = n_t + 2^r$. Now applying (D-M) we pick, for $Y = X$, $N = 2^r - 1$, $E = \text{span}(e_1, e_2, \dots, e_{n_t}, y)$, $\delta = \varepsilon/4$, $Y_1 = \ker y^* \cap \bigcap_{1 \leq j \leq n_t} \ker e_j^*$,

a linear subspace F of X which satisfies conditions (i)–(iii). Let $(v_j)_{1 \leq j \leq N}$ denote the unit vector basis of ℓ_N^2 and let $(v_j^*)_{1 \leq j \leq N}$ denote the coordinate functionals on ℓ_N^2 . Let us set $f_j = T(v_j)$ and $f_j^* = (T^{-1})^*(v_j^*)$ for $1 \leq j \leq N$. (By S^* we denote the adjoint of a linear operator S .) Furthermore, let $(w_{i,j}^r)_{1 \leq i, j \leq 2^r}$ be the $2^r \times 2^r$ Walsh orthogonal matrix, i.e. $w_{i,j}^r = 2^{-r/2} w_i(2^{-r-1}(2j-1))$, where $(w_i)_{1 \leq i < \infty}$ denotes the Walsh orthonormal system (cf. [7], Kapitel IV, § 6). For $i = 1, 2, \dots, 2^r$ we put

$$e_{n_t+i} = w_{i,1}^r y + \sum_{j=2}^{2^r} w_{i,j}^r f_{j-1},$$

and we define $e_{n_t+i}^* \in X^*$ to be any norm preserving extension of the linear functional g_i^* defined on $G = E + F$ by

$$g_i^* = w_{i,1}^r y_G^* + \sum_{j=2}^{2^r} w_{i,j}^r P^*(f_{j-1}^*),$$

where y_G^* denotes the restriction of y^* to G .

To complete the induction in case 1° it remains to verify that the sequences $(e_j)_{0 \leq j \leq n_{t+1}}$ and $(e_j^*)_{1 \leq j \leq n_{t+1}}$ satisfy conditions (2)–(5).

By the inductive hypothesis, $e_i^*(e_k) = \delta_i^k$ for $i, k \leq n_t$. If $i \leq n_t$ and $n_t < k \leq n_{t+1}$, then $e_i^*(e_k) = 0$ because $e_i^*(y) = 0$ (by (7)) and for $1 \leq j \leq N$, $e_i^*(f_j) = 0$ (because $f_j \in F \subset \ker e_i^*$). If $n_t < i \leq n_{t+1}$ and $k \leq n_t$, then $e_i^*(e_k) = g_{i-n_t}^*(e_k) = 0$ because $y^*(e_k) = 0$ (by (7)) and for $1 \leq j \leq N$, $P^*(f_j^*)(e_k) = f_j^*(P(e_k)) = 0$ (because $e_k \in E = \ker P$). If $n_t < i, k \leq n_{t+1}$, then $e_i^*(e_k) = g_{i-n_t}^*(e_k) = \delta_i^k$ because the sequence

$$(y, f_1, f_2, \dots, f_N; y_G^*, P^*(f_1^*), P^*(f_2^*), \dots, P^*(f_N^*))$$

is biorthogonal and the matrix $(w_{i,j}^r)_{1 \leq i, j \leq 2^r}$ is orthogonal. This proves (2).

To check (3) observe that $\|w_{i,j}^r\| = 2^{-r/2}$ for $i, j = 1, 2, \dots, 2^r$. Thus, by (8),

$$\begin{aligned} \|e_{n_t+i}\| &\leq \|y\| 2^{-r/2} + \left\| T \left(\sum_{j=2}^{2^r} w_{i,j}^r v_{j-1} \right) \right\| < \varepsilon/4 + \|T\| (1-2^{-r})^{1/2} \\ &< \varepsilon/4 + 1 + \varepsilon/4 < 1 + \varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \|e_{n_t+i}^*\| &= \|g_i^*\| \leq \|y^*\| 2^{-r/2} + \|P^*\| \|(T^{-1})^*\| \cdot \left\| \sum_{j=2}^{2^r} w_{i,j}^r v_{j-1}^* \right\| \\ &< \varepsilon/4 + (1+\varepsilon/4)^2 < 1 + \varepsilon. \end{aligned}$$

The above inequalities together with the inductive hypothesis prove (3).

Finally, since the matrix $(w_{i,j}^r)_{1 \leq i, j \leq 2^r}$ is invertible and since the elements y, f_1, f_2, \dots, f_N are linearly independent, we infer that

$$\text{span}(y, f_1, f_2, \dots, f_N) = \text{span}(e_{n_t+1}, e_{n_t+2}, \dots, e_{n_t+1}).$$

Hence, remembering that $t+1 = 2q-1$, we get, by (6),

$$\text{span}(e_j)_{1 \leq j \leq n_{2q-1}} \supset \text{span}(x_p)_{1 \leq p \leq q}.$$

This proves (4) for $t+1 = 2q-1$. Condition (5) for $t+1 = 2q-1$ follows directly from the inductive hypothesis.

2° $t+1 = 2q$ for some $q = 1, 2, \dots$ Similarly as in case 1° , we first pick $y \in X$ and $y^* \in X^*$ so that

$$(6^*) \quad \text{span}(e_1^*, e_2^*, \dots, e_{n_t}^*, y^*) \supset \text{span}(x_p^*)_{1 \leq p \leq q}$$

and

(7*) the sequence $(e_1, e_2, \dots, e_{n_t}, y; e_1^*, e_2^*, \dots, e_{n_t}^*, y^*)$ is biorthogonal.

Next we put $n_{t+1} = n_t + 2^r$, where $r \geq 1$ is an integer satisfying (8). Now, by (D-M), we pick for $Y = X^*$, $N = 2^r - 1$, $E = \text{span}(e_1^*, e_2^*, \dots, e_{n_t}^*, y^*)$, $Y_1 = \text{ker } y \cap \text{ker } e_j$ (where $\text{ker } x = \{x^* \in X^* : x^*(x) = 0\}$) and $\delta = \varepsilon/4$ a linear subspace F of X^* which satisfies conditions (i)-(iii). Let us set, as in case 1° , $T(v_j) = f_j \in F \subset X^*$ and $(T^{-1})^*(v_j^*) = f_j^* \in F^*$ for $j = 1, 2, \dots, N$ and define for $i = 1, 2, \dots, 2^r$ the functionals $e_{n_t+i}^*$ by

$$e_{n_t+i}^* = w_{i,1}^r y^* + \sum_{j=2}^{2^r} w_{i,j}^r f_{j-1}^*.$$

Let $e_{n_t+i}^{**} \in X^{**}$ be any norm preserving extension of the linear functional g_i^* defined on G by

$$g_i^* = w_{i,1}^r y_G + \sum_{j=2}^{2^r} w_{i,j}^r P^*(f_{j-1}^*)$$

(where $y_G \in G^*$ is defined by $y_G(g) = g(y)$ for $g \in G \subset X^*$). Finally, for $i = 1, 2, \dots, 2^r$, using Goldstine's Theorem we define e_{n_t+i} to be an arbitrary element of X such that

$$\begin{aligned} e_k^*(e_{n_t+i}) &= e_{n_t+i}^{**}(e_k^*) \quad \text{for } k = 1, 2, \dots, n_{t+1}, \\ \|e_{n_t+i}\| &\leq \|e_{n_t+i}^{**}\| (1 + \varepsilon/16). \end{aligned}$$

We omit the verification (similar to that of case 1°) that such defined sequences $(e_j)_{0 \leq j \leq n_{t+1}}$ and $(e_j^*)_{0 \leq j \leq n_{t+1}}$ satisfy conditions (2)-(5). This completes the induction and the proof of the theorem.

Remarks. A. The proof of Theorem 1 gives also:

If X^* is separable, then for every $\varepsilon > 0$ there exists in X a biorthogonal sequence (e_n, e_n^*) bounded by $1 + \varepsilon$ which is fundamental (equivalently the e_n^* 's are linearly dense in X) and such that the e_n^* 's are linearly dense in X^* .

B. We do not know any example of a separable Banach space which does not admit any fundamental and total biorthogonal sequence bounded by 1.

2. We begin with some notation (cf. Dvoretzky [5] and Milman [8], [9]). By S_K we denote the unit sphere of the K -dimensional real Hilbert space \mathcal{E}_K . By $\mu_{K,m}$ we denote the rotation invariant normalized Borel measure on the Grassman manifold \mathcal{G}_K^m of all m -dimensional linear subspaces of \mathcal{E}_K . Clearly, $\mu_{K,1}$ can be regarded as the normalized Lebesgue measure of symmetric Borel subsets of S_K . Given a symmetric Borel subset A of S_K a $t \geq 0$ and $m = 1, 2, \dots, K$, we denote by A_t the set of all points of S_K whose geodesic distance from A is $\leq t$ and we put

$$A^m = \{H \in \mathcal{G}_K^m : H \cap A \neq \emptyset\}.$$

Clearly, if A is measurable, so are A_t and $A_t^m = (A_t)^m$.

The basic tool for the proof of (D-M) is the following result due to Dvoretzky [5], Theorem 2(B) (for some details of the proof cf. [6]).

For every $t > 0$ and for every symmetric Borel subset A of S_K , we have

$$(D_1) \quad \mu_{K,1}(A_t) \geq [\mu_{K,2}(A^2)]^{1/2} (1 - e^{-2t((K-2)\mu_{K,2}(A^2)/2\pi)^{1/2}})^2, \quad K = 3, 4, \dots$$

$$(D_2) \quad \mu_{K,2}(A_t^2) \geq \mu_{K,m}(A^m) (1 - e^{-\frac{t}{m-2}((K-m)/2\pi)^{1/2}})^{m-2}, \quad K = 4, 5, \dots, \\ m = 3, 4, \dots, K-1.$$

The next lemma shows how the above inequalities work.

LEMMA 2.1. Let $\varepsilon > 0$ and let positive integers s and $m \geq 3$ be given. Then there exists a $C = C(\varepsilon, s, m)$ such that, for every integer $K > C$ and for every symmetric Borel subset A of S_K with $A^m = \mathcal{G}_K^m$, there exists an s -dimensional linear subspace H of \mathcal{E}_K such that $S_K \cap H \subset A$.

Proof⁽¹⁾. First observe that, given $\varepsilon > 0$ and a positive integer s , there exists an $\eta = \eta(\varepsilon, s) < 1$ such that if B is a symmetric Borel subset of S_s with $\mu_{s,1}(B) > 1 - \eta$, then B is an $\varepsilon/3$ -net for S_s . This is obvious because the Lebesgue measure of a geodesic sphere (in S_s) of radius $\varepsilon/3$ is positive. Since $A^m = \mathcal{G}_K^m$, it follows from (D₂) that for every t with $\varepsilon/3 > t > 0$ and for K large enough (precisely $K > C_1(\eta, m)$),

$$\mu_{K,2}(A_t^2) \geq (1 - e^{-\frac{t}{m-2}((K-m)/2\pi)^{1/2}})^{m-2} > (1 - \eta)^2.$$

Thus, by (D₁) applied for A_t , we get

$$\mu_{K,1}(A_t) > (1 - \eta)(1 - e^{-2t(K-2)(1-\eta)/2\pi})^2.$$

for K large enough. Since $(A_t)_t = A_{2t}$, we infer that $\mu_{K,1}(A_{2t}) > 1 - \eta$, for K large enough (precisely for $K > C_2(\eta, m)$).

Now, for each $H \in \mathcal{G}_K^s$, let $\mu_{s,1}^H$ be the normalized rotation invariant Borel measure on $S_K \cap H$. By the uniqueness of the normalized rotation invariant Borel measure on S_K , for every symmetric Borel subset B of S_K , we have

$$\mu_{K,1}(B) = \int_{\mathcal{G}_K^s} \mu_{s,1}^H(B \cap H) \cdot \mu_{K,s}(dH).$$

Applying this formula to $B = A_{2t}$, for $K > C_2(\eta, m)$, we get

$$1 - \eta < \int_{\mathcal{G}_K^s} \mu_{s,1}^H(A_{2t} \cap H) \mu_{K,s}(dH).$$

Hence, for some $H \in \mathcal{G}_K^s$, $1 - \eta < \mu_{s,1}^H(A_{2t} \cap H)$. Thus, by the choice of η , $A_{2t} \cap H$ is an $\varepsilon/3$ -net for $H \cap S_K$. Thus $S_K \cap H \subset A_\varepsilon \cap H$ because $t < \varepsilon/3$. Hence $S_K \cap H \subset A_\varepsilon$ for $K > C(\varepsilon, s, m) = C_2(\eta(\varepsilon, s), m)$.

Our next lemma goes back to Krasnosel'skii, Krein, Milman [16] (cf. also [3]). We state it for a strictly convex Banach space, i.e. such a space whose unit sphere does not contain intervals.

LEMMA 2.2. Let $1 \geq \varepsilon > 0$ and let s and $m \geq 3$ be positive integers. Let Y be a finite dimensional strictly convex Banach space which is a direct sum of its subspaces E and Z . Assume that $\dim E = m$ and Z is isometrically isomorphic to the Hilbert space \mathbb{V}_L^s with $L > C(\varepsilon/6, s, m+1)$ in the case of real scalars and $L > C(\varepsilon/6, 2s, 2m+1)$ in the case of complex scalars, where $C(\cdot, \cdot, \cdot)$ is the function of Lemma 2.1. Then there exists a linear subspace H of Z with $\dim H = s$ such that the projection $P: E + H \xrightarrow{\text{onto}} H$ with $\ker P = E$ has the norm $\|P\| \leq 1 + \varepsilon$.

⁽¹⁾ The author has learned this proof from T. Figiel.

Proof. If B is a Banach space, then $\text{rdim} B$ denotes the dimension of B over the reals. It will be convenient for us to identify the unit sphere of Z with the metric space $S_{\text{rdim} Z}$. A point $z \in S_{\text{rdim} Z}$ is E -supporting if $\|z + e\| \geq 1$ for every $e \in E$. We denote by A the (closed and symmetric) subset of $S_{\text{rdim} Z}$ consisting of all E -supporting points of $S_{\text{rdim} Z}$. First we show that, for every $H \in \mathcal{G}_{\text{rdim} Z}^{\text{rdim} E+1}$, the intersection $A \cap H$ is non-empty. To this end, fix a normalized basis for E over the real scalars, say $e_1, e_2, \dots, e_{\text{rdim} E}$, and a real subspace H of Z with $\text{rdim} H = \text{rdim} E + 1$. For $j = 1, 2, \dots, \text{rdim} E$ and for $z \in H \cap S_{\text{rdim} Z}$ we put

$$a_j(z) = \begin{cases} 0, & \text{if } \|te_j + z\| \geq 1 \text{ for every real } t, \\ \text{the unique real number } t \neq 0 \text{ such that } \|te_j + z\| = 1, & \text{otherwise.} \end{cases}$$

Since Y is strictly convex, all $a_j(\cdot)$ are well-defined continuous functions on $S_{\text{rdim} Z} \cap H$. Now we define $\Phi: S_{\text{rdim} Z} \cap H \rightarrow \mathcal{C}_{\text{rdim} E}$ by

$$\Phi(z) = (a_1(z), a_2(z), \dots, a_{\text{rdim} E}(z)) \quad \text{for } z \in S_{\text{rdim} Z} \cap H.$$

Clearly, Φ is a continuous antipodic function. Thus, by the Borsuk Theorem [2], there exists a $z_0 \in S_{\text{rdim} Z} \cap H$ such that $\Phi(z_0) = 0$. Evidently, the point z_0 is E -supporting because there are $\text{rdim} E$ straight lines passing through z_0 supporting the sphere $S_{\text{rdim} Z} \cap H$ at z_0 and parallel to the linearly independent vectors $e_1, e_2, \dots, e_{\text{rdim} E}$, respectively.

We have just proved that $A^{\text{rdim} E+1} \subset \mathcal{G}_{\text{rdim} Z}^{\text{rdim} E+1}$. Thus, by Lemma 2.1, the assumption imposed on L yields that there exists a real subspace H_R of Z such that $H_R \cap S_{\text{rdim} Z} \subset A_{\varepsilon/6}$, and $\text{rdim} H_R = s$ in the real case and $\text{rdim} H_R = 2s$ in the complex case. Since in the case of complex scalars A is circled, it follows that if H is the smallest complex subspace containing H_R , then $H \cap S_{\text{rdim} Z} \subset A_{\varepsilon/6}$. Thus in both cases of the real and of the complex scalars there exists an s -dimensional subspace H of Z such that $H \cap S_{\text{rdim} Z} \subset A_{\varepsilon/2}$. To complete the proof we show that such chosen H has the desired property. To this end, pick $z \in H$ with $\|z\| = 1$. Since $H \cap S_{\text{rdim} Z} \subset A_{\varepsilon/2}$, there exists a $z_0 \in A$ such that $\|z - z_0\| \leq \varepsilon/2$. Thus, for every $e \in E$, we have

$$\|z + e\| \geq \|z_0 + e\| - \|z - z_0\| \geq 1 - \varepsilon/2 > (1 + \varepsilon)^{-1}.$$

Hence, by the homogeneity of the norm, we get

$$\|z + e\| > (1 + \varepsilon)^{-1} \|z\| \quad \text{for every } 0 \neq z \in H \text{ and every } e \in E.$$

The last inequality implies that the projection $P: H + E \xrightarrow{\text{onto}} H$ with $\ker P = E$ has the norm $\|P\| < 1 + \varepsilon$.

Proof of (D-M). First observe that given a positive integer n and $\delta > 0$, there exists an integer $d = d(n, \delta)$ such that the unit sphere of any n -dimensional Banach space admits a δ -net consisting of less than

d points. Thus the standard Mazur's technique of constructing basic sequences (cf. e.g. [13]) yields the existence of a subset B of the unit sphere of Y^* such that B consists of exactly d points and if $Y_B = \bigcap_{y^* \in B} \ker y^*$,

then the projection $Q: E + Y_B \xrightarrow{\text{onto}} E$ with $\ker Q = Y_B$ has the norm $\|Q\| < 1 + \delta$. Now, by the Dvoretzky Theorem (cf. [5], [14]), for every positive integer L , there exists a $D = D(L, \delta)$ such that if $\dim Y > D + d + m \geq D + \text{codim } Y_B + \text{codim } Y_1$, then there exists a linear subspace W of $Y_B \cap Y_1$ which admits an isomorphism $T_W: W \xrightarrow{\text{onto}} \ell_L^2$ with $\|w\| \leq \|T_W(w)\| \leq (1 + \delta)^{1/3} \|w\|$ for every $w \in W$. Using Lemma 1 of [12] we can construct a Banach space Y_W which contains ℓ_L^2 isometrically and has the property that the isomorphism T_W admits an extension to an isomorphism T from $E + W$ onto Y_W satisfying the condition $\|x\| \leq \|T(x)\| \leq (1 + \delta)^{1/3} \|x\|$ for every $x \in E + W$. The finite dimensional space Y_W admits a renorming such that in the new norm Y_W is strictly convex and the ratio of the new norm and the original norm of every element belongs to the interval $[1, (1 + \delta)^{1/3}]$. Being a little bit careful we can do the renorming so that in addition the new norm coincides with the original norm on $T(W) = \ell_L^2$.

In the sequel we denote by \tilde{Y} the space Y_W under this new strictly convex norm and we regard T as an isomorphism from $E + W$ onto \tilde{Y} . Now, assuming that $L > C(((1 + \delta)^{1/3} - 1)/6, N, n + 1)$ in the case of real scalars and $L > C(((1 + \delta)^{1/3} - 1)/6, 2N, 2n + 1)$ in the case of complex scalars, we can apply Lemma 2.2 to choose an N -dimensional linear subspace of ℓ_L^2 so that the projection \tilde{P} from $T(E) + H$ onto H with $\ker \tilde{P} = T(E)$ has the norm $\|\tilde{P}\| < (1 + \delta)^{1/3}$. Finally, we put $F = T^{-1}(H)$ and $P = T^{-1}\tilde{P}T$. We admit

$$\begin{aligned} K &= K(n, m, N, \delta) \\ &= d(n, \delta) + m + D \left(1 + \left[C \left(\frac{(1 + \delta)^{1/3} - 1}{6}, 2N, 2n + 1 \right) \right], \delta \right). \end{aligned}$$

Added in proof. After this paper was submitted for publication, the author discovered the following simple derivation of (D-M) from the Dvoretzky Theorem. The essential part of (D-M) is:

(*) Given positive integers k and n and $1 > \varepsilon > 0$, there is an integer $N = N(k, n, \varepsilon)$ such that if X is a Banach space with $\dim X > N$ and E is a linear subspace of X with $\dim E = k$, then there exists an ε -Euclidean subspace F of X with $\dim F = n$ such that

$$\|e + f\| \geq (1 - \varepsilon) \|f\| \quad \text{for every } e \in E \text{ and } f \in F.$$

An n -dimensional Banach space F is ε -Euclidean iff there is an isomorphism $U: \ell_n^2 \rightarrow F$ with $\|U\| \|U^{-1}\| < 1 + \varepsilon$.

Proof. The Dvoretzky Theorem says: given $\delta > 0$ and a positive integer q , there exists a positive integer $d(q, \delta)$ such that every Banach space B with $\dim B \geq d(q, \delta)$ contains a δ -Euclidean subspace C with $\dim C = q$. Put $\delta = \min(\varepsilon, (1 - \varepsilon)^{-1/4} - 1)$, $N = d(d((2n + 2k - 1), \delta) - k, \delta) + k$. Let $h: X \rightarrow X/E$ be the quotient map. Since

$\dim X/E > N - k$, there is a δ -Euclidean subspace C_1 of X/E with $\dim C_1 = d(2n + 2k - 1, \delta) - k$. Thus $\dim h^{-1}(C_1) = d(2n + 2k - 1, \delta)$. Hence there is a δ -Euclidean subspace C_2 of $h^{-1}(C_1)$ with $\dim C_2 = 2n + 2k - 1$. Since $\dim E = k$, there is a δ -Euclidean subspace C of C_2 with $\dim C = 2n + k - 1$ and such that $C \cap E = \{0\}$, i.e. h restricted to C is an isomorphism. Thus there are isomorphisms $U: \ell_{2n+k-1}^2 \rightarrow C$ and $V: h(C) \rightarrow \ell_{2n+k-1}^2$ such that $\max(\|U\| \|U^{-1}\|, \|V\| \|V^{-1}\|) < 1 + \delta$.

Next recall the Krasnosel'skii-Krein-Milman Lemma [16] (cf. Lemma 2.2 in this paper) which says: If E is a linear subspace of a Banach space X with $\dim E = k$, then for every linear subspace Z of X with $\dim Z = k + 1$ and $E \cap Z = \{0\}$ there exists a $z \in Z$ such that $\|z\| = \|h(z)\| = 1$ ($h: X \rightarrow X/E$ is the quotient map). The lemma yields that the operator $T = VhU: \ell_{2n+k-1}^2 \rightarrow \ell_{2n+k-1}^2$ satisfies the condition

(+) If G is a linear subspace of ℓ_{2n+k-1}^2 with $\dim G = k + 1$, then there is a $g \in G$ with $\|g\| = 1$ and $\|Tg\| \geq (1 + \delta)^{-2}$.

Using (+) and the observation that if H is a subspace of ℓ_{2n+k-1}^2 with $\dim H = j - 1$ ($1 < j < n$), then

$$\dim(H^\perp \cap T^{-1}(T(H)^\perp)) \geq 2n + k - 1 - 2(j - 1) \geq k + 1$$

(A^\perp denotes the orthogonal complement of A in ℓ_{2n+k-1}^2), we define inductively a sequence $(g_j)_{1 \leq j \leq n}$ in ℓ_{2n+k-1}^2 so that, for $j = 1, 2, \dots, n$, $\|g_j\| = 1$, $\|T(g_j)\| < (1 + \delta)^{-2} \cap (g_j \in H_{j-1} \cap T^{-1}(T(H_{j-1})^\perp)$, where $H_0 = \{0\}$, $H_j = \text{span}\{g_1, g_2, \dots, g_j\}$. Clearly, $(g_j)_{1 \leq j \leq n}$ and $(T(g_j))_{1 \leq j \leq n}$ are orthogonal sequences. Hence, for $g = \sum_{j=1}^n t_j g_j \in H_n$,

$$\|T(g)\|^2 = \sum_{j=1}^n |t_j|^2 \|T(g_j)\|^2 \geq (1 + \delta)^{-4} \sum_{j=1}^n |t_j|^2 = (1 + \delta)^{-4} \|g\|^2.$$

Thus if $F = U(H_n)$ then, for every $f \in F$, we have

$$\begin{aligned} \|h(f)\| &= \|V^{-1} T U^{-1}(f)\| \geq \|V^{-1}\|^{-1} \|T U^{-1}(f)\| \geq \|V^{-1}\|^{-1} (1 + \delta)^{-2} \|U^{-1}(f)\| \\ &\geq \|V^{-1}\|^{-1} \|U^{-1}\|^{-1} (1 + \delta)^{-2} \|f\| \geq (1 + \delta)^{-4} \|f\|. \end{aligned}$$

Thus, for every $e \in E$, the definition of δ yields

$$\|e + f\| \geq \inf_{e \in E} \|e + f\| = \|h(f)\| \geq (1 - \varepsilon) \|f\|.$$

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