

All spacetimes with vanishing curvature invariants

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Abstract

All Lorentzian spacetimes with vanishing invariants constructed from the Riemann tensor and its covariant derivatives are determined. A subclass of the Kundt spacetimes results and we display the corresponding metrics in local coordinates. Some potential applications of these spacetimes are discussed.

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1. Introduction

A curvature invariant of order n is a scalar obtained by contraction from a polynomial in the Riemann tensor and its covariant derivatives up to the order n . In general, there are 14 algebraically independent curvature invariants of zeroth order, the simplest being the Ricci scalar. Many papers are devoted to studying the properties of the zeroth-order curvature invariants (see [1–5], and references therein) but higher-order curvature invariants remain largely unexplored. Recently, it was shown that for spacetimes in which the Ricci tensor does not possess a null eigenvector, an appropriately chosen set of zeroth-order curvature invariants contains all the information that is present in the Riemann tensor [5]. This is certainly not true for vacuum Petrov type N spacetimes with nonzero expansion or twist, all of whose zeroth- and first-order curvature invariants vanish, but for which there are non-vanishing curvature invariants of the second order [6]; and for some non-flat spacetimes in which all curvature invariants of all orders vanish [6, 7].

In this paper we shall determine all Lorentzian spacetimes for which all curvature invariants of all orders are zero. Indeed, we shall prove the following:

Theorem 1. *All curvature invariants of all orders vanish if and only if the following two conditions are satisfied:*

(A) *The spacetime possesses a non-diverging SFR (shear-free, geodesic null congruence).*

(B) *Relative to the above null congruence, all curvature scalars with non-negative boost weight vanish.*

The analytic form of condition (A), expressed relative to any spin basis where o^A is aligned with the null congruence in question, is simply

$$\kappa = \rho = \sigma = 0, \quad (1)$$

and the analytic form of condition (B) is

$$\Psi_0 = \Psi_1 = \Psi_2 = 0, \quad (2)$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = 0, \quad (3)$$

$$\Lambda = 0. \quad (4)$$

Previous equations are expressed in terms of quantities used in the Newman–Penrose formalism.

Spacetimes that satisfy condition (A) belong to Kundt’s class [8, 9] (also see section 4 and appendix A). Condition (B) implies that the spacetime is of Petrov type III, N or O (see equation (2)) with the Ricci tensor restricted by (3) and (4). (Note: throughout this paper we follow the notation of [10]; Λ is not the cosmological constant, it is the Ricci scalar up to a constant factor.)

The GHP formalism [10] assigns an integer, called the boost weight, to curvature scalars and certain connection coefficients and operators. This is important for this work and we shall summarize some of the key details of this notion in section 2.

The outline for the rest of the paper is as follows. Section 2 is devoted to the proof that the above conditions are sufficient for vanishing of curvature invariants. The ‘necessary’ part of theorem 1 is proved in section 3. The curvature invariants constructed in this section may also be useful for computer-aided classification of spacetimes. Kundt’s class of spacetimes admits a conveniently specialized system of coordinates and so it is possible to classify and explicitly describe all spacetimes with vanishing curvature invariants. This is briefly summarized in section 4, and some of the details are presented in appendix A. We conclude with a discussion.

Perhaps the best known class of spacetimes with vanishing curvature invariants are the pp-waves (or plane-fronted gravitational waves with parallel rays), which are characterized as Ricci-flat (vacuum) type N spacetimes that admit a covariantly constant null vector field. The vanishing of curvature invariants in pp-wave spacetimes has been known for a long time [11] (see also [12]), and the spacetimes obtained here can perhaps be regarded as extensions and generalizations of these important spacetimes. In many applications (e.g., in vacuum pp-wave spacetimes), the resulting exact solutions have a five-dimensional isometry group acting on three-dimensional null orbits (which includes translations in the transverse direction along the wavefront) and, hence, the solutions are plane waves. However, the solutions studied here need not be plane waves and are not necessarily vacuum solutions. In particular, non-vacuum spacetimes with a covariantly constant null vector are often referred to as generalized pp-wave and typically have no further symmetries (the arbitrary function in the metric is not subject to a further differential equation, namely, Laplace’s equation, when the Ricci tensor has the form of null radiation). The pp-wave spacetimes have a number of remarkable symmetry properties and have been the subject of much research [8]. For example, the existence of a homothety in spacetimes with plane-wave symmetry and the scaling properties of generally covariant field equations has been used to show that all generally covariant scalars are constant [13] and that metrics with plane-wave symmetry trivially satisfy every system of generally covariant vacuum field equations *except* the Einstein equations [14].

In addition to pp-waves, presently there are known to be three classes of metrics with vanishing curvature invariants: the conformally flat pure radiation spacetime given in [7]; the vacuum Petrov type-N non-expanding and non-twisting spacetimes [6] (this class contains the pp-waves); and vacuum Petrov type-III non-expanding and non-twisting spacetimes [15]. Naturally all of these spacetimes are subcases of the class studied here. The spacetimes studied in [16, 17] also belong to our class.

There are two important applications of the class of spacetimes obtained in this paper. Knowledge of all Lorentzian spacetimes for which all the curvature invariants constructed from the Riemann tensor and its covariant derivatives are zero, which implies that all covariant 2-tensors thus constructed are zero except for the Ricci tensor, will be of potential relevance in the equivalence problem and the classification of spacetimes, and may be a useful first step towards addressing the important question of when a spacetime can be uniquely characterized by its curvature invariants. More importantly perhaps, the spacetimes obtained in this paper are also of physical interest. For example, pp-wave spacetimes are exact solutions in string theory (to all perturbative orders in the string tension) [18, 19] and they are of importance in quantum gravity [20]. It is likely that all spacetimes for which all curvature invariants vanish will have similar applications and it would be worthwhile investigating these metrics further.

Finally, we note that it is possible to generalize theorem 1 by including spacetimes with non-vanishing cosmological constant. The assumptions regarding the Weyl and traceless Ricci tensors remain the same. Even in this general case, the invariants constructed from the Weyl tensor, the traceless Ricci tensor and their arbitrary covariant derivatives vanish. The only non-vanishing curvature invariants are order-zero curvature invariants constructed as various polynomials of the cosmological constant. It must be noted, however, that there may exist other types of spacetimes with constant curvature invariants.

2. Sufficiency of the conditions

Before tackling the proof of the main theorem we make some necessary definitions and establish a number of auxiliary results. We shall make use of the Newman–Penrose (NP) and the compacted (GHP) formalisms [10]. Throughout we work with a normalized spin basis o^A, ι^A , i.e.,

$$o_A \iota^A = 1.$$

The corresponding null tetrad is given by

$$l^\alpha \longleftrightarrow o^A \bar{\sigma}^{\dot{A}}, \quad n^\alpha \longleftrightarrow \iota^A \bar{\tau}^{\dot{A}}, \quad m^\alpha \longleftrightarrow o^A \bar{\tau}^{\dot{A}}, \quad \bar{m}^\alpha \longleftrightarrow \iota^A \bar{\sigma}^{\dot{A}}$$

with the only nonzero scalar products

$$l_\alpha n^\alpha = -m_\alpha \bar{m}^\alpha = 1.$$

We also recall that

$$o_A o^A = 0 = \iota_A \iota^A. \quad (5)$$

The spinorial form of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ is

$$R_{\alpha\beta\gamma\delta} \longleftrightarrow X_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} + \bar{X}_{\dot{A}\dot{B}\dot{C}\dot{D}} \varepsilon_{AB} \varepsilon_{CD} + \Phi_{AB\dot{C}\dot{D}} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{CD} + \bar{\Phi}_{\dot{A}\dot{B}CD} \varepsilon_{AB} \varepsilon_{\dot{C}\dot{D}}, \quad (6)$$

where

$$X_{ABCD} = \Psi_{ABCD} + \Lambda(\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{AD} \varepsilon_{BC})$$

and $\Lambda = R/24$, with R the scalar curvature. The Weyl spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$ is related to the Weyl tensor $C_{\alpha\beta\gamma\delta}$ by

$$C_{\alpha\beta\gamma\delta} \longleftrightarrow \Psi_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \varepsilon_{AB} \varepsilon_{CD}. \quad (7)$$

Projections of Ψ_{ABCD} onto the basis spinors o^A, ι^A give five complex scalar quantities $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$. The Ricci spinor $\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})} = \bar{\Phi}_{AB\dot{C}\dot{D}}$ is connected to the traceless Ricci tensor $S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4}Rg_{\alpha\beta}$

$$\Phi_{AB\dot{A}\dot{B}} \longleftrightarrow -\frac{1}{2}S_{ab}. \quad (8)$$

The projections of $\Phi_{AB\dot{A}\dot{B}}$ onto the basis spinors o^A, ι^A are denoted by $\Phi_{00} = \bar{\Phi}_{00}, \Phi_{01} = \bar{\Phi}_{10}, \Phi_{02} = \bar{\Phi}_{20}, \Phi_{11} = \bar{\Phi}_{11}, \Phi_{12} = \bar{\Phi}_{21}$ and $\Phi_{22} = \bar{\Phi}_{22}$.

Equations (2) and (3) imply

$$\Psi_{ABCD} = \Psi_4 o_A o_B o_C o_D - 4\Psi_3 o_{(A} o_B o_C \iota_{D)}, \quad (9)$$

$$\Phi_{AB\dot{C}\dot{D}} = \Phi_{22} o_A o_B \bar{o}_{\dot{C}} \bar{o}_{\dot{D}} - 2\Phi_{12} \iota_{(A} o_B) \bar{o}_{\dot{C}} \bar{o}_{\dot{D}} - 2\Phi_{21} o_A o_B \bar{\iota}_{(\dot{C}} \bar{o}_{\dot{D})}. \quad (10)$$

Following the convention established in [10], we say that η is a weighted quantity (a scalar, a spinor, a tensor or an operator) of type $\{p, q\}$, if for every non-vanishing scalar field λ a transformation of the form

$$o^A \mapsto \lambda o^A, \quad \iota^A \mapsto \lambda^{-1} \iota^A,$$

representing a boost in the $l^\alpha - n^\alpha$ plane and a spatial rotation in the $m^\alpha - \bar{m}^\alpha$ plane, transforms η according to

$$\eta \mapsto \lambda^p \bar{\lambda}^q \eta.$$

The boost weight b of a weighted quantity is defined by

$$b = \frac{1}{2}(p + q).$$

Directional derivatives are defined by

$$\begin{aligned} D &= l^\alpha \nabla_\alpha = o^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}}, & \delta &= m^\alpha \nabla_\alpha = o^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}}, \\ D' &= n^\alpha \nabla_\alpha = \iota^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}}, & \delta' &= \bar{m}^\alpha \nabla_\alpha = \iota^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}} \end{aligned}$$

and thus

$$\nabla^\alpha \longleftrightarrow \nabla^{A\dot{A}} = \iota^A \bar{\iota}^{\dot{A}} D + o^A \bar{o}^{\dot{A}} D' - \iota^A \bar{o}^{\dot{A}} \delta - o^A \bar{\iota}^{\dot{A}} \delta'. \quad (11)$$

In the GHP formalism, new derivative operators $\flat, \flat', \delta, \delta'$, which are additive and obey the Leibniz rule, are introduced. They act on a scalar, spinor or tensor η of type $\{p, q\}$ as follows:

$$\begin{aligned} \flat\eta &= (D + p\gamma' + q\bar{\gamma}')\eta, & \delta\eta &= (\delta - p\beta + q\bar{\beta}')\eta, \\ \flat'\eta &= (D' - p\gamma - q\bar{\gamma})\eta, & \delta'\eta &= (\delta' + p\beta' - q\bar{\beta})\eta. \end{aligned} \quad (12)$$

Let us explicitly write down how the operators $\flat, \flat', \delta, \delta'$ act on the basis spinors:

$$\begin{aligned} \flat o^A &= -\kappa \iota^A, & \flat \bar{o}^{\dot{A}} &= -\bar{\kappa} \bar{\iota}^{\dot{A}}, & \flat \iota^A &= -\tau' o^A, & \flat \bar{\iota}^{\dot{A}} &= -\bar{\tau}' \bar{o}^{\dot{A}}, \\ \flat' o^A &= -\tau \iota^A, & \flat' \bar{o}^{\dot{A}} &= -\bar{\tau} \bar{\iota}^{\dot{A}}, & \flat' \iota^A &= -\kappa' o^A, & \flat' \bar{\iota}^{\dot{A}} &= -\bar{\kappa}' \bar{o}^{\dot{A}}, \\ \delta o^A &= -\sigma \iota^A, & \delta \bar{o}^{\dot{A}} &= -\bar{\rho} \bar{\iota}^{\dot{A}}, & \delta \iota^A &= -\rho' o^A, & \delta \bar{\iota}^{\dot{A}} &= -\bar{\sigma}' \bar{o}^{\dot{A}}, \\ \delta' o^A &= -\rho \iota^A, & \delta' \bar{o}^{\dot{A}} &= -\bar{\sigma} \bar{\iota}^{\dot{A}}, & \delta' \iota^A &= -\sigma' o^A, & \delta' \bar{\iota}^{\dot{A}} &= -\bar{\rho}' \bar{o}^{\dot{A}}. \end{aligned} \quad (13)$$

The types and boost weights of various weighted quantities encountered in the GHP formalism are summarized in table 1.

Henceforth we shall assume that conditions (A) and (B) of theorem 1 hold, and by implication that equations (1)–(4) also hold. Without loss of generality, we also assume that o^A and ι^A are parallelly propagated along l^α . Analytically, this condition takes the form of the

Table 1. Boost weights of weighted quantities.

	p	q	b		p	q	b
o^A	1	0	$\frac{1}{2}$	ι^A	-1	0	$-\frac{1}{2}$
κ	3	1	2	κ'	-3	-1	-2
σ	3	-1	1	σ'	-3	1	-1
ρ	1	1	1	ρ'	-1	-1	-1
τ	1	-1	0	τ'	-1	1	0
\mathfrak{b}	1	1	1	\mathfrak{b}'	-1	-1	-1
δ	1	-1	0	δ'	-1	1	0
Ψ_r	$4 - 2r$	0	$2 - r$	Φ_{rt}	$2 - 2r$	$2 - 2t$	$2 - r - t$
				Λ	0	0	0

following two additional relations:

$$\gamma' = 0, \quad \tau' = 0. \tag{14}$$

Assumptions (A), (B) and conditions (14) greatly simplify the form of the spin-coefficient equations, the Bianchi and the commutators identities [10]. Most of these relations assume the form $0 = 0$. Some of the non-trivial relations are as follows:

$$\mathfrak{b}\tau = 0, \tag{15}$$

$$\mathfrak{b}\sigma' = 0, \tag{16}$$

$$\mathfrak{b}\rho' = 0, \tag{17}$$

$$\mathfrak{b}\kappa' = \bar{\tau}\rho' + \tau\sigma' - \Psi_3 - \Phi_{21}, \tag{18}$$

$$\mathfrak{b}\Phi_{21} = 0, \tag{19}$$

$$\mathfrak{b}\Psi_3 = 0, \tag{20}$$

$$\mathfrak{b}\Phi_{22} = \delta\Phi_{21} + (\delta - 2\tau)\Psi_3, \tag{21}$$

$$\mathfrak{b}\Psi_4 = \delta'\Psi_3 + (\delta' - 2\bar{\tau})\Phi_{21}, \tag{22}$$

$$\mathfrak{b}\mathfrak{b}' - \mathfrak{b}'\mathfrak{b} = \bar{\tau}\delta + \tau\delta', \tag{23}$$

$$\mathfrak{b}\delta - \delta\mathfrak{b} = 0. \tag{24}$$

Extending an idea introduced in [6], we make the following key definition.

Definition 2. We shall say that a weighted scalar η with the boost weight b is balanced if $\mathfrak{b}^{-b}\eta = 0$ for $b < 0$ and $\eta = 0$ for $b \geq 0$.

We can now prove the following.

Lemma 3. If η is a balanced scalar, then $\bar{\eta}$ is also balanced.

Proof. By definition, a weighted scalar η of type $\{p, q\}$ is changed by complex conjugation to a weighted scalar $\bar{\eta}$ of type $\{q, p\}$. The boost weight, however, remains unchanged. Let us also recall that

$$\bar{\mathfrak{b}} = \mathfrak{b}$$

and, hence, that

$$\mathfrak{p}^{-b}\bar{\eta} = \overline{\mathfrak{p}^{-b}\eta} = 0,$$

as desired. \square

Lemma 4. *If η is a balanced scalar, then*

$$\tau\eta, \rho'\eta, \sigma'\eta, \kappa'\eta, \quad (25)$$

$$\mathfrak{p}\eta, \delta\eta, \delta'\eta, \mathfrak{p}'\eta \quad (26)$$

are all balanced as well.

Proof. Let b be the boost weight of a balanced scalar η . From table 1 we see that the scalars listed in (25) have boost weights $b, b-1, b-1, b-2$, respectively. Hence, it suffices to show that the following quantities are all zero:

$$\mathfrak{p}^{-b}(\tau\eta), \quad \mathfrak{p}^{1-b}(\rho'\eta), \quad \mathfrak{p}^{1-b}(\sigma'\eta), \quad \mathfrak{p}^{2-b}(\kappa'\eta).$$

This follows from the Leibniz rule and from equations (15)–(20).

Next we show that the scalars in (26) are balanced as well. These scalars have boost weights $b+1, b, b, b-1$, respectively. Hence, it suffices to show that the following scalars are all zero:

$$\mathfrak{p}^{-1-b}(\mathfrak{p}\eta), \quad \mathfrak{p}^{-b}(\delta\eta), \quad \mathfrak{p}^{-b}(\delta'\eta), \quad \mathfrak{p}^{1-b}(\mathfrak{p}'\eta).$$

The vanishing of the first quantity follows immediately from definition 2. Using the commutator relation (24) we have

$$\mathfrak{p}^{-b}\delta\eta = \delta\mathfrak{p}^{-b}\eta = 0, \quad (27)$$

as desired. Vanishing of the quantity involving δ' follows by considering the complex conjugate of (27) and using the relation $\bar{\delta} = \delta'$ and lemma 3.

To show that the quantity involving \mathfrak{p}' vanishes, we employ (15), (23), and (27) to obtain

$$\mathfrak{p}^{1-b}(\mathfrak{p}'\eta) = \mathfrak{p}^{-b}(\mathfrak{p}'\mathfrak{p}\eta) + \bar{\tau}(\mathfrak{p}^{-b}\delta\eta) + \tau(\mathfrak{p}^{-b}\delta'\eta) = \mathfrak{p}^{-b}(\mathfrak{p}'\mathfrak{p}\eta).$$

We now proceed inductively and conclude that

$$\mathfrak{p}^{1-b}\mathfrak{p}'\eta = \mathfrak{p}'\mathfrak{p}^{1-b}\eta = 0. \quad \square$$

Lemma 5. *If η_1, η_2 are balanced scalars both of type $\{p, q\}$, then $\eta_1 + \eta_2$ is a balanced scalar of type $\{p, q\}$ as well.*

Proof. The sum $\eta_1 + \eta_2$ satisfies

$$\eta_1 + \eta_2 \mapsto \lambda^p \bar{\lambda}^q (\eta_1 + \eta_2), \quad \mathfrak{p}^{-b}(\eta_1 + \eta_2) = \mathfrak{p}^{-b}\eta_1 + \mathfrak{p}^{-b}\eta_2 = 0$$

and thus is a balanced scalar of type $\{p, q\}$. \square

Lemma 6. *If η_1, η_2 are balanced scalars, then $\eta_1\eta_2$ is also balanced.*

Proof. Let b_1, b_2 be the respective boost weights. Boost weights are additive and hence the boost weight of the product is $b_1 + b_2$. Setting $n = -b_1 - b_2$ and applying the Leibniz rule gives

$$\mathfrak{p}^n(\eta_1\eta_2) = \sum_{i=0}^n \binom{n}{i} \mathfrak{p}^i(\eta_1)\mathfrak{p}^{n-i}(\eta_2).$$

For $-b_1 \leq i \leq n$, the factor $\mathfrak{p}^i(\eta_1)$ vanishes. For $0 \leq i \leq -1 - b_1$, we have $n - i > -b_2$ and hence the other factor vanishes. Therefore the entire sum vanishes. \square

Definition 7. A balanced spinor is a weighted spinor of type $\{0, 0\}$ whose components are all balanced scalars.

Lemma 8. If S_1, S_2 are balanced spinors, then $S_1 S_2$ is also a balanced spinor.

Proof. The product $S_1 S_2$ is a weighted spinor of type $\{0, 0\}$ and its components are balanced scalars, thanks to lemma 6. \square

Lemma 9. A covariant derivative of an arbitrary order of a balanced spinor S is again a balanced spinor.

Proof. Applying the covariant derivative (11) to a balanced spinor S , we obtain

$$\nabla^{A\dot{A}} S = \iota^A \bar{\iota}^{\dot{A}} \mathfrak{p} S + o^A \bar{o}^{\dot{A}} \mathfrak{p}' S - \iota^A \bar{o}^{\dot{A}} \delta S - o^A \bar{\iota}^{\dot{A}} \delta' S.$$

From table 1, it follows that $\nabla^{A\dot{A}} S$ is again a weighted spinor of type $\{0, 0\}$ and its components are balanced scalars due to (13) and lemmas 3–5. \square

Lemma 10. A scalar constructed as a contraction of a balanced spinor is equal to zero.

Proof. A scalar constructed as a contraction of a balanced spinor also has zero boost weight, and therefore vanishes by definition 2.

Let us explain more intuitively how this works. A balanced spinor has the form $\sum C_i B_i$ where C_i are balanced scalars and B_i are the basis spinors (products of o, ι, \bar{o} and $\bar{\iota}$). Since the boost weight of each C_i is negative and the boost weight of each $C_i B_i$ is zero, it follows that the boost weight of each B_i is positive, i.e., there are more o and \bar{o} than ι and $\bar{\iota}$ in B_i . As a consequence of (5), a full contraction of each B_i vanishes. In a nutshell, all scalars constructed as a contraction of a balanced spinor vanish because each term contains more o than ι . \square

We are now ready to prove that conditions (A) and (B) of theorem 1 are sufficient for vanishing of all curvature invariants.

Proof. From table 1 and equations (19)–(22), it follows that the Weyl spinor (9) and the Ricci spinor (10) and their complex conjugates (lemma 3) are balanced spinors. Their products and covariant derivatives of arbitrary orders are balanced spinors as well (lemmas 8 and 9).

Finally, due to lemma 10 and equations (6)–(8), all curvature invariants constructed from the Riemann tensor and its covariant derivatives of arbitrary order vanish. \square

3. Necessity of the conditions

In this section we consider a spacetime with vanishing curvature invariants and prove that this spacetime satisfies the conditions listed in theorem 1. The Ricci scalar, being a curvature invariant, must vanish. To prove the other conditions, we consider various zeroth-, first- and second-order invariants formed from the Weyl and the Ricci spinors, as well as the Newman–Penrose equations and the Bianchi identities.

In the following, we will employ these Newman–Penrose equations:

$$\begin{aligned} \mathfrak{p}\rho - \delta'\kappa &= \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau' + \Phi_{00}, \\ \delta\rho - \delta'\sigma &= \tau(\rho - \bar{\rho}) + \kappa(\bar{\rho}' - \rho') - \Psi_1 + \Phi_{01} \end{aligned} \quad (28)$$

and the Bianchi identities

$$\begin{aligned}
\flat\Psi_3 - \delta'\Psi_2 - \flat\Phi_{21} + \delta\Phi_{20} - 2\delta'\Lambda &= 2\sigma'\Psi_1 - 3\tau'\Psi_2 + 2\rho\Psi_3 - \kappa\Psi_4 \\
&\quad - 2\rho'\Phi_{10} + 2\tau'\Phi_{11} + \bar{\tau}'\Phi_{20} - 2\bar{\rho}\Phi_{21} + \bar{\kappa}\Phi_{22}, \\
\flat\Psi_4 - \delta'\Psi_3 + \flat'\Phi_{20} - \delta'\Phi_{21} &= 3\sigma'\Psi_2 - 4\tau'\Psi_3 + \rho\Psi_4 \\
&\quad - 2\kappa'\Phi_{10} + 2\sigma'\Phi_{11} + \bar{\rho}'\Phi_{20} - 2\bar{\tau}\Phi_{21} + \bar{\sigma}\Phi_{22}, \\
\flat\Phi_{22} + \flat'\Phi_{11} - \delta\Phi_{21} - \delta'\Phi_{12} + 3\flat'\Lambda &= (\rho + \bar{\rho})\Phi_{22} + 2(\rho' + \bar{\rho}')\Phi_{11} \\
&\quad - (\tau + 2\bar{\tau}')\Phi_{21} - (2\tau' + \bar{\tau})\Phi_{12} - \bar{\kappa}'\Phi_{10} - \kappa'\Phi_{01} + \sigma'\Phi_{02} + \bar{\sigma}'\Phi_{20}, \\
\flat'\Psi_2 - \delta\Psi_3 + \flat\Phi_{22} - \delta\Phi_{21} + 2\flat'\Lambda &= \sigma\Psi_4 - 2\tau\Psi_3 + 3\rho'\Psi_2 - 2\kappa'\Psi_1 \\
&\quad + \bar{\rho}\Phi_{22} - 2\bar{\tau}'\Phi_{21} - 2\tau'\Phi_{12} + 2\rho'\Phi_{11} + \bar{\sigma}'\Phi_{20}.
\end{aligned} \tag{29}$$

First, we consider the well-known invariants

$$I = \Psi_{AB}{}^{CD}\Psi_{CD}{}^{AB}, \quad J = \Psi_{AB}{}^{CD}\Psi_{CD}{}^{EF}\Psi_{EF}{}^{AB}. \tag{30}$$

It is generally known that these invariants vanish if and only if the Petrov type is III, N or O. In the following, we choose the spinor basis o^A and ι^A in such a way that for the Petrov types III and N, o^A is the multiple eigenspinor of the Weyl spinor. Thus, condition (2) is satisfied.

We consider the three Petrov types case by case.

(a) *Petrov type N.*

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0.$$

Demanding that the invariant

$$\mathcal{I}_1 = \nabla_D{}^{\dot{E}}\Psi^{ABCD}\nabla_C{}^{\dot{D}}\Psi_{ABKL}\nabla_K{}^L\bar{\Psi}^{\dot{R}\dot{S}\dot{T}\dot{K}}\nabla_T{}^K\bar{\Psi}_{\dot{R}\dot{S}\dot{D}\dot{E}} = (2\Psi_4\bar{\Psi}_4\kappa\bar{\kappa})^2 \tag{31}$$

vanishes, we obtain

$$\kappa = 0. \tag{32}$$

In further calculations we assume that (32) is valid.

Vanishing of another invariant

$$\mathcal{I}_2 = K^{F\dot{F}E\dot{E}}{}_{M\dot{M}L\dot{L}}\bar{K}^{M\dot{M}L\dot{L}}{}_{F\dot{F}E\dot{E}} = (24\Psi_4\bar{\Psi}_4)^2(\rho\bar{\rho} + \sigma\bar{\sigma})^4, \tag{33}$$

where

$$K^{F\dot{F}E\dot{E}}{}_{M\dot{M}L\dot{L}} = \nabla^{F\dot{F}}\nabla^{E\dot{E}}\Psi^{ABCD}\nabla_{M\dot{M}}\nabla_{L\dot{L}}\Psi_{ABCD}, \tag{34}$$

implies that

$$\sigma = \rho = 0 \tag{35}$$

and therefore condition (1), i.e., condition (A) of theorem 1, holds.

Substituting (32) and (35) into equations (28), we get

$$\Phi_{00} = \Phi_{01} = \Phi_{10} = 0. \tag{36}$$

And finally from the vanishing of the invariant

$$\Phi^{A\dot{B}\dot{A}\dot{B}}\Phi_{A\dot{B}\dot{A}\dot{B}} = 4\Phi_{11}^2 + 2\Phi_{02}\Phi_{20} + 2\Phi_{00}\Phi_{22} - 4\Phi_{10}\Phi_{12} - 4\Phi_{01}\Phi_{21}, \tag{37}$$

using (36), it follows that

$$\Phi_{11} = \Phi_{02} = \Phi_{20} = 0$$

and thus condition (3), i.e., condition (B) of theorem 1, is also satisfied.

- (b) *Petrov type III*. Providing that the Weyl spinor Ψ_{ABCD} is of Petrov type III, we can construct another spinor $\tilde{\Psi}_{ABCD}$

$$\tilde{\Psi}_{ABCD} = \Psi_{ABEF}\Psi_{CD}{}^{EF} = -2\Psi_3^2 o_A o_B o_C o_D \quad (38)$$

which is of Petrov type N. Now we can construct analogical curvature invariants from $\tilde{\Psi}_{ABCD}$ as we did from Ψ_{ABCD} for type N and again conclude that $\kappa = \sigma = \rho = 0$ and $\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = 0$ for metrics with all curvature invariants vanishing.

- (c) *Petrov type 0*. Recall that the totally symmetric Plebański spinor is defined by

$$\chi_{ABCD} = \Phi_{(AB}{}^{\dot{C}\dot{D}}\Phi_{CD)\dot{C}\dot{D}}. \quad (39)$$

Its components are

$$\begin{aligned} \chi_0 &= 2(\Phi_{00}\Phi_{02} - \Phi_{01}^2), \\ \chi_1 &= \Phi_{00}\Phi_{12} + \Phi_{10}\Phi_{02} - 2\Phi_{11}\Phi_{01}, \\ \chi_2 &= \frac{1}{3}(\Phi_{00}\Phi_{22} - 4\Phi_{11}^2 + \Phi_{02}\Phi_{20} + 4\Phi_{10}\Phi_{12} - 2\Phi_{01}\Phi_{21}), \\ \chi_3 &= \Phi_{22}\Phi_{10} + \Phi_{12}\Phi_{20} - 2\Phi_{11}\Phi_{21}, \\ \chi_4 &= 2(\Phi_{22}\Phi_{20} - \Phi_{21}^2). \end{aligned} \quad (40)$$

In analogy with the Petrov classification of the Weyl tensor, it is possible to define the Plebański–Petrov type (PP-type) of the Plebański spinor [21]. Thus vanishing of curvature invariants analogous to I and J (30) constructed from the Plebański spinor implies that the PP-type is III, N or O.

For the PP-types III and N, we can argue as we did for the cases of the Petrov types III and N and conclude that $\kappa = \sigma = \rho = 0$ and $\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = 0$. Substituting these results into (40) we obtain $\chi_0 = \chi_1 = \chi_2 = \chi_3 = 0$ and thus the PP-type III is excluded.

It remains to consider the PP-type O case.

Without loss of generality we can choose a null tetrad so that

$$\Phi_{00} = 0.$$

Then $\chi_0 = 0$ in (40) implies

$$\Phi_{01} = \Phi_{10} = 0.$$

Hence, the vanishing of the invariant (37) gives

$$\Phi_{11} = \Phi_{02} = \Phi_{20} = 0.$$

Demanding $\chi_4 = 0$ in (40) we get

$$\Phi_{12} = \Phi_{21} = 0.$$

Thus, the only non-vanishing component of the Ricci spinor is Φ_{22} . The Bianchi identities (29) take the form

$$\bar{\kappa}\Phi_{22} = 0, \quad \bar{\sigma}\Phi_{22} = 0, \quad \flat\Phi_{22} = (\rho + \bar{\rho})\Phi_{22}, \quad \flat\Phi_{22} = \bar{\rho}\Phi_{22},$$

which implies

$$\kappa = \sigma = \rho = 0.$$

Up to now we have proved that spacetimes with vanishing invariants constructed from the Weyl and the Ricci *spinors*, and their arbitrary derivatives satisfy conditions (A) and (B) of theorem 1. Invariants constructed from the Riemann *tensor* and its derivatives are combinations of corresponding spinorial invariants. Thus, one could argue that there might exist a very special class of spacetimes for which all tensorial invariants vanish even though there exist nonzero spinorial invariants. To prove that this does not happen we now construct several tensorial curvature invariants. They may also be useful for computer-aided classification of spacetimes.

The curvature invariants shown in (30) can be given as

$$C_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\alpha\beta} - iC_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\alpha\beta}, \quad C_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\varepsilon\phi}C_{\varepsilon\phi}{}^{\alpha\beta} - iC_{\alpha\beta}{}^{\gamma\delta}C_{\gamma\delta}{}^{\varepsilon\phi}C_{\varepsilon\phi}{}^{\alpha\beta},$$

where

$$C_{\alpha\beta}{}^{\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\varepsilon\phi}C^{\varepsilon\phi}{}_{\gamma\delta}$$

denotes the dual of the Weyl tensor. Their vanishing implies that the Petrov type is III, N or O.

(a) *Petrov type N*. From the vanishing of

$$I_1 = C^{\alpha\beta\gamma\delta;\varepsilon}C_{\alpha\beta\kappa\lambda;\gamma}C^{\rho\sigma\tau\kappa;\lambda}C_{\rho\sigma\delta\varepsilon;\tau} = 8\mathcal{I}_1 = 2(4\Psi_4\bar{\Psi}_4\kappa\bar{\kappa})^2, \quad (41)$$

$\kappa = 0$ follows.

To obtain $\sigma = \rho = 0$ we have to construct the tensorial invariants I_2, I_3, I_4 :

$$I_2 = C^{\alpha\beta\gamma\delta;\varepsilon\phi}C_{\alpha\mu\gamma\nu;\varepsilon\phi}C^{\lambda\mu\rho\nu;\sigma\tau}C_{\lambda\beta\rho\delta;\sigma\tau} = 4\mathcal{I}_2 + 2\mathcal{I}_3\bar{\mathcal{I}}_3 + 4(\mathcal{I}_4 + \bar{\mathcal{I}}_4), \quad (42)$$

where

$$\mathcal{I}_3 = K^{F\dot{F}E\dot{E}}{}_{F\dot{F}E\dot{E}} = 6(4\rho\sigma\Psi_4)^2, \quad (43)$$

$$\begin{aligned} \mathcal{I}_4 &= \nabla^{F\dot{F}}\nabla^{E\dot{E}}\Psi^{ABCD}\nabla_{F\dot{F}}\nabla_{E\dot{E}}\Psi_{ACMN}\nabla^{T\dot{T}}\nabla^{S\dot{S}}\Psi^{LMNR}\nabla_{T\dot{T}}\nabla_{S\dot{S}}\Psi_{BDLR} \\ &= 18(4\rho\sigma\Psi_4)^4, \end{aligned} \quad (44)$$

and thus I_2 is equal to

$$I_2 = 2^83^2[(\Psi_4\bar{\Psi}_4)^2(\rho\bar{\rho} + \sigma\bar{\sigma})^4 + 8(\rho\bar{\rho}\sigma\bar{\sigma}\Psi_4\bar{\Psi}_4)^2 + 8(\rho\sigma\Psi_4)^4 + 8(\bar{\rho}\bar{\sigma}\bar{\Psi}_4)^4].$$

I_3 is defined by

$$I_3 = C^{\alpha\beta\gamma\delta;\varepsilon\phi}C_{\alpha\beta\gamma\delta;\lambda\mu}C^{\rho\sigma\tau\nu;\lambda\mu}C_{\rho\sigma\tau\nu;\varepsilon\phi} = 16(2\mathcal{I}_2 + \mathcal{I}_5 + \bar{\mathcal{I}}_5), \quad (45)$$

where

$$\mathcal{I}_5 = K^{F\dot{F}E\dot{E}}{}_{M\dot{M}L\dot{L}}K^{M\dot{M}L\dot{L}}{}_{F\dot{F}E\dot{E}} = 2^{10}3^2(\rho\sigma\Psi_4)^4, \quad (46)$$

and so it takes the form

$$I_3 = 2^{11}3^2[(\Psi_4\bar{\Psi}_4)^2(\rho\bar{\rho} + \sigma\bar{\sigma})^4 + 8(\rho\sigma\Psi_4)^4 + 8(\bar{\rho}\bar{\sigma}\bar{\Psi}_4)^4].$$

The curvature invariant I_4 is a linear combination of I_2 and I_3

$$I_4 = 8I_2 - I_3 = 2^{14}3^2(\rho\bar{\rho}\sigma\bar{\sigma}\Psi_4\bar{\Psi}_4)^2.$$

Demanding $I_4 = 0$ we obtain

$$\rho\sigma = 0$$

and then the vanishing of I_3 implies

$$\rho = \sigma = 0.$$

As in the spinorial case, the Newman–Penrose equations (28) imply $\Phi_{00} = \Phi_{01} = 0$, and finally the invariant constructed from the traceless Ricci tensor corresponding to (37)

$$S^{\alpha\beta}S_{\alpha\beta} = 4\Phi^{A\dot{B}\dot{A}\dot{B}}\Phi_{A\dot{B}\dot{A}\dot{B}} = 4(4\Phi_{11}^2 + 2\Phi_{02}\Phi_{20})$$

is zero if $\Phi_{11} = \Phi_{02} = \Phi_{20} = 0$. Thus conditions (A) and (B) of theorem 1 are satisfied.

- (b) *Petrov type III*. In analogy to (38), the tensor $D_{\alpha\beta\gamma\delta}$ can be defined in terms of the Petrov type-III Weyl tensor

$$D^{\alpha\beta\gamma\delta} = C^{\alpha\beta}_{\lambda\mu} C^{\gamma\delta\lambda\mu}$$

which is traceless, has the same symmetries as the Weyl tensor and is of the Petrov type N

$$D^{\alpha\beta\gamma\delta} \longleftrightarrow -4\Psi_3^2 o^A o^B o^C o^D \varepsilon^{\dot{A}\dot{B}} \varepsilon^{\dot{C}\dot{D}} - 4\bar{\Psi}_3^2 \bar{o}^{\dot{A}} \bar{o}^{\dot{B}} \bar{o}^{\dot{C}} \bar{o}^{\dot{D}} \varepsilon^{AB} \varepsilon^{CD}.$$

We can construct curvature invariants from $D^{\alpha\beta\gamma\delta}$ similar to those made from $C^{\alpha\beta\gamma\delta}$ for type N and again show that their vanishing leads to $\kappa = \sigma = \rho = 0$ and $\Phi_{00} = \Phi_{01} = \Phi_{02} = \Phi_{11} = 0$.

- (c) *Petrov type O*. It is possible to define the traceless Plebański tensor corresponding to (39) which is endowed with the same symmetries as the Weyl tensor in terms of the traceless Ricci tensor $S_{\alpha\beta}$ (see [22])

$$P^{\alpha\beta}_{\gamma\delta} = S^{[\alpha}_{[\gamma} S^{\beta]}_{\delta]} + \delta^{[\alpha}_{[\gamma} S_{\delta]\lambda} S^{\beta]\lambda} - \frac{1}{6} \delta^{[\alpha}_{[\gamma} \delta^{\beta]}_{\delta]} S_{\mu\nu} S^{\mu\nu}.$$

With the Plebański tensor we can proceed in the same way as in the spinorial case.

3.1. Alternative proof

Another way to prove the necessity of conditions (1)–(4) for the vanishing of all curvature invariants is to use the result from paper [23] that the invariants I_6 , I_7 and I_8 constructed from the Ricci spinor are equal to zero only if all four eigenvalues of the Ricci tensor are equal to zero. Consequently, the Segre types of the Ricci tensor are $\{(31)\}$ (i.e., PP-type N with the only non-vanishing components Φ'_{12} and Φ'_{22} [24]), $\{(211)\}$ (i.e., PP-type O with the only non-vanishing component Φ'_{22}) or $\{(1111)\}$ (i.e., vacuum). In non-vacuum cases, the multiple null eigenvector l' of the Ricci tensor may, in general, differ from the repeated null vector of the Weyl tensor l ; however, by demanding vanishing of the mixed invariants m_1, m_4, m_6 [5] constructed from both the Weyl and the Ricci tensors, we arrive at the condition $l' = l$. Then the Bianchi identities for non-vanishing $\Psi_3, \Psi_4, \Phi_{12}$ and Φ_{22} imply $\kappa = 0$. And finally, the vanishing of the invariant (33) for P-types III and N results in $\rho = \sigma = 0$.

4. Local description of the spacetimes with vanishing curvature invariants

Let us describe the metric, written in an adapted coordinate form, of all of the spacetimes with vanishing curvature invariants (i.e., those satisfying theorem 1). We recall that spacetimes with vanishing curvature invariants satisfy (1) (i.e., belong to the Kundt class [8, 9, 25]), are of the Petrov type III, N or O (i.e., the Weyl spinor Ψ_{ABCD} is of the form (9)) and the Ricci spinor $\Phi_{\dot{A}\dot{B}\dot{C}\dot{D}}$ has the form (10) that corresponds to the Ricci tensor

$$R_{\alpha\beta} = -2\Phi_{22} l_{\alpha} l_{\beta} + 4\Phi_{21} l_{(\alpha} m_{\beta)} + 4\Phi_{12} l_{(\alpha} \bar{m}_{\beta)}. \quad (47)$$

Consequently, the Plebański spinor (39) has the form

$$\chi_{ABCD} = -2\Phi_{21}^2 o_A o_B o_C o_D \quad (48)$$

and the Plebański–Petrov type (PP-type) is N for $\Phi_{12} \neq 0$ or O for $\Phi_{12} = 0$. We note that for PP-type N, using a null rotation about l^{α} we can transform away the Ricci component Φ_{22} and using further a boost in the $l^{\alpha}-n^{\alpha}$ plane and a spatial rotation in the $m^{\alpha}-\bar{m}^{\alpha}$ plane set $\Phi_{12} = \Phi_{21} = 1$. For PP-type O, it is possible to set $\Phi_{22} = 1$ by performing a boost in the $l^{\alpha}-n^{\alpha}$ plane.

The Ricci tensor (47) has all four eigenvalues equal to zero and its Segre type is $\{(31)\}$ ($\Phi_{12} \neq 0$), $\{(211)\}$ ($\Phi_{12} = 0$ and $\Phi_{22} \neq 0$) or $\{(1111)\}$ (for vacuum $\Phi_{12} = \Phi_{22} = 0$). The most physically interesting non-vacuum case $\{(211)\}$ corresponds to a pure null-radiation field [8]. It can be shown that an electromagnetic field compatible with (47) has to be null. Other energy–momentum tensors, including a fluid with anisotropic pressure and heat flux, can correspond to a Ricci tensor of PP-type O. Indeed, it is known that no energy–momentum tensor for a spacetime corresponding to a Ricci tensor of Segre type $\{31\}$ (or its degeneracies) can satisfy the weak energy conditions (see [8], p 72), and, hence, spacetimes of PP-type N are not regarded as physical in classical general relativity and, hence, attention is usually restricted to PP-type O models. However, for mathematical completeness we will discuss all of the models here. In addition, in view of possible applications in high energy physics in which the energy conditions are not necessarily satisfied, these models may have physical applications [26].

The most general form of the Kundt metric in adapted coordinates $u, v, \zeta, \bar{\zeta}$ [8] is

$$ds^2 = 2du[Hdu + dv + Wd\zeta + \bar{W}d\bar{\zeta}] - 2P^{-2}d\zeta d\bar{\zeta}, \quad (49)$$

where the metric functions

$$H = H(u, v, \zeta, \bar{\zeta}), \quad W = W(u, v, \zeta, \bar{\zeta}), \quad P = P(u, \zeta, \bar{\zeta})$$

satisfy the Einstein equations (see [8] and appendix A). For the spacetimes considered here, we may, without loss of generality, put $P = 1$. Tables 2–4 summarize the Kundt metrics for different subcases in the studied class.

It is of interest to find the conditions for which the repeated null eigenvector l^α of the Weyl tensor is recurrent for the Kundt class. The vector l^α satisfies

$$l^\alpha l_\alpha = 0, \quad l^\alpha{}_{;\beta} l^\beta = 0, \quad l^\alpha{}_{;\alpha} = 0, \quad l_{(\alpha;\beta)} l^{\alpha;\beta} = 0, \quad l_{[\alpha;\beta]} l^{\alpha;\beta} = 0$$

and its covariant derivative has, in general, the form

$$l_{\alpha;\beta} = (\gamma + \bar{\gamma})l_\alpha l_\beta + (\beta' - \bar{\beta})l_\alpha m_\beta + (\bar{\beta}' - \beta)l_\alpha \bar{m}_\beta - \bar{\tau}m_\alpha l_\beta - \tau\bar{m}_\alpha l_\beta.$$

Performing a boost in the l^α – n^α plane

$$\tilde{l}_\alpha = Al_\alpha, \quad \tilde{m}_\alpha = m_\alpha, \quad \tilde{n}_\alpha = A^{-1}n_\alpha$$

with A satisfying

$$A_{,\alpha} = A(\bar{\beta} - \beta' + \bar{\tau})m_\alpha + A(\beta - \bar{\beta}' + \tau)\bar{m}_\alpha,$$

i.e., putting $\tilde{\beta}' - \bar{\beta} = \beta' - \bar{\beta} - \delta' A/A = \bar{\tau}$, $\tilde{\tau} = \tau$ (see, e.g., [27] for transformation properties of NP quantities), we obtain

$$\tilde{l}_{\alpha;\beta} = (\tilde{\gamma} + \bar{\tilde{\gamma}})\tilde{l}_\alpha \tilde{l}_\beta + \bar{\tilde{\tau}}(\tilde{l}_\alpha m_\beta - \tilde{l}_\beta m_\alpha) + \tau(\tilde{l}_\alpha \bar{m}_\beta - \tilde{l}_\beta \bar{m}_\alpha) \quad (50)$$

with \tilde{l}^α satisfying

$$\mathfrak{L}_{\tilde{l}} g_{\alpha\beta} = \tilde{l}_{\alpha;\beta} + \tilde{l}_{\beta;\alpha} = 2(\tilde{\gamma} + \bar{\tilde{\gamma}})\tilde{l}_\alpha \tilde{l}_\beta. \quad (51)$$

This normalization is called ‘an almost Killing normalization’ in [28]. As τ cannot be transformed away by any transformation of the tetrad preserving the l -direction and one can even show that $\tau\bar{\tau}$ is invariant with respect to all tetrad transformations preserving the l -direction, the repeated null eigenvector l^α of the Weyl tensor is proportional to a recurrent vector \tilde{l}^α if and only if $\tau = 0$. To summarize, all Kundt spacetimes with $\tau = 0$ admit a recurrent null vector.

Finally, let us present the relation between quantities L and L' given in [28] and NP-quantities when l^α satisfies (51)

$$L = \gamma + \bar{\gamma}, \quad L' = DL = -2\tau\bar{\tau}.$$

Table 2. All spacetimes with vanishing invariants with $\Phi_{12} \neq 0$ and $\Phi_{22} \neq 0$, i.e., PP-N, are displayed. For details and references, see appendix A.

τ	P-type	Metric functions
= 0	III	$W = W_0(u, \zeta, \bar{\zeta})$ $H = v h_1(u, \zeta, \bar{\zeta}) + h_0(u, \zeta, \bar{\zeta})$ Equations (A.16), (A.17)
	N	$\Psi_3 = 0$ (A.18) Equations (A.16), (A.17)
	O	$\Psi_3 = \Psi_4 = 0$ (A.18), (A.19) Equations (A.16), (A.17)
$\neq 0$	III	$W = \frac{-2v}{\zeta + \bar{\zeta}} + W_0(u, \zeta, \bar{\zeta})$ $H = \frac{-v^2}{(\zeta + \bar{\zeta})^2} + v h_1(u, \zeta, \bar{\zeta}) + h_0(u, \zeta, \bar{\zeta})$ Equations (A.16), (A.17)
	N	$\Psi_3 = 0$ (A.18) Equations (A.16), (A.17)
	O	$\Psi_3 = \Psi_4 = 0$ (A.18), (A.19) Equations (A.16), (A.17)

5. Discussion

The pp-wave spacetimes have a number of important physical applications, many of which also apply to the other spacetimes obtained in this paper. As mentioned earlier, pp-wave spacetimes are exact vacuum solutions to string theory to all orders in α' , the scale set by the string tension [18]. Horowitz and Steif [19] generalized this result to include the dilaton field and antisymmetric tensor fields which are also massless fields of string theory using a more geometrical approach. They showed that pp-wave metrics satisfy all other field equations that are symmetric rank two tensors covariantly constructed from curvature invariants and polynomials in the curvature and their covariant derivatives, and since the curvature is null, all higher-order corrections to Einstein's equation constructed from higher powers of the Riemann tensor automatically vanish. Therefore, all higher-order terms in the string equations of motion are automatically zero. Many of the spacetimes obtained here will have similar properties.

In addition, solutions of classical field equations for which the counter terms required to regularize quantum fluctuations vanish are of particular importance because they offer insights into the behaviour of the full quantum theory. The coefficients of quantum corrections to Ricci flat solutions of Einstein's theory of gravity in four dimensions have been calculated up to two loops. In particular, a class of Ricci flat (vacuum) Lorentzian 4-metrics, which includes the pp-wave spacetimes and some special Petrov type III or N spacetimes, have vanishing counter terms up to and including two loops. Thus these Lorentzian metrics suffer no quantum corrections to all loop orders [20]. In view of the vanishing of all quantum corrections it is possible that all of the metrics summarized in tables 2–4 are of physical import and merit further investigation.

String theory in pp-wave backgrounds has been studied by many authors [18, 29], partly in a search for a connection between quantum gravity and gauge theory dynamics. Such string backgrounds are technically tractable and have direct applications to the four-dimensional conformal theories from the point of view of a duality between string and gauge theories. Indeed, pp-waves provide exact solutions of string theory [19, 18] and type-IIB superstrings in this background were shown to be exactly solvable even in the presence of the RR 5-form field strength [30]. As a result, the spectrum of the theory can be explicitly obtained.

Table 3. All spacetimes with vanishing invariants with $\Phi_{12} = 0$ and $\Phi_{22} \neq 0$, i.e., PP-O, null radiation, are displayed. For details and references, see appendix A.

τ	P-type	Metric functions
= 0	III	$W = W_0(u, \zeta)$
		$H = \frac{1}{2}v(W_{0,\bar{\zeta}} + \bar{W}_{0,\zeta}) + h_0(u, \zeta, \bar{\zeta})$
		$\Phi_{22} = h_{0,\zeta\bar{\zeta}} - \Re(W_0 W_{0,\bar{\zeta}\bar{\zeta}} + W_{0,u\bar{\zeta}} + W_{0,\zeta^2})$
	N	$W = 0$
		$H = h_0(u, \zeta, \bar{\zeta})$
		$\Phi_{22} = h_{0,\zeta\bar{\zeta}}$
O	$W = 0$	
	$H = h_{02}(u)\zeta\bar{\zeta} + h_{01}(u)\zeta + \bar{h}_{01}(u)\bar{\zeta} + h_{00}(u)$	
	$\Phi_{22} = h_{02}(u)$	
$\neq 0$	III	$W = \frac{-2v}{\zeta+\bar{\zeta}} + W_0(u, \zeta)$
		$H = \frac{-v^2}{(\zeta+\bar{\zeta})^2} + v\frac{W_0+\bar{W}_0}{\zeta+\bar{\zeta}} + h_0(u, \zeta, \bar{\zeta})$
		$\Phi_{22} = (\zeta + \bar{\zeta})\left(\frac{h_0+W_0\bar{W}_0}{\zeta+\bar{\zeta}}\right)_{,\zeta\bar{\zeta}} - W_{0,\zeta}\bar{W}_{0,\bar{\zeta}}$
	N	$W = \frac{-2v}{\zeta+\bar{\zeta}}$
		$H = \frac{-v^2}{(\zeta+\bar{\zeta})^2} + h_0(u, \zeta, \bar{\zeta})$
		$\Phi_{22} = (\zeta + \bar{\zeta})\left(\frac{h_0}{\zeta+\bar{\zeta}}\right)_{,\zeta\bar{\zeta}}$
	O	$W = \frac{-2v}{\zeta+\bar{\zeta}}$
		$H = \frac{-v^2}{(\zeta+\bar{\zeta})^2} + h_{00}(u)[1 + h_{01}(u)\zeta + \bar{h}_{01}(u)\bar{\zeta} + h_{02}(u)\zeta\bar{\zeta}](\zeta + \bar{\zeta})$
		$\Phi_{22} = h_{00}(u)h_{02}(u)(\zeta + \bar{\zeta})$

Table 4. All spacetimes with vanishing invariants with $\Phi_{12} = \Phi_{22} = 0$, i.e., PP-O, vacuum, are displayed. For details and references, see appendix A.

τ	P-type	Metric functions
= 0	III	$W = W_0(u, \zeta)$
		$H = \frac{1}{2}v(W_{0,\bar{\zeta}} + \bar{W}_{0,\zeta}) + h_0(u, \zeta, \bar{\zeta})$
		$h_{0,\zeta\bar{\zeta}} = \Re(W_0 W_{0,\bar{\zeta}\bar{\zeta}} + W_{0,u\bar{\zeta}} + W_{0,\zeta^2})$
	N pp-waves	$W = 0$
		$H = h_{00}(u, \zeta) + \bar{h}_{00}(u, \bar{\zeta})$
$\neq 0$	III	$W = \frac{-2v}{\zeta+\bar{\zeta}} + W_0(u, \zeta)$
		$H = \frac{-v^2}{(\zeta+\bar{\zeta})^2} + v\frac{W_0+\bar{W}_0}{\zeta+\bar{\zeta}} + h_0(u, \zeta, \bar{\zeta})$
		$(\zeta + \bar{\zeta})\left(\frac{h_0+W_0\bar{W}_0}{\zeta+\bar{\zeta}}\right)_{,\zeta\bar{\zeta}} = W_{0,\zeta}\bar{W}_{0,\bar{\zeta}}$
	N	$W = \frac{-2v}{\zeta+\bar{\zeta}}$
		$H = \frac{-v^2}{(\zeta+\bar{\zeta})^2} + [h_{00}(u, \zeta) + \bar{h}_{00}(u, \bar{\zeta})](\zeta + \bar{\zeta})$

This model is expected to provide some hints for the study of superstrings on more general backgrounds. There is also an interesting connection between pp-wave backgrounds and gauge field theories. It is known that any solution of Einstein gravity admits plane-wave backgrounds in the Penrose limit [31]. This was extended to solutions of supergravities in [32]. It was shown that the super-pp-wave background can be derived by the Penrose limit from the $AdS_p \times S^q$ backgrounds in [33]. The Penrose limit was recognized to be important in an exploration of the AdS/CFT correspondence beyond massless string modes in

[34, 35]. Maximally supersymmetric pp-wave backgrounds of supergravity theories in 11 and 10 dimensions have attracted great interest [36].

Recently, the idea that our universe is embedded in a higher-dimensional world has received a great deal of renewed attention [37]. Due to the importance of branes in understanding the non-perturbative dynamics of string theories, a number of classical solutions of branes in the background of a pp-wave have been studied; in particular a new brane-world model has been introduced in which the bulk solution consists of outgoing plane waves (only), which avoids the problem that the evolution requires initial data specified in the bulk [38].

Finally, in [13] an example of non-isometric spacetimes with non-vanishing curvature scalars which cannot be distinguished by curvature invariants was presented. This example represents a solution of Einstein's equation with a negative cosmological term and a minimally coupled massless scalar field. In this paper we have noted the existence of a class of spacetimes in which all of the curvature invariants are constants (depending on the cosmological constant). These results and their extensions to higher dimensions are consequently also of physical interest.

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Appendix. The Kundt metrics with all curvature invariants vanishing

Let us present here more details on the spacetimes with all curvature invariants vanishing, which were briefly summarized in section 4. Appendix A.1, appendix A.2 and appendix A.3 correspond to tables 2, 3 and 4, respectively.

Since all of the spacetimes satisfying condition (A) of theorem 1 (i.e., that satisfy (1)), belong to the Kundt class we start with the metric given by (49) in coordinates $u, v, \zeta, \bar{\zeta}$ [8], where the null tetrad is given by

$$l = \partial_v, \quad n = \partial_u - (H + P^2 W \bar{W}) \partial_v + P^2 (\bar{W} \partial_\zeta + W \partial_{\bar{\zeta}}), \quad m = P \partial_\zeta. \quad (\text{A.1})$$

Only certain coordinate transformations and tetrad rotations can be performed which preserve the form of the metric (49) and the null tetrad (A.1) (see [8])

$$\begin{aligned} \text{(I)} \quad \zeta' &= f(\zeta, u), \\ P'^2 &= P^2 f_{,\zeta} \bar{f}_{,\bar{\zeta}}, \quad W' = \frac{W}{f_{,\zeta}} - \frac{\bar{f}_{,u}}{P^2 f_{,\zeta} \bar{f}_{,\bar{\zeta}}}, \end{aligned} \quad (\text{A.2})$$

$$H' = H - \frac{1}{f_{,\zeta} \bar{f}_{,\bar{\zeta}}} \left(\frac{f_{,u} \bar{f}_{,u}}{P^2} + W f_{,u} \bar{f}_{,\bar{\zeta}} + \bar{W} \bar{f}_{,u} f_{,\zeta} \right);$$

$$\begin{aligned} \text{(II)} \quad v' &= v + g(\zeta, \bar{\zeta}, u), \\ P' &= P, \quad W' = W - g_{,\zeta}, \quad H' = H - g_{,u}; \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \text{(III)} \quad u' &= h(u), \quad v' = v/h_{,u}, \\ P' &= P, \quad W' = \frac{W}{h_{,u}}, \quad H' = \frac{1}{h_{,u}^2} \left(H + v \frac{h_{,uu}}{h_{,u}} \right). \end{aligned} \quad (\text{A.4})$$

In particular, in these coordinates it is not possible, in general, to simultaneously simplify the forms of the Ricci spinor components in (47) in PP-types N and O by boosts and null and spatial rotations.

In most cases it is possible to specialize the solution form by an appropriate choice of coordinates, thereby narrowing the range of allowed coordinate transformations. The remaining coordinate freedom will be described below on a case-by-case basis.

A.1. Plebański–Petrov type N, i.e., $\Phi_{12} \neq 0$ and $\Phi_{22} \neq 0$

- *Petrov type III.* The functions H , W and P have to satisfy equations which follow from the fact that we assume the Petrov types III, N or O ($\Psi_0 = \Psi_1 = \Psi_2 = 0$) and have the Ricci tensor of the form (47).

For the Kundt class, Ψ_0 vanishes identically and

$$\Psi_1 = \frac{1}{2} P R_{v\zeta} = -\frac{1}{4} P W_{,vv} = 0$$

and thus

$$W_{,vv} = 0. \quad (\text{A.5})$$

Then $\Psi_2 = 0 = \bar{\Psi}_2$ and $R = 0$ reduce to

$$\Psi_2 = -\frac{1}{6} [H_{,vv} + 2(P_{,\zeta} P_{,\bar{\zeta}} - P P_{,\zeta\bar{\zeta}}) + P^2 (2W_{,v\bar{\zeta}} - \bar{W}_{,v\zeta})] = 0, \quad (\text{A.6})$$

$$W_{,v\bar{\zeta}} = \bar{W}_{,v\zeta}, \quad (\text{A.7})$$

$$2W_{,v\bar{\zeta}} = W_{,v} \bar{W}_{,v} \quad (\text{A.8})$$

and

$$R_{\zeta\bar{\zeta}} = -2(\ln P)_{,\zeta\bar{\zeta}} = 0. \quad (\text{A.9})$$

The Gaussian curvature, $K = 2P^2(\ln P)_{,\zeta\bar{\zeta}} = \Delta(\ln P)$, of wave surfaces determined uniquely by the spacetime geometry is a spacetime invariant and since it vanishes for the studied class of spacetimes, they are characterized by plane-wave surfaces [8].

From (A.9), using a type I coordinate transformation (A.2), we can put

$$P = 1. \quad (\text{A.10})$$

This restricts the type I transformations to

$$\zeta' = e^{i\theta(u)} \zeta + f(u). \quad (\text{A.11})$$

Then, equations (A.5), (A.7) and (A.8), together with

$$R_{\zeta\zeta} = -W_{,v\zeta} + \frac{1}{2} W_{,v}^2 = 0, \quad (\text{A.12})$$

after another type I coordinate transformation (A.11), give without loss of generality [8]

$$W(u, v, \zeta, \bar{\zeta}) = \frac{-2v}{\zeta + \bar{\zeta}} n + W_0(u, \zeta, \bar{\zeta}) \quad (\text{A.13})$$

with $n = 0$ or 1 . If $n = 1$, the wave surfaces are polarized, and consequently type I transformations are further restricted to

$$\zeta' = \zeta + f(u), \quad \bar{f} + f = 0.$$

Finally, equations (A.6) and

$$R_{uv} = -H_{,vv} - \frac{1}{2} W_{,v} \bar{W}_{,v} = 0 \quad (\text{A.14})$$

are identical and have the solution [8]

$$H(u, v, \zeta, \bar{\zeta}) = \frac{-v^2}{(\zeta + \bar{\zeta})^2}n + v h_1(u, \zeta, \bar{\zeta}) + h_0(u, \zeta, \bar{\zeta}). \tag{A.15}$$

Employing (A.5)–(A.8), (A.10), (A.12) and (A.14), the remaining Einstein equations are

$$\begin{aligned} R_{uu} &= 2(\bar{W}H_{,v\zeta} + WH_{,v\bar{\zeta}}) - 2H_{,\zeta\bar{\zeta}} + H_{,v}(W_{,\bar{\zeta}} + \bar{W}_{,\zeta}) - (H_{,\zeta}\bar{W}_{,v} + H_{,\bar{\zeta}}W_{,v}) \\ &\quad - HW_{,v}\bar{W}_{,v} - (W\bar{W}_{,uv} + \bar{W}W_{,uv}) + W_{,u\bar{\zeta}} + \bar{W}_{,u\zeta} - W_{,\bar{\zeta}}\bar{W}_{,\zeta} \\ &\quad + (W\bar{W}_{,v} - \bar{W}W_{,v})(-\bar{W}_{,\zeta} + W_{,\bar{\zeta}}) + \frac{1}{2}(W_{,\bar{\zeta}}^2 + \bar{W}_{,\zeta}^2 + W^2\bar{W}_{,v}^2 + \bar{W}^2W_{,v}^2) \\ &= -2[\Phi_{22} - 2(W\Phi_{21} + \bar{W}\Phi_{12})] \end{aligned} \tag{A.16}$$

$$\begin{aligned} R_{u\zeta} &= -H_{,v\zeta} + \frac{1}{2}(W_{,uv} - W_{,\zeta\bar{\zeta}} + \bar{W}_{,\zeta\zeta} + W_{,v}W_{,\bar{\zeta}} - \bar{W}_{,v}W_{,\zeta}) - \frac{1}{4}W_{,v}(W\bar{W}_{,v} + \bar{W}W_{,v}) \\ &= -2\Phi_{12}. \end{aligned} \tag{A.17}$$

The NP quantities read

$$\begin{aligned} \rho &= \sigma = \kappa = \varepsilon = 0, \\ \tau &= -\tau' = 2\beta = -2\beta' = -\frac{1}{\zeta + \bar{\zeta}}n, \\ \sigma' &= -\frac{2v}{(\zeta + \bar{\zeta})^2}n - \bar{W}_{0,\bar{\zeta}}, \\ \rho' &= -\frac{2v}{(\zeta + \bar{\zeta})^2}n - \frac{1}{2}(W_{0,\bar{\zeta}} + \bar{W}_{0,\zeta}), \\ \kappa' &= \frac{6v^2}{(\zeta + \bar{\zeta})^3}n - v \left[h_{1,\bar{\zeta}} + 2\frac{W_0 + \bar{W}_0}{(\zeta + \bar{\zeta})^2}n - 2\frac{W_{0,\bar{\zeta}} + \bar{W}_{0,\bar{\zeta}}}{\zeta + \bar{\zeta}}n \right] - h_{0,\bar{\zeta}} - (W_0\bar{W}_0)_{,\bar{\zeta}}, \\ \gamma &= \frac{3v}{(\zeta + \bar{\zeta})^2}n + \frac{1}{2}h_1 - \frac{W_0 + \bar{W}_0}{\zeta + \bar{\zeta}}n + \frac{1}{4}(W_{0,\bar{\zeta}} - \bar{W}_{0,\zeta}), \\ \Psi_3 &= -2h_{1,\bar{\zeta}} + \bar{W}_{0,\zeta\bar{\zeta}} - W_{0,\bar{\zeta}\bar{\zeta}} + 2\frac{W_{0,\bar{\zeta}} - \bar{W}_{0,\bar{\zeta}}}{\zeta + \bar{\zeta}}n - 2\frac{W_0 + \bar{W}_0}{(\zeta + \bar{\zeta})^2}n, \end{aligned} \tag{A.18}$$

$$\begin{aligned} \Psi_4 &= v \left[-h_{1,\bar{\zeta}\bar{\zeta}} + 2\frac{h_{1,\bar{\zeta}} - \bar{W}_{0,\zeta\bar{\zeta}} + W_{0,\bar{\zeta}\bar{\zeta}}}{\zeta + \bar{\zeta}}n + 2\frac{\bar{W}_{0,\bar{\zeta}} - 2W_{0,\bar{\zeta}}}{(\zeta + \bar{\zeta})^2}n + 4\frac{W_0 + \bar{W}_0}{(\zeta + \bar{\zeta})^3}n \right] \\ &\quad + h_1\bar{W}_{0,\bar{\zeta}} - h_{0,\bar{\zeta}\bar{\zeta}} + \bar{W}_{0,u\bar{\zeta}} + \bar{W}_0(\bar{W}_{0,\zeta\bar{\zeta}} - W_{0,\bar{\zeta}\bar{\zeta}}) \\ &\quad + 2\frac{h_{0,\bar{\zeta}} + \bar{W}_0(W_{0,\bar{\zeta}} - \bar{W}_{0,\bar{\zeta}})}{\zeta + \bar{\zeta}}n - 2\frac{h_0 + W_0\bar{W}_0}{(\zeta + \bar{\zeta})^2}n. \end{aligned} \tag{A.19}$$

The remaining coordinate freedom for the case $n = 1$ is

$$\begin{aligned} \text{(I)} \quad &\zeta' = \zeta + f(u), \quad \bar{f} + f = 0, \quad W_0' = W_0 - f_{,u}, \\ &h_0' = h_0 - f_{,u}\bar{f}_{,u} + (W_0 - \bar{W}_0)f_{,u}, \quad h_1' = h_1; \\ \text{(II)} \quad &v' = v + g, \quad W_0' = W_0 - g_{,\zeta} + 2g/(\zeta + \bar{\zeta}), \\ &h_0' = h_0 - g_{,u} - gh_1 - g^2/(\zeta + \bar{\zeta})^2, \quad h_1' = h_1 + 2g/(\zeta + \bar{\zeta})^2; \\ \text{(III)} \quad &u' = h(u), \quad v' = v/h_{,u}, \\ &W_0' = W_0/h_{,u}, \quad h_0' = h_0/h_{,u}^2, \quad h_1' = h_1/h_{,u} + h_{,uu}/h_{,u}^2. \end{aligned} \tag{A.20}$$

One could, without loss of generality, take $h_1 = 0$.

The remaining coordinate freedom for the $n = 0$ case is

$$\begin{aligned}
 \text{(I)} \quad & \zeta' = e^{i\theta(u)}\zeta + f(u), \\
 & W_0' = e^{-i\theta}W_0 + \bar{f}_{,u} - ie^{-i\theta}\theta_{,u}\bar{\zeta}, \quad h_1' = h_1, \\
 & h_0' = h_0 - f_{,u}\bar{f}_{,u} - ie^{i\theta}\theta_{,u}\bar{f}_{,u}\zeta + ie^{-i\theta}\theta_{,u}f_{,u}\bar{\zeta} - \theta_{,u}^2\zeta\bar{\zeta} \\
 & \quad - W_0(e^{-i\theta}f_{,u} + i\theta_{,u}\zeta) - \bar{W}_0(e^{i\theta}\bar{f}_{,u} - i\theta_{,u}\bar{\zeta}); \\
 \text{(II)} \quad & v' = v + g, \\
 & W_0' = W_0 - g_{,\zeta}, \quad h_0' = h_0 - g_{,u} - gh_1, \quad h_1' = h_1; \\
 \text{(III)} \quad & u' = h(u), \quad v' = v/h_{,u}, \\
 & W_0' = W_0/h_{,u}, \quad h_0' = h_0/h_{,u}^2, \quad h_1' = h_1/h_{,u} + h_{,uu}/h_{,u}^2.
 \end{aligned} \tag{A.21}$$

One could therefore without loss of generality take $h_0 = 0$.

In general, we cannot make any further progress unless we identify a specific source, e.g., null radiation or null electromagnetic field, which then yields additional field equations through equations (A.16) and (A.17) (and, for example, the Maxwell equations).

- *Petrov type N*. In this case $\Psi_3 = 0$ and equation (A.18) constitutes an additional differential equation that must be satisfied. This equation can be integrated to obtain a more specialized form of the metric.
- *Petrov type O*. In this case $\Psi_3 = \Psi_4 = 0$, i.e. right-hand sides of equations (A.18), (A.19) must vanish. These equations can be integrated to obtain a fully specified form of the metric.

A.2. Plebański–Petrov type O, $\Phi_{12} = 0$ and $\Phi_{22} \neq 0$ —pure radiation

Conformally Ricci-flat pure radiation metrics, studied in [17], all belong to this class. In fact, in [17] the authors present all pure radiation solutions belonging to Kundt's class of Petrov types N and O for $\tau \neq 0$ and of Petrov types III, N and O for $\tau = 0$. For pure radiation, one of the remaining Einstein equations simply serves to define the radiation energy–density. For specific sources, such as a null electromagnetic field, these equations (e.g., equations (A.16) and (A.17)) lead to additional differential equations. In the case of vacuum, all solutions can be explicitly written (see the following subsection).

A.2.1. $n = 0$ ($\tau = 0$)

- *Petrov type III*. For $n = 0$ the Einstein equation $R_{u\zeta} = 0$ (A.17) becomes

$$[h_1 + \frac{1}{2}(W_{0,\bar{\zeta}} - \bar{W}_{0,\zeta})]_{,\zeta} = 0.$$

Using a type II transformation (A.3), (A.21) (see [8] for a discussion) and (A.13), (A.15), its solution turns out to be

$$\begin{aligned}
 W &= W_0(u, \bar{\zeta}), \\
 H &= \frac{1}{2}v(W_{0,\bar{\zeta}} + \bar{W}_{0,\zeta}) + h_0(u, \zeta, \bar{\zeta}).
 \end{aligned} \tag{A.22}$$

The metric functions are subject to the only remaining Einstein equation (A.16)

$$\Phi_{22} = h_{0,\zeta\bar{\zeta}} - \Re(W_0W_{0,\bar{\zeta}\bar{\zeta}} + W_{0,u\bar{\zeta}} + W_{0,\bar{\zeta}}^2). \tag{A.23}$$

The NP quantities take the form

$$\begin{aligned}
 \rho &= \sigma = \kappa = \varepsilon = \tau = \sigma' = \tau' = \beta = \beta' = 0, \\
 \rho' &= -\frac{1}{2}(W_{0,\bar{\zeta}} + \bar{W}_{0,\zeta}), \quad \kappa' = -\frac{1}{2}vW_{0,\bar{\zeta}\bar{\zeta}} - h_{0,\bar{\zeta}} - \bar{W}_0W_{0,\bar{\zeta}}, \quad \gamma = \frac{1}{2}W_{0,\bar{\zeta}}, \\
 \Psi_3 &= -2W_{0,\bar{\zeta}\bar{\zeta}}, \quad \Psi_4 = -\frac{1}{2}vW_{0,\bar{\zeta}\bar{\zeta}\bar{\zeta}} - h_{0,\bar{\zeta}\bar{\zeta}} - \bar{W}_0W_{0,\bar{\zeta}\bar{\zeta}}.
 \end{aligned} \tag{A.24}$$

This choice of metric form restricts the type I, II and III transformations (A.21) to the following three cases:

$$\begin{aligned}\zeta' &= \zeta + f(u); \\ v' &= v + g_1(u)\zeta + \bar{g}_1(u)\bar{\zeta} + g_0(u); \\ u' &= h(u), \quad v' = v/h_{,u} - (h_{,uu}/h_{,u}^2)\zeta\bar{\zeta},\end{aligned}$$

where f , g_1 , g_0 and h are arbitrary functions of u and a_1 , a_0 are real constants.

- *Petrov type N*. For type-N spacetimes ($\Psi_3 = 0 \rightarrow W_{0,\bar{\zeta}\bar{\zeta}} = 0$), W_0 can be transformed away (A.21) [8] and thus the metric functions (A.22) are

$$W = 0, \quad H = h_0(u, \zeta, \bar{\zeta}) \quad (\text{A.25})$$

and the NP quantities (A.24) read

$$\begin{aligned}\rho = \sigma = \kappa = \varepsilon = \tau = \sigma' = \tau' = \beta = \beta' = \rho' = \gamma = 0, \quad \kappa' &= -h_{0,\bar{\zeta}}, \\ \Psi_3 = 0, \quad \Psi_4 &= -h_{0,\bar{\zeta}\bar{\zeta}}.\end{aligned} \quad (\text{A.26})$$

The only remaining Einstein equation (A.23) now becomes

$$\Phi_{22} = h_{0,\zeta\bar{\zeta}}. \quad (\text{A.27})$$

The remaining coordinate freedom comes from a mixed type I and II transformation:

$$\begin{aligned}\zeta' &= e^{i\theta}(\zeta + f(u)), \quad v' = v + \bar{f}_{,u}\zeta + f_{,u}\bar{\zeta} + g(u), \\ h_0' &= h_0 - g_{,u} + f_{,u}\bar{f}_{,u} - \bar{f}_{,uu}\zeta - f_{,uu}\bar{\zeta},\end{aligned} \quad (\text{A.28})$$

where θ is a real constant and u is determined up to an affine transformation.

These spacetimes are known as generalized pp-wave solutions. In the case of a null electromagnetic field, energy momentum tensor, equation (A.27) and Maxwell's equations lead to a further differential equation for h_0 , whose solution is known [8].

- *Petrov type O*. All metrics belonging to this class are given in [16] (see (12) therein). The condition $\Psi_4 = 0$ from (A.26) is $h_{0,\bar{\zeta}\bar{\zeta}} = 0$ with the solution

$$h_0 = h_{02}(u)\zeta\bar{\zeta} \quad (\text{A.29})$$

after a transformation (A.28). The metric functions are thus given by (A.25) with (A.29) and the Einstein equation (A.27) becomes $\Phi_{22} = h_{02}$.

The coordinates are fixed up to an eight-parameter group of transformations:

$$\begin{aligned}\zeta' &= e^{i\theta}(\zeta + f(u)), \\ v' &= v/a_1 + \bar{f}_{,u}\zeta + f_{,u}\bar{\zeta} + \frac{1}{2}(f\bar{f})_{,u} + g_0, \\ u' &= a_1u + a_0,\end{aligned}$$

where $f(u)$ is a complex-valued solution of

$$f_{,uu} + fh_{02} = 0,$$

and θ , a_1 , a_0 , g_0 are real constants.

A.2.2. $n = 1, \tau \neq 0$.

- *Petrov type III*. For $n = 1$, the Einstein equation $R_{u\zeta} = 0$ (A.17) is

$$\left[h_1 + \frac{1}{2}(W_{0,\bar{\zeta}} - \bar{W}_{0,\zeta}) - \frac{W_0 + \bar{W}_0}{\zeta + \bar{\zeta}} \right]_{,\zeta} = -\frac{W_{0,\bar{\zeta}} + \bar{W}_{0,\zeta}}{\zeta + \bar{\zeta}}.$$

Again using a type II transformations (A.20) (as in [8]), we obtain the solution (A.13), (A.15)

$$W = \frac{-2v}{\zeta + \bar{\zeta}} + W_0(u, \zeta), \quad H = \frac{-v^2}{(\zeta + \bar{\zeta})^2} + v \frac{W_0 + \bar{W}_0}{\zeta + \bar{\zeta}} + h_0(u, \zeta, \bar{\zeta}). \quad (\text{A.30})$$

The remaining Einstein equation (A.16) then reads

$$\Phi_{22} = (\zeta + \bar{\zeta}) \left(\frac{h_0 + W_0 \bar{W}_0}{\zeta + \bar{\zeta}} \right)_{,\zeta\bar{\zeta}} - W_{0,\zeta} \bar{W}_{0,\bar{\zeta}}. \quad (\text{A.31})$$

The NP quantities are as follows

$$\begin{aligned} \rho &= \sigma = \kappa = \varepsilon = 0, & \tau &= -\tau' = 2\beta = -2\beta' = -\frac{1}{\zeta + \bar{\zeta}}, \\ \sigma' &= -\frac{2v}{(\zeta + \bar{\zeta})^2} - \bar{W}_{0,\bar{\zeta}}, & \rho' &= -\frac{2v}{(\zeta + \bar{\zeta})^2}, \\ \kappa' &= \frac{6v^2}{(\zeta + \bar{\zeta})^3} - v \left[\frac{W_0 + \bar{W}_0}{(\zeta + \bar{\zeta})^2} - \frac{\bar{W}_{0,\bar{\zeta}}}{\zeta + \bar{\zeta}} \right] - h_{0,\bar{\zeta}} - W_0 \bar{W}_{0,\bar{\zeta}}, \\ \gamma &= \frac{3v}{(\zeta + \bar{\zeta})^2} - \frac{1}{2} \frac{W_0 + \bar{W}_0}{\zeta + \bar{\zeta}}, \\ \Psi_3 &= -\frac{4\bar{W}_{0,\bar{\zeta}}}{\zeta + \bar{\zeta}}, \\ \Psi_4 &= v \left[-\frac{\bar{W}_{0,\bar{\zeta}}}{\zeta + \bar{\zeta}} + \frac{6\bar{W}_{0,\bar{\zeta}}}{(\zeta + \bar{\zeta})^2} \right] + \frac{2h_{0,\bar{\zeta}} + \bar{W}_{0,\bar{\zeta}}(W_0 - \bar{W}_0)}{\zeta + \bar{\zeta}} - 2\frac{h_0 + W_0 \bar{W}_0}{(\zeta + \bar{\zeta})^2} \\ &\quad - h_{0,\bar{\zeta}\bar{\zeta}} + \bar{W}_{0,u\bar{\zeta}}. \end{aligned} \quad (\text{A.32})$$

The remaining coordinate freedom is

$$\begin{aligned} \zeta' &= \zeta + f(u), & \bar{f} + f &= 0; \\ v' &= v + (\zeta + \bar{\zeta})g(u); \\ u' &= h(u), & v' &= \frac{v}{h_{,u}} - (\zeta + \bar{\zeta})^2 \frac{h_{,uu}}{2h_{,u}^2}. \end{aligned} \quad (\text{A.33})$$

- *Petrov type N*. All type-N pure radiation metrics were found in [17]. For type-N spacetimes ($\Psi_3 = 0 \rightarrow \bar{W}_{0,\bar{\zeta}} = 0$), W_0 can be transformed away again using (A.21), (A.33) and the metric functions (A.30) take the form

$$W = \frac{-2v}{\zeta + \bar{\zeta}}, \quad H = \frac{-v^2}{(\zeta + \bar{\zeta})^2} + h_0(u, \zeta, \bar{\zeta}) \quad (\text{A.34})$$

with h_0 satisfying (A.31)

$$\Phi_{22} = (\zeta + \bar{\zeta}) \left(\frac{h_0}{\zeta + \bar{\zeta}} \right)_{,\zeta\bar{\zeta}}. \quad (\text{A.35})$$

The NP quantities are as follows

$$\begin{aligned} \rho = \sigma = \kappa = \varepsilon = 0, \quad \tau = -\tau' = 2\beta = -2\beta' = -\frac{1}{\zeta + \bar{\zeta}}, \\ \sigma' = \rho' = -\frac{2v}{(\zeta + \bar{\zeta})^2}, \quad \kappa' = \frac{6v^2}{(\zeta + \bar{\zeta})^3} - h_{0,\bar{\zeta}}, \quad \gamma = \frac{3v}{(\zeta + \bar{\zeta})^2}, \\ \Psi_3 = 0, \quad \Psi_4 = -(\zeta + \bar{\zeta}) \left(\frac{h_0}{\zeta + \bar{\zeta}} \right)_{,\bar{\zeta}\bar{\zeta}}. \end{aligned} \tag{A.36}$$

The remaining coordinate freedom is given by

$$\begin{aligned} \zeta' = \zeta + if_0, \quad u' = h(u), \quad v' = \frac{v}{h_{,u}} - (\zeta + \bar{\zeta})^2 \frac{h_{,uu}}{2h_{,u}^2}, \\ h_0' = \frac{h_0}{h_{,u}^2} + \frac{(\zeta + \bar{\zeta})^2}{4h_{,u}^4} (-3h_{,uu}^2 + 2h_{,u} h_{,uuu}), \end{aligned} \tag{A.37}$$

where f_0 is a real constant, and $h = h(u)$ is an arbitrary real function.

- *Petrov type O.* All conformally flat pure radiation metrics (both with $\tau = 0$ and $\tau \neq 0$), generalizing the solutions found in [39] and [7], were given in [16]. The physical interpretation of this class of spacetimes is discussed in [40].

The equation $\Psi_4 = 0$ in (A.36) has the solution (see (16) in [16])

$$h_0 = h_{00}(u)[1 + h_{01}(u)\zeta + \bar{h}_{01}(u)\bar{\zeta} + h_{02}(u)\zeta\bar{\zeta}](\zeta + \bar{\zeta}) \tag{A.38}$$

which is to be substituted into the metric functions (A.34).

The Einstein equation (A.35) turns to be

$$\Phi_{22} = (\zeta + \bar{\zeta})h_{00}(u)h_{02}(u).$$

The remaining coordinate freedom is a four-parameter transformation group consisting of translations of ζ by an imaginary constant and fractional linear transformations of u . More precisely,

$$\begin{aligned} \zeta' = \zeta + if_0, \quad u' = \frac{au + b}{cu + d}, \quad v' = \frac{cu + d}{ad - bc} [v(cu + d) + c(\zeta + \bar{\zeta})^2], \\ h_{00}' = \frac{(cu + d)^4}{(ad - bc)^2} h_{00}, \end{aligned}$$

where a, b, c, d, f_0 are all real constants (note: one of the constants is redundant). Einstein–Maxwell null fields, massless scalar fields and neutrino fields *do not* exist for this class of metrics [16].

A.3. The vacuum case, i.e., $\Phi_{12} = \Phi_{22} = 0$.

The vacuum Petrov types-III, N and O Kundt metrics are reviewed in [8] (ch 27.5). The form of the metric, and the remaining coordinate freedom are as in appendix A.2, with the vacuum condition imposing an additional constraint on the metric parameters.

A.3.1. $n = 0, \tau = 0$.

- *Petrov type III.* For vacuum Petrov type-III spacetimes, the metric and the NP quantities are given by (A.22) and (A.24), respectively, where h_0 satisfies the Einstein equation (A.23)

$$h_{0,\zeta\bar{\zeta}} - \Re(W_0 W_{0,\bar{\zeta}\bar{\zeta}} + W_{0,u\bar{\zeta}} + W_{0,\bar{\zeta}}^2) = 0. \tag{A.39}$$

Petrov [41] found an example belonging to this class (in different coordinates)

$$ds^2 = x(v - e^x) du^2 - 2 du dv + e^x(dx^2 + e^{-2u} dz^2). \quad (\text{A.40})$$

- *Petrov type N—pp waves.* The metric functions (A.25) and the NP quantities (A.26) of vacuum Petrov type-N spacetimes satisfy (A.27)

$$h_{0,\zeta\bar{\zeta}} = 0, \quad \text{i.e.,} \quad h_0 = h_{00}(u, \zeta) + \bar{h}_{00}(u, \bar{\zeta}). \quad (\text{A.41})$$

These spacetimes belong to the class of pp-wave spacetimes (see chapter 21.5 in [8]) which admit a covariantly constant null vector that is consequently also a null Killing vector.

- *Petrov type O—flat spacetime.* For flat spacetime, equation (A.41) reduces the solution (A.29) to $h_0 = 0$, a flat metric.

A.3.2. $n = 1, \tau \neq 0$.

- *Petrov type III.* For Petrov type-III vacuum spacetimes with non-vanishing τ , the remaining Einstein equation (A.31) turns out to be

$$(\zeta + \bar{\zeta}) \left(\frac{h_0 + W_0 \bar{W}_0}{\zeta + \bar{\zeta}} \right)_{,\zeta\bar{\zeta}} = W_{0,\zeta} \bar{W}_{0,\bar{\zeta}}. \quad (\text{A.42})$$

Its solution determines the metric (A.30) and the NP quantities (A.32).

An example from this class, which was originally found by Kundt [9], with $W_0 = \bar{W}_0 = \psi/(\zeta + \bar{\zeta})$ satisfying $\psi_{,\zeta\bar{\zeta}} = 0$, is known (see [8]).

- *Petrov type N.* For Petrov type-N vacuum spacetimes, the Einstein equation (A.42) simplifies to

$$\left(\frac{h_0}{\zeta + \bar{\zeta}} \right)_{,\zeta\bar{\zeta}} = 0$$

with the solution

$$h_0 = [h_{00}(u, \zeta) + \bar{h}_{00}(u, \bar{\zeta})](\zeta + \bar{\zeta}). \quad (\text{A.43})$$

The metric and NP quantities are then given by (A.34) and (A.36) with (A.43).

- *Petrov type O—flat spacetime.* For the flat spacetime the condition $\Psi_4 = 0$ in (A.36), i.e.,

$$\left(\frac{h_0}{\zeta + \bar{\zeta}} \right)_{,\bar{\zeta}\bar{\zeta}} = 0,$$

has the solution

$$h_0 = h_{00}(u)[1 + h_{01}(u)\zeta + \bar{h}_{01}(u)\bar{\zeta}](\zeta + \bar{\zeta}).$$

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