

Research Article

Almost α -Cosymplectic Pseudo Metric Manifolds

Sermin Öztürk ¹ and Hakan Öztürk ²

¹Department of Mathematics, Afyon Kocatepe University, Faculty of Science and Literature, Afyonkarahisar 03200, Turkey

²Program of Hybrid and Electric Vehicles Technology, Afyon Kocatepe University, Afyon Vocational School, Afyonkarahisar 03200, Turkey

Correspondence should be addressed to Hakan Öztürk; hakser23@gmail.com

Received 21 April 2021; Accepted 25 June 2021; Published 30 July 2021

Academic Editor: Ljubisa Kocinac

Copyright © 2021 Sermin Öztürk and Hakan Öztürk. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main purpose of this paper is to study almost α -cosymplectic pseudo metric manifold satisfying certain η -parallel tensor fields. We first focus on the concept of almost α -cosymplectic pseudo metric manifold and its curvature properties. Then, we obtain some results related to the η -parallelity of h , φh , and τ . Moreover, the deformation of almost α -Kenmotsu pseudo metric structure is given. We conclude the paper with an illustrative example of almost α -cosymplectic pseudo metric manifold.

1. Introduction

Manifolds known as almost contact metric manifolds have been studied in [1–3]. The class of almost contact metric manifolds which are called almost Kenmotsu manifolds is firstly introduced by Kenmotsu. These manifolds appear for the first time in [4], where they have been locally classified. Kenmotsu defined a structure closely related to the warped product which was characterized by tensor equations.

Recently, Kim and Pak have introduced a wide subclass of almost contact metric manifolds called almost α -cosymplectic manifolds [5]. The authors investigated canonical foliations of an almost α -cosymplectic manifold. Later, most of the research is devoted to this topic [6–10]. However, the classical papers related to almost contact metric manifolds are assumed to have a Riemannian metric, and we notice that the almost contact manifolds furnished with a pseudo Riemannian metric are introduced in [11–14].

On that account, Wang and Liu introduced the geometry of almost Kenmotsu pseudo metric manifolds [12]. They emphasized the analogies and differences in connection with the Riemann metric tensor and obtained certain classification results related to locally symmetry and nullity condition. Also, Naik et al. studied Kenmotsu pseudo metric manifolds. In particular, the authors established necessary

and sufficient conditions for Kenmotsu pseudo metric manifolds satisfying certain tensor conditions [13].

Furthermore, Boeckx and Cho studied η -parallel contact metric spaces in [15]. They considered a milder condition that h is η -parallel, i.e.,

$$g((\nabla_X h)Y, Z) = 0, \quad (1)$$

in contact metric manifolds for all $X, Y, Z \in D$.

In [16], Ghosh et al. studied the η -parallelity of the torsion tensor τ for a contact metric manifold M^{2n+1} . The torsion tensor field τ defined as

$$g(\tau X, Y) = (L_\xi g)(X, Y), \quad (2)$$

for any vector fields X and Y on M^{2n+1} was firstly introduced by Hamilton and Chern [17].

In this paper, we consider the almost α -cosymplectic pseudo metric manifold which is a wide subclass of almost contact pseudo metric manifolds. We first give the concept of almost α -cosymplectic pseudo metric manifolds and state general curvature properties. We derive several formulas on almost α -cosymplectic pseudo metric manifolds. These formulas would enable us to find the geometrical properties of almost α -cosymplectic pseudo metric manifolds with η -parallel tensor h and φh . We study the η -parallelity of the

tensor fields h and ϕh . Next, we obtain some results related to the η -parallelity and η -cyclic parallelity of the torsion tensor τ . Moreover, we investigate the deformation of almost α -Kenmotsu pseudo metric structure. Finally, we give an illustrative example of almost α -cosymplectic pseudo metric manifolds.

2. Preliminaries

Let M^{2n+1} be a $(2n + 1)$ -dimensional differentiable manifold equipped with a triple (ϕ, ξ, η) , where ϕ is a type of $(1, 1)$ tensor field, ξ is a vector field, and η is a 1-form on M^{2n+1} such that

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad (3)$$

which implies

$$\begin{aligned} \phi(\xi) &= 0, \\ \eta \circ \phi &= 0, \\ \text{rank}(\phi) &= 2n. \end{aligned} \quad (4)$$

A pseudo Riemannian metric g on M^{2n+1} is said to be compatible with the almost contact structure (ϕ, ξ, η) if $g(\phi X, \phi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y)$ where $\varepsilon = \pm 1$.

A smooth manifold M^{2n+1} furnished with an almost contact structure (ϕ, ξ, η) and a compatible pseudo Riemannian metric g is called an almost contact pseudo metric manifold which is denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. It is clear that $g(\phi X, Y) = -g(X, \phi Y)$, $\eta(X) = \varepsilon g(X, \xi)$, and $g(\xi, \xi) = \varepsilon$.

On such a manifold, the fundamental 2-form Φ of M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y on M^{2n+1} [18]. An almost contact pseudo metric manifold

satisfying the conditions $d\eta = 0$ and $d\Phi = 2\alpha(\eta \wedge \Phi)$ is said to be an almost α -Kenmotsu pseudo metric manifold for $\alpha \neq 0$ and $\alpha \in R$. It is well known that the normality of almost contact structure is expressed by the vanishing of the tensor as follows:

$$\begin{aligned} N_\phi(X, Y) &= [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y] \\ &\quad + 2d\eta(X, Y)\xi, \end{aligned} \quad (5)$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [19].

An almost contact pseudo metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be almost cosymplectic pseudo metric manifold if $d\eta = 0$ and $d\Phi = 0$, where d is the exterior differential operator.

If we join these two classes, we obtain the notion of an almost α -cosymplectic pseudo metric manifold, defined by $d\eta = 0$ and $d\Phi = 2\alpha(\eta \wedge \Phi)$, for any real number α [5]. When an almost α -cosymplectic pseudo metric manifold M^{2n+1} has a normal almost contact structure, we can say that M^{2n+1} is an α -cosymplectic pseudo metric manifold. In this paper, we shall denote by $\Gamma(TM)$ and ∇ the Lie algebra of all tangent vector fields on M^{2n+1} and the Levi Civita connection of pseudo Riemannian metric g , respectively.

3. Certain Properties

In this section, we give the basic relations on almost α -cosymplectic pseudo metric manifolds.

Proposition 1. *Let M^{2n+1} be an almost contact metric manifold and ∇ be the Riemannian connection. Then, the following equations are held [3]:*

$$(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \phi)Z), \quad (6)$$

$$(\nabla_X \Phi)(Y, Z) + (\nabla_X \Phi)(\phi Y, \phi Z) = \eta(Z)(\nabla_X \eta)\phi Y - \eta(Y)(\nabla_X \eta)\phi Z, \quad (7)$$

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi) = (\nabla_X \Phi)(\xi, \phi Y), \quad (8)$$

$$2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X, \quad (9)$$

$$3d\Phi(X, Y, Z) = \bigoplus_{X, Y, Z} (\nabla_X \Phi)(Y, Z). \quad (10)$$

Here, $\bigoplus_{X, Y, Z}$ denotes the cyclic sum over the vector fields X, Y , and Z [1].

Lemma 1. *Let M^{2n+1} be an almost contact pseudo metric manifold. Then, the following equation is held:*

$$\begin{aligned} 2g((\nabla_X \phi)Y, Z) &= 3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z) \\ &\quad + g(N^{(0)}(Y, Z), \phi X) + \varepsilon N^{(1)}(Y, Z)\eta(X) \\ &\quad + 2\varepsilon d\eta(\phi Y, X)\eta(Z) - 2\varepsilon d\eta(\phi Z, X)\eta(Y), \end{aligned} \quad (11)$$

for any tangent vector fields $X, Y, Z \in \Gamma(TM)$ where $N^{(0)}$ and $N^{(1)}$ are defined by

$$N^{(0)}(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi, \quad (12)$$

$$N^{(1)}(X, Y) = (L_{\varphi X}\eta)Y - (L_{\varphi Y}\eta)X, \quad (13)$$

respectively. Here, L_X denotes the Lie derivative in the direction of X [20].

Proposition 2. Let M^{2n+1} be an almost α -cosymplectic pseudo metric manifold. Then, we have

$$hX = \frac{1}{2}(L_\xi\varphi)X, \quad (14)$$

$$h(\xi) = 0,$$

$$\nabla_X\xi = -\alpha\varphi^2X - \varphi hX, \quad (15)$$

$$\begin{aligned} \nabla_\xi\xi &= 0, \\ \nabla_\xi\varphi &= 0, \end{aligned} \quad (16)$$

$$(\varphi \circ h)X + (h \circ \varphi)X = 0, \quad (17)$$

$$\begin{aligned} (\nabla_X\eta)Y &= \alpha[\varepsilon g(X, Y) - \eta(X)\eta(Y)] \\ &\quad + \varepsilon g(\varphi Y, hX), \end{aligned} \quad (18)$$

$$\text{tr}(h) = 0, \quad h = 0 \iff \nabla\xi = -\alpha\varphi^2, \quad (19)$$

for any tangent vector fields $X, Y, Z \in \Gamma(TM)$.

Proof. Considering the Koszul formula (11), we have

$$\begin{aligned} 2g((\nabla_X\varphi)Y, Z) &= 3d\Phi(X, \varphi Y, \varphi Z) - 3d\Phi(X, Y, Z) \\ &\quad + g(N^{(0)}(Y, Z), \varphi X) + \varepsilon N^{(1)}(Y, Z)\eta(X). \end{aligned} \quad (20)$$

In view of (6), (7), and (10) for $Y = \xi$, we deduce

$$\begin{aligned} 2g((\nabla_X\varphi)\xi, Z) &= 2g(\varphi Z, X) + g(N^{(0)}(\xi, Z), \varphi X) \\ &\quad + \varepsilon N^{(1)}(\xi, Z)\eta(X). \end{aligned} \quad (21)$$

Taking into account of (12) and (13), we get

$$\begin{aligned} N^{(0)}(\xi, Z) &= -[\xi, Z] + \eta([\xi, Z])\xi, \\ N^{(1)}(\xi, Z) &= \eta([\varphi Z, \xi]). \end{aligned} \quad (22)$$

Then, making use of (22) in (21) and (20) reduces to

$$\begin{aligned} 2g((\nabla_X\varphi)\xi, Z) &= 2g(\varphi Z, X) - g((L_\xi\varphi)Z, X) \\ &\quad + \eta(X)\varepsilon(\eta[\xi, \varphi Z] + \eta[\xi, \varphi Z]) \\ &= -2g(\varphi X, Z) - g((L_\xi\varphi)Z, X). \end{aligned} \quad (23)$$

Here, if we choose the symmetric (1, 1)-type tensor field h as follows:

$$hZ = \frac{1}{2}(L_\xi\varphi)Z, \quad (24)$$

for any vector fields Z , and Equation (24) takes the form

$$g((\nabla_X\varphi)\xi, Z) = -g(\alpha\varphi X, Z) - g(hZ, X), \quad (25)$$

which completes the proof of (15). From (14) and (15), the first equation of (16) is obvious. Moreover, using (20) for $X = \xi$ and putting $X = \varphi X$ and $Y = \varphi Y$ in (13), we obtain the following:

$$2g((\nabla_\xi\varphi)Y, Z) = -2g(\varphi Y, Z) - 2g(Y, \varphi Z) + \varepsilon N^{(1)}(Y, Z). \quad (26)$$

This means that $\nabla_\xi\varphi = 0$ for any nonzero vector field Z . Now, considering the sum of $(\varphi \circ h)$ and $(h \circ \varphi)$ for any vector field X , we have

$$\begin{aligned} \varphi(hX) + h(\varphi X) &= \frac{1}{2}(\varphi[\xi, \varphi X] - \varphi^2[\xi, X] + [\xi, \varphi^2 X] - \varphi[\xi, \varphi X]) \\ &= \frac{1}{2}(-\eta(\nabla_\xi X)\xi + \eta(\nabla_X\xi)\xi + \varepsilon\nabla_\xi g(X, \xi)\xi) \\ &= \eta(\nabla_X\xi)\xi. \end{aligned} \quad (27)$$

Equation (27) shows that the sum of $(\varphi \circ h)$ and $(h \circ \varphi)$ vanishes identically. In addition, from (8), we get

$$(\nabla_X\eta)Y = \varepsilon\alpha g(X, Y) - \varepsilon^2\alpha\eta(X)\eta(Y) - \varepsilon g(Y, \varphi hX), \quad (28)$$

where $\varepsilon^2 = 1$. Thus, we can complete the proof of (18). Also, from (14) and (15), we can easily obtain (19). Here, if h^* is defined by $h^* = -\varphi h$, then we have $h^*\varphi = -\varphi h^*$, $h^2 = h^{*2}$, and $h\xi = \text{tr}(h) = 0$.

Now, we investigate the curvature properties of almost contact pseudo metric manifolds. First, we have the following propositions. \square

Proposition 3. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. Then, we have

$$\begin{aligned} R(X, Y)\xi &= \alpha^2[\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\varphi hY - \eta(Y)\varphi hX] \\ &\quad + (\nabla_Y\varphi h)X - (\nabla_X\varphi h)Y, \end{aligned} \quad (29)$$

for any tangent vector fields $X, Y \in \Gamma(TM)$.

Proof. Making use of the Riemannian curvature tensor and (15), we obtain (29) such that $\eta(X) = \varepsilon g(\xi, X)$. \square

Proposition 4. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. Then, the following relations are held:

$$R(X, \xi)\xi = \alpha^2 \varphi^2 X + 2\alpha \varphi h X - h^2 X + \varphi(\nabla_\xi h)X, \quad (30)$$

$$(\nabla_\xi h)X = -\varphi R(X, \xi)\xi - \alpha^2 \varphi X - 2\alpha h X - \varphi h^2 X, \quad (31)$$

$$R(X, \xi)\xi - \varphi R(\varphi X, \xi)\xi = 2[\alpha^2 \varphi^2 X - h^2 X], \quad (32)$$

$$S(X, \xi) = -2n\alpha^2 \eta(X) - (\operatorname{div}(\varphi h))X, \quad (33)$$

$$S(\xi, \xi) = -[2n\alpha^2 + \operatorname{tr}(h^2)], \quad (34)$$

$$\operatorname{div} \xi = 2\alpha n, \quad (35)$$

$$\operatorname{div} \eta = -2\alpha n \varepsilon, \quad (36)$$

for any tangent vector fields $X, Y \in \Gamma(TM)$.

Proof. By the hypothesis, using (29) with $Y = \xi$ and considering the following equations:

$$(\nabla_\xi \varphi h)X = \varphi(\nabla_\xi h)X, \quad (37)$$

$$(\nabla_X \varphi h)\xi = h^2 X - \varphi h X,$$

we obtain (30). Applying φ to (30) and remarking that $g((\nabla_\xi h)X, \xi) = 0$, we get (31). Also, with the help of (30) for φX , we have

$$R(\varphi X, \xi)\xi = -\alpha^2 \varphi X - 2\alpha \varphi^2 h X - \varphi h^2 X + \varphi(\nabla_\xi h)(\varphi X). \quad (38)$$

Then, we get

$$R(X, \xi)\xi - \varphi R(\varphi X, \xi)\xi = 2\alpha^2 \varphi^2 X - 2h^2 X + \varphi(\nabla_\xi h)X + \varphi^2(\nabla_\xi h)(\varphi X), \quad (39)$$

which reduces to (32) where $(\nabla_\xi h) \circ \varphi = -\varphi \circ (\nabla_\xi h)$.

Now, we may take a local orthonormal φ -basis as follows:

$$\{E_1, \dots, E_{2n}, \xi\} = \{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi\}. \quad (40)$$

From (29) and the Ricci curvature tensor, we have

$$\begin{aligned} S(X, \xi) &= \sum_{i=1}^{2n+1} \varepsilon_i g(R(E_i, X)\xi, E_i) \\ &= \sum_{i=1}^n \varepsilon_i g(R(e_i, X)\xi, e_i) + \sum_{i=1}^n \varepsilon_{i+n} g(R(\varphi e_i, X)\xi, \varphi e_i) \\ &\quad + \varepsilon_{2n+1} g(R(\xi, X)\xi, \xi), \end{aligned} \quad (41)$$

where $\varepsilon_{2n+1} g(R(\xi, X)\xi, \xi) = 0$. It follows that

$$\begin{aligned} S(X, \xi) &= -2n\alpha^2 \eta(X) + \alpha \eta(X) \sum_{i=1}^n \varepsilon_i g(\varphi h e_i, e_i) \\ &\quad + \varepsilon_{i+n} g(\varphi h \varphi e_i, \varphi e_i) \\ &\quad + \sum_{i=1}^n [\varepsilon_i g((\nabla_X \varphi h)e_i, e_i) + \varepsilon_{i+n} g((\nabla_X \varphi h)\varphi e_i, \varphi e_i)] \\ &\quad - \sum_{i=1}^n [\varepsilon_i g((\nabla_{e_i} \varphi h)X, e_i) + \varepsilon_{i+n} g((\nabla_{\varphi e_i} \varphi h)X, \varphi e_i)]. \end{aligned} \quad (42)$$

Then, we have

$$\begin{aligned} S(X, \xi) &= -2n\alpha^2 \eta(X) - \sum_{i=1}^{2n+1} \varepsilon_i g(R(E_i, X)\xi, E_i) \\ &= -2n\alpha^2 \eta(X) - (\operatorname{div}(\varphi h))X, \end{aligned} \quad (43)$$

such that

$$\operatorname{div}(\varphi h) = \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{E_i} \varphi h)X, E_i). \quad (44)$$

Since $\operatorname{tr}(\varphi h) = 0$, we deduce

$$0 = \sum_{i=1}^n \varepsilon_i g(\varphi h e_i, e_i) + \varepsilon_{i+n} g(\varphi h \varphi e_i, \varphi e_i). \quad (45)$$

Thus, the proof of (33) completes. Moreover, putting $X = \xi$ in (33), we obtain (34) where $(\operatorname{div}(\varphi h))\xi = \operatorname{tr}(h^2)$. This proof can also be given in another way. Consider the local orthonormal φ -basis on M^{2n+1} . The sectional curvatures of nondegenerate planes spanned by $\{\xi, e_i\}$ and $\{\xi, \varphi e_i\}$, respectively, are defined as

$$K(\xi, e_i) = \varepsilon \varepsilon_i R(\xi, e_i, \xi, e_i) = \varepsilon \varepsilon_i g(l(e_i), e_i), \quad (46)$$

$$K(\xi, \varphi e_i) = \varepsilon \varepsilon_i R(\xi, \varphi e_i, \xi, \varphi e_i) = -\varepsilon \varepsilon_i g(\varphi l \varphi(e_i), e_i), \quad (47)$$

where $\varepsilon_i = g(e_i, e_i) = g(\varphi e_i, \varphi e_i) = \pm 1$ for all indices $i = 1, \dots, n$ and l is the Jacobi operator defined by $l = R(\cdot, \xi)\xi$. Thus, we have

$$lX - \varphi l \varphi X = 2(\alpha^2 \varphi^2 X - h^2 X), \quad (48)$$

and from (46) and (47), it follows that

$$\begin{aligned} S(\xi, \xi) &= \sum_{i=1}^n R(\xi, e_i, \xi, e_i) + \sum_{i=1}^n R(\xi, \varphi e_i, \xi, \varphi e_i) \\ &= \varepsilon \sum_{i=1}^n \varepsilon_i [K(\xi, e_i) + K(\xi, \varphi e_i)] \\ &= \sum_{i=1}^n g(l(e_i) - \varphi l \varphi(e_i), e_i) \\ &= -2n\alpha^2 - \operatorname{tr}(h^2). \end{aligned} \quad (49)$$

It is well known that

$$\operatorname{div}\eta = -\operatorname{tr}(\nabla\eta) = \delta\eta = -\sum_{i=1}^n \{(\nabla_{e_i}\eta)e_i + (\nabla_{\varphi e_i}\eta)\varphi e_i\}. \quad (50)$$

From Equation (50), we obtain

$$\begin{aligned} \operatorname{div}\xi &= \alpha \sum_{i=1}^n \varepsilon_i [g(e_i - \varphi h e_i, e_i) + g(\varphi e_i - \varphi h \varphi e_i, \varphi e_i)], \\ \operatorname{div}\eta &= -\operatorname{tr}(\nabla\eta) = -\varepsilon \operatorname{div}\xi, \end{aligned} \quad (51)$$

which completes the proof. \square

4. Main Results

In this section, we consider some certain parallel tensor conditions on almost α -cosymplectic pseudo metric manifolds. Also, we study the deformation of almost α -Kenmotsu pseudo metric manifolds with $\alpha > 0$. Firstly, we study the η -parallelity of the tensor fields h and φh on almost α -cosymplectic pseudo metric manifolds. As we know that

we can take $X = X^T + \eta(X)\xi$ where X^T is tangentially part of X and $\eta(X)\xi$ is the normal part of X . So, the symmetric (1, 1)-type tensor field B on a Riemannian manifold (M, g) is said to be a η -parallel tensor if it holds the following:

$$g((\nabla_{X^T} B)Y^T, Z^T) = 0, \quad (52)$$

for all tangent vectors X^T, Y^T , and Z^T orthogonal to ξ [15].

Proposition 5. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. If h satisfies the η -parallelity condition, then we have*

$$\begin{aligned} (\nabla_X h)Y &= \eta(X)(\nabla_\xi h)Y - \eta(Y)[\alpha hX + \varphi h^2 X] \\ &\quad - \varepsilon g(Y, \alpha hX + \varphi h^2 X)\xi, \end{aligned} \quad (53)$$

for any tangent vector fields $X, Y \in \Gamma(TM)$.

Proof. Assume that h is η -parallel. From (52), we have

$$\begin{aligned} 0 &= g((\nabla_X h)Y, Z) - \eta(X)g((\nabla_\xi h)Y, Z) - \eta(Y)g((\nabla_X h)\xi, Z) \\ &\quad - \eta(Z)g((\nabla_X h)Y, \xi) + \eta(X)\eta(Y)g((\nabla_\xi h)\xi, Z) + \eta(Y)\eta(Z)g((\nabla_X h)\xi, \xi) \\ &\quad + \eta(Z)\eta(X)g((\nabla_\xi h)Y, \xi) - \eta(X)\eta(Y)\eta(Z)g((\nabla_\xi h)\xi, \xi), \end{aligned} \quad (54)$$

for any $X, Y, Z \in \Gamma(TM)$. It follows that

$$\begin{aligned} 0 &= g((\nabla_X h)Y, -\varphi^2 Z) - \eta(X)g((\nabla_\xi h)Y, Z) \\ &\quad - \eta(Y)g((\nabla_X h)\xi, Z). \end{aligned} \quad (55)$$

From Equation (55), we deduce

$$\begin{aligned} (\nabla_X h)Y &= -\eta(X)[\varphi lY + \alpha^2 \varphi Y + 2\alpha hY + \varphi h^2 Y] \\ &\quad - \eta(Y)[- \alpha \varphi^2 hX + \varphi h^2 X] + \varepsilon g((\nabla_X h)Y, \xi)\xi. \end{aligned} \quad (56)$$

Thus, it completes the proof. \square

Proposition 6. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. If φh satisfies the η -parallelity condition, then we have*

$$\begin{aligned} (\nabla_X \varphi h)Y &= \eta(X)[lY - \alpha^2 \varphi^2 Y - 2\alpha \varphi hY + h^2 Y] \\ &\quad - \eta(Y)[\alpha \varphi hX - h^2 X] - \varepsilon g(Y, \alpha \varphi hX - h^2 X)\xi, \end{aligned} \quad (57)$$

for any $X, Y \in \Gamma(TM)$.

Proof. By the hypothesis, we suppose that φh is η -parallel. Then, we have

$$\begin{aligned} 0 &= g((\nabla_X \varphi h)Y, Z) - \eta(X)g((\nabla_\xi \varphi h)Y, Z) \\ &\quad - \eta(Y)g((\nabla_X \varphi h)\xi, Z) - \eta(Z)g((\nabla_X \varphi h)Y, \xi) \\ &\quad + \eta(X)\eta(Y)g((\nabla_\xi \varphi h)\xi, Z) \\ &\quad + \eta(Y)\eta(Z)g((\nabla_X \varphi h)\xi, \xi) + \eta(Z)\eta(X)g((\nabla_\xi \varphi h)Y, \xi) \\ &\quad - \eta(X)\eta(Y)\eta(Z)g((\nabla_\xi \varphi h)\xi, \xi), \end{aligned} \quad (58)$$

for any $X, Y, Z \in \Gamma(TM)$. By a straightforward computation, we obtain

$$\begin{aligned} g((\nabla_X \varphi h)Y, \varphi^2 Z) &= -\eta(X)g((\nabla_\xi \varphi h)Y, Z) \\ &\quad - \eta(Y)g((\nabla_X \varphi h)\xi, Z). \end{aligned} \quad (59)$$

With the help of (15) and (17), Equation (59) reduces to

$$\begin{aligned} (\nabla_X \varphi h)Y &= -\eta(Y)[\alpha \varphi hX - h^2 X] + \eta(X)(\nabla_\xi \varphi h)Y \\ &\quad + \varepsilon [g(hY, hX)\xi + \alpha g(\varphi Y, hX)\xi]. \end{aligned} \quad (60)$$

From Equation (60) and $(\nabla_\xi \varphi h)Y = \varphi(\nabla_\xi h)Y$, (57) is easily seen. Then, the proof is completed. \square

Theorem 1. *An almost α -cosymplectic pseudo metric manifold with η -parallel tensor φh holds the following equation:*

$$R(X, Y)\xi = -\eta(X)lY + \eta(Y)lX, \quad (61)$$

for any $X, Y \in \Gamma(TM)$ where $l = R(\cdot, \xi)\xi$ is the Jacobi operator with respect to ξ .

Proof. Making use of (29), we have

$$\begin{aligned} R(X, Y)\xi &= \alpha^2\eta(X)Y - \alpha^2\eta(Y)X - \alpha\eta(X)\phi hY + \alpha\eta(Y)\phi hX \\ &\quad - \eta(X)(\nabla_\xi\phi h)Y + \eta(Y)[\alpha\phi hX - h^2X] \\ &\quad + \varepsilon g(Y, \alpha\phi hX - h^2X)\xi \\ &\quad + \eta(Y)(\nabla_\xi\phi h)X - \eta(X)[\alpha\phi hY - h^2Y] \\ &\quad - \varepsilon g(X, \alpha\phi hY - h^2Y)\xi. \end{aligned} \quad (62)$$

Then, simplifying Equation (62), we obtain

$$\begin{aligned} R(X, Y)\xi &= -\eta(X)lY + \eta(Y)lX + \eta(X)\alpha^2\varphi^2Y \\ &\quad - \eta(Y)\alpha^2\varphi^2X + \eta(X)\alpha^2Y - \eta(Y)\alpha^2X, \end{aligned} \quad (63)$$

which is desired result. \square

Theorem 2. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold with η -parallel tensor ϕh . Then, ξ is the eigenvector of Ricci operator on M^{2n+1} .

Proof. Let $\{e_1, \dots, e_{2n}, \xi\}$ be an orthonormal basis of the tangent space at any point. Taking the inner product of both sides of (61) with respect to Z and contracting (61) for $1 \leq i \leq 2n+1$ with $X = W = e_i$, we have

$$\sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i, Y)\xi, e_i) = \sum_{i=1}^{2n+1} \varepsilon_i [\eta(Y)g(l e_i, e_i) - \eta(e_i)g(l Y, e_i)], \quad (64)$$

for any $X, Y \in \Gamma(TM)$. This means that

$$S(\xi, \xi) = \sum_{i=1}^{2n+1} \varepsilon_i g(l e_i, e_i). \quad (65)$$

Thus, it completes the proof. \square

Theorem 3. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. If h satisfies the η -parallelity condition and $\nabla_\xi h = 0$, then the eigenvalues of h are constant.

Proof. Let $Z \in D$ be an eigen unit vector field such that $h(Z) = \mu Z$ where μ is an eigen function corresponding to the vector field Z . Then, (53) can be written as

$$g((\nabla_X h)Z, Z) = \eta(X)g((\nabla_\xi h)Z, Z) = \eta(X)\xi(\mu), \quad (66)$$

for $Z \in D$. Also, we have

$$g((\nabla_X h)Z, Z) = X(\mu). \quad (67)$$

Taking into account of (66) and (67), we also get

$$\begin{aligned} X(\mu) &= \eta(X)\xi(\mu), \\ d\mu &= \eta \otimes \xi(\mu). \end{aligned} \quad (68)$$

Furthermore, since $\nabla_\xi h = 0$, we obtain

$$g((\nabla_\xi h)Z, Z) = 0 = \xi(\mu). \quad (69)$$

Follows from (68) and (69), we have $d\mu = 0$. Thus, it completes the proof. \square

Proposition 7. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. Then, the torsion tensor field τ holds the following:

$$\tau X = 2\nabla_X \xi, \quad (70)$$

for any $X \in \Gamma(TM)$.

Proof. From the definition of τ , we get

$$(L_\xi g)(X, Y) = g(-\alpha\varphi^2 X - \phi hX, Y) + g(X, -\alpha\varphi^2 Y - \phi hY), \quad (71)$$

which completes the proof. \square

Proposition 8. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. If τ is η -parallel, then we have

$$\begin{aligned} (\nabla_X \phi h)Y &= \eta(X)(\nabla_\xi \phi h)Y - \eta(Y)\phi h(\nabla_X \xi) \\ &\quad + \varepsilon g((\nabla_X \phi h)\xi, Y)\xi, \end{aligned} \quad (72)$$

for any $X, Y \in \Gamma(TM)$.

Proof. The hypothesis is essentially same as

$$g((\nabla_X \tau)Y^T, Z^T) = 0, \quad (73)$$

for all tangent vectors orthogonal to ξ . Putting $X^T = X - \eta(X)\xi$ and using the definition of τ , we obtain

$$(\nabla_X \tau)Y = \eta(X)(\nabla_\xi \tau)Y + \eta(Y)(\nabla_X \tau)\xi + g((\nabla_X \tau)Y, \xi)\xi. \quad (74)$$

It follows that

$$(\nabla_X \tau)Y = -2\alpha\eta(Y)\nabla_X \xi - 2\alpha\eta(Y)g(\nabla_X \xi, Y)\xi - 2(\nabla_X \phi h)Y. \quad (75)$$

Putting $Y = \xi$ in (75), we have

$$(\nabla_X \tau)\xi = -2\alpha\nabla_X \xi + 2\phi h\nabla_X \xi. \quad (76)$$

Also, it is noted that

$$(\nabla_\xi \tau)Y = -2(\nabla_\xi \phi h)Y. \quad (77)$$

Finally, taking into account of (74)–(77), (72) holds. Then, we complete the proof. \square

Theorem 4. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. If τ is η -parallel, then ξ is the eigenvector of Ricci operator on M^{2n+1} .

Proof. From (72), we have

$$\begin{aligned} (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y &= \eta(Y)(\nabla_\xi \varphi h)X - \eta(X)\varphi h(\nabla_Y \xi) \\ &\quad + \varepsilon g((\nabla_Y \varphi h)\xi, X)\xi \\ &\quad - \eta(X)(\nabla_\xi \varphi h)Y + \eta(Y)\varphi h(\nabla_X \xi) \\ &\quad - \varepsilon g((\nabla_X \varphi h)\xi, Y)\xi. \end{aligned} \tag{78}$$

Simplifying Equation (78), we get

$$\begin{aligned} (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y &= \eta(Y)(\nabla_\xi \varphi h)X - \eta(X)\varphi h\nabla_Y \xi \\ &\quad - \eta(X)(\nabla_\xi \varphi h)Y + \eta(Y)\varphi h\nabla_X \xi. \end{aligned} \tag{79}$$

With the help of (29), (31), and (79), we obtain

$$\begin{aligned} R(X, Y)\xi &= \eta(Y)[- \varphi^2 lX - \alpha^2 \varphi^2 X - 2\alpha \varphi hX - \varphi^2 h^2 X] \\ &\quad - \eta(X)[- \varphi^2 lY - \alpha^2 \varphi^2 Y - 2\alpha \varphi hY - \varphi^2 h^2 Y] - \eta(X)\varphi h(-\alpha \varphi^2 Y - \varphi hY) \\ &\quad + \alpha^2 \eta(X)Y + \eta(Y)\varphi h(-\alpha \varphi^2 X - \varphi hX) - \alpha^2 \eta(Y)X - \alpha \eta(X)\varphi hY + \alpha \eta(Y)\varphi hX. \end{aligned} \tag{80}$$

By a direct calculation, the desired result is achieved. \square

Theorem 5. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -cosymplectic pseudo metric manifold. If τ is cyclically η -parallel, then ξ is the eigenvector of Ricci operator on M^{2n+1} .

Proof. According to the hypothesis, it means that

$$0 = g((\nabla_{X^T} \tau)Y^T, Z^T) + g((\nabla_{Y^T} \tau)Z^T, X^T) + g((\nabla_{Z^T} \tau)X^T, Y^T), \tag{81}$$

for all tangent vectors orthogonal to ξ . Simplifying Equation (81), we obtain

$$\begin{aligned} 0 &= \bigoplus_{X, Y, Z} g((\nabla_X \tau)Y, Z) - \eta(X)\bigoplus_{Y, Z, \xi} g((\nabla_Y \tau)Z, \xi) \\ &\quad - \eta(Y)\bigoplus_{Z, X, \xi} g((\nabla_Z \tau)X, \xi) - \eta(Z)\bigoplus_{X, Y, \xi} g((\nabla_X \tau)Y, \xi), \end{aligned} \tag{82}$$

where $\bigoplus_{X, Y, Z}$ denotes the cyclic sum over X, Y, Z . It follows that

$$\begin{aligned} \bigoplus_{X, Y, Z} g((\nabla_X \varphi h)Y, Z) - \eta(X)[g((\nabla_\xi \varphi h)Y, Z) + g((\nabla_Y \varphi h)Z, \xi) - g(\varphi h(\nabla_Z \xi), Y)] \\ - \eta(Y)[g((\nabla_\xi \varphi h)Z, X) + g((\nabla_Z \varphi h)X, \xi) - g(\varphi h(\nabla_X \xi), Z)] \\ - \eta(Z)[g((\nabla_X \varphi h)Y, \xi) + g((\nabla_\xi \varphi h)X, Y) - g(\varphi h(\nabla_Y \xi), X)] = 0. \end{aligned} \tag{83}$$

Contracting (83) with respect to Y and Z , we have

$$\eta(X)[2n\alpha^2 + 3\text{tr}(h^2) + \text{tr}(l)] - 2(\text{div}(\varphi h))X = 0. \tag{84}$$

From (33), (84) reduces to

$$Q\xi = -\frac{3}{2}[\text{tr}(h^2) + \frac{1}{3}\text{tr}(l) + 2\alpha^2 n]\xi. \tag{85}$$

From (85), the proof is clearly seen.

Now, we investigate the deformation of almost contact pseudo metric manifold. Here, our main goal is to study the relationship between pseudo Riemannian metrics with different signatures associated to the same almost contact pseudo metric manifold.

Let (φ, ξ, η, g) be an almost contact pseudo metric structure associated to a compatible pseudo Riemannian metric where $g(\xi, \xi) = \varepsilon$ on a smooth manifold M^{2n+1} . With

the help of [20], we have the following pseudo Riemannian metric formula:

$$g^*(X, Y) = \alpha g(X, Y) - 2\varepsilon \eta(X)\eta(Y), \tag{86}$$

for any $X, Y \in \Gamma(TM)$ where $g^*(\xi, \xi) = \varepsilon^* = \varepsilon(\alpha - 2)$, $\alpha > 0$. This means that (86) is still compatible pseudo metric with the same almost contact pseudo metric structure (φ, ξ, η, g) .

Thus, we give the following results. Here, we denote by ∇^* and R^* as the semi Riemannian connection and the curvature tensor of g^* on almost α -Kenmotsu pseudo metric manifold, respectively. \square

Proposition 9. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -Kenmotsu pseudo metric manifold. Then, $(M^{2n+1}, \varphi, \xi, \eta, g^*)$ is also an almost α -Kenmotsu pseudo metric manifold in the sense of (86).

Proof. According to (86), we denote by Φ the fundamental 2-form with respect to $(M^{2n+1}, \varphi, \xi, \eta, g^*)$ and then we have $\Phi^*(X, Y) = g^*(X, \varphi Y) = \alpha \Phi(X, Y)$ with $\alpha > 0$. Furthermore, from $\Phi^*(X, Y) = \alpha \Phi(X, Y)$, we obtain $d\Phi^*(X, Y) = 2\alpha(\eta \wedge \Phi^*(X, Y))$ for any $X, Y \in \Gamma(TM)$. Thus, it completes the proof. \square

Proposition 10. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost α -Kenmotsu pseudo metric manifold and g^* the pseudo Riemannian metric given by (86). Then, the following equations are held:*

$$\nabla_X^* Y = \nabla_X Y + \frac{2\varepsilon}{\alpha} g(\nabla_X \xi, Y) \xi, \tag{87}$$

$$R^*(X, Y)Z = R(X, Y)Z - \frac{2\varepsilon}{\alpha} [g(\nabla_X \xi, Z) \nabla_Y \xi - g(\nabla_Y \xi, Z) \nabla_X \xi] \\ + \frac{2\varepsilon}{\alpha} [g((\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y, Z) \xi] + 2\varepsilon [g((\nabla_Y \varphi^2)X - (\nabla_X \varphi^2)Y, Z) \xi], \tag{88}$$

for any $X, Y \in \Gamma(TM)$.

Proof. According to the Koszul formula, we have

$$2g^*(\nabla_X^* Y, Z) = X(g^*(Y, Z)) + Y(g^*(X, Z)) - Z(g^*(X, Y)) \\ + g^*([X, Y], Z) + g^*([Z, X], Y) \\ + g^*([Z, Y], X). \tag{89}$$

Using (86) and (89), we get

$$g^*(\nabla_X^* Y, Z) = \alpha g(\nabla_X Y, Z) - 2\varepsilon \eta(Z) g(\nabla_X \xi, Y), \tag{90}$$

where $X(\eta(Y)) = g(\nabla_X \xi, Y)$.

On the other hand, making use of (86) in (90), it follows that

$$\alpha g(\nabla_X^* Y - \nabla_X Y, Z) - 2\varepsilon \eta(Z) [\eta(\nabla_X^* Y) - X(\eta(Y))] = 0. \tag{91}$$

Then, taking into account of (90) and (91), we obtain (87). Moreover, using the definition of Riemannian curvature tensor and (87), we have

$$R^*(X, Y)Z = \nabla_X^* (\nabla_Y Z + \frac{2\varepsilon}{\alpha} g(\nabla_Y \xi, Z) \xi) \\ - \nabla_Y^* (\nabla_X Z + \frac{2\varepsilon}{\alpha} g(\nabla_X \xi, Z) \xi) \\ - \nabla_{[X, Y]}^* Z - \frac{2\varepsilon}{\alpha} (g(\nabla_{[X, Y]} \xi, Z) \xi). \tag{92}$$

From (92), we obtain

$$R^*(X, Y)Z = R(X, Y)Z + \frac{2\varepsilon}{\alpha} \nabla_X g(\nabla_Y \xi, Z) \xi + \frac{2\varepsilon}{\alpha} g(\nabla_X \xi, \nabla_Y Z) + \frac{4}{\alpha^2} g(\nabla_X \xi, \xi) g(\nabla_Y \xi, Z) \xi \\ - \frac{2\varepsilon}{\alpha} \nabla_Y g(\nabla_X \xi, Z) \xi - \frac{2\varepsilon}{\alpha} g(\nabla_Y \xi, \nabla_X Z) - \frac{4}{\alpha^2} g(\nabla_Y \xi, \xi) g(\nabla_X \xi, Z) \xi - \frac{2\varepsilon}{\alpha} (g(\nabla_{[X, Y]} \xi, Z) \xi). \tag{93}$$

Finally, substituting (15) into (92), it reduces to (88). Then, we complete the proof. \square

5. An Example

Consider the $M \subset R_1^3$ manifold such that $M = \{(u, v, w) \in R_1^3: w \neq 0\}$, where (u, v, w) are the standard coordinates in R_1^3 . The vector fields are

$$e_1 = e^{2w} \frac{\partial}{\partial u}, \\ e_2 = e^{2w} \frac{\partial}{\partial v}, \\ e_3 = \frac{\partial}{\partial w}. \tag{94}$$

Moreover, the following equations hold:

$$g = e^{-4w} (\varepsilon_1 du^2 + \varepsilon_2 dv^2) + \varepsilon dw^2, \quad \eta = dw, \\ \varphi(\xi) = 0, \\ \varphi(e_1) = e_2, \\ \varphi(e_2) = -e_1, \\ \varphi^2 X = -X + \eta(X) e_3, \\ \eta(X) = \varepsilon g(e_3, X), \\ \eta(e_3) = g(e_3, e_3) = \varepsilon = \varepsilon_3, \\ g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y), \quad \varepsilon_i = g(e_i, e_i), i = 1, 2, 3. \tag{95}$$

From Equation (95), there exists an almost contact pseudo metric structure (φ, ξ, η, g) on M . In order to check,

whether it is almost α -cosymplectic pseudo metric or not, we verify the condition $d\Phi = 2\alpha(\eta \wedge \Phi)$. On the other hand, all $\Phi_{i,j}$ s vanish except for $\Phi(e_1, e_2) = -\varepsilon_i$. Hence, we have

$$\Phi\left(\left(\frac{\partial}{\partial u}\right), \left(\frac{\partial}{\partial v}\right)\right) = -\varepsilon_i e^{-4w}, \quad (96)$$

$$\Phi = -\varepsilon_i e^{-4w} (du \wedge dv).$$

It follows that

$$d\Phi = 4\varepsilon_i e^{-4w} (du \wedge dv \wedge dw). \quad (97)$$

Since $\eta = dw$, we have

$$d\Phi = 2(-2\varepsilon_i)(\eta \wedge \Phi). \quad (98)$$

Here, it is noted that $N_\varphi = 0$. Therefore, M is an α -cosymplectic pseudo metric manifold.

6. Conclusion and Discussion

Since Kenmotsu introduced the notion of Kenmotsu structures in [4] which can be regarded as an analogy of almost contact metric structures, numerous authors studied such structures under some certain conditions [1–3, 7–10]. In particular, Dileo and Pastore studied certain parallel tensors, local symmetry, and nullity distribution on almost Kenmotsu manifolds [21]. Also, Kim and Pak introduced a new definition which combines almost Kenmotsu and almost cosymplectic manifold called almost cosymplectic manifold [5].

On the other hand, a systematic study of almost α -cosymplectic pseudo manifolds has not been undertaken yet. The main purpose of this paper is to contribute to future studies on this subject. We introduce the geometry of almost α -cosymplectic pseudo metric manifolds and underline the differences and similarities in the sense of Riemannian metric tensor. For this purpose, many results are given in the third and fourth sections. This study will shed light on our future investigations. Our further studies will be devoted to nullity distributions, local symmetry, semisymmetric conditions, and the other curvature tensor fields on almost α -cosymplectic pseudo metric manifolds.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this study.

Acknowledgments

This work was supported by Afyon Kocatepe University Scientific Research Coordination Unit with the project no. 17.FEN.BİL.11.

References

- [1] I. Vaisman, "Conformal changes of almost contact metric structures," *Lecture Notes in Math.* Springer, Berlin, Germany, pp. 435–443, 1980.
- [2] D. Janssens and L. Vanhecke, "Almost contact structures and curvature tensors," *Kodai Mathematical Journal*, vol. 4, pp. 1–27, 1981.
- [3] D. Chinea and C. Gonzalez, "A classification of almost contact metric manifolds," *Annali di Matematica Pura ed Applicata*, vol. 156, no. 4, pp. 15–36, 1990.
- [4] K. Kenmotsu, "A class of contact Riemannian manifold," *Tohoku Mathematical Journal*, vol. 24, pp. 93–103, 1972.
- [5] T. W. Kim and H. K. Pak, "Canonical foliations of certain classes of almost contact metric structures," *Acta Mathematica Sinica, English Series*, vol. 21, no. 4, pp. 841–846, 2005.
- [6] N. Aktan, M. Yildirim, and C. Murathan, "Almost f -cosymplectic manifolds," *Mediterranean Journal of Mathematics*, vol. 11, no. 2, pp. 775–787, 2014.
- [7] D. M Naik, V. Venkatesha, and H. A. Kumara, "Some results on almost Kenmotsu manifolds," *Mathematical notes*, vol. 40, pp. 87–100, 2020.
- [8] V. Venkatesha, H. A. Kumara, and D. M. Naik, "Riemann solitons and almost Riemann solitons on almost Kenmotsu manifolds," *International Journal of Geometric Methods in Modern Physics*, vol. 17, Article ID 2050105, 2020.
- [9] V. Venkatesha, D. M. Naik, and H. A. Kumara, "*-Ricci solitons and gradient almost *-Ricci solitons on Kenmotsu manifolds," *Mathematica Slovaca*, vol. 69, no. 6, pp. 1447–1458, 2019.
- [10] V. Venkatesha, D. M. Naik, and A. T. Vanli, "Second order parallel tensor on almost Kenmotsu manifolds," *Kyungpook Mathematical Journal*, vol. 61, pp. 191–203, 2021.
- [11] S. Öztürk and H. Öztürk, "Alfa Kenmotsu pseudo metrik manifoldlar Üzerine," *Afyon Kocatepe University Journal of Sciences and Engineering*, vol. 20, no. 6, pp. 975–982, 2020.
- [12] Y. Wang and X. Liu, "Almost Kenmotsu pseudo-metric manifolds," *Annals of the Alexandru Ioan Cuza University-Mathematics*, vol. 62, pp. 241–256, 2016.
- [13] D. M. Naik, V. Venkatesha, and D. G. Prakasha, "Certain results on Kenmotsu pseudo-metric manifolds," *Miskolc Mathematical Notes*, vol. 20, no. 2, pp. 1083–1099, 2019.
- [14] V. Venkatesha, D. M. Naik, and M. M. Tripathi, "Certain results on almost contact pseudo-metric manifolds," *Journal of Geometry*, vol. 110, no. 41, 2019.
- [15] E. Boeckx and J. T. Cho, " η -parallel contact metric spaces," *Differential Geometry and Its Applications*, vol. 22, no. 3, pp. 275–285, 2005.
- [16] A. Ghosh, R. Sharma, and J. T. Cho, "Contact metric manifolds with η -parallel torsion tensor," *Annals of Global Analysis and Geometry*, vol. 34, no. 3, pp. 287–299, 2008.
- [17] S. S. Chern and R. S. Hamilton, "On riemannian metrics adapted to three-dimensional contact manifolds," *Lecture Notes in Mathematics*, Springer, Berlin, Germany, pp. 279–308, 1985.
- [18] K. Yano and M. Kon, "Structures on manifolds," *Series in Pure Mathematics*, World Scientific Publishing Corporation, Singapore, 1984.
- [19] D. E. Blair, "Riemann geometry of contact and symplectic manifolds," *Progress in Mathematics*, Birkhäuser, Boston, MA, USA, 2002.

- [20] G. Calvaruso and D. Perrone, "Contact pseudo-metric manifolds," *Differential Geometry and Its Applications*, vol. 28, no. 5, pp. 615–634, 2010.
- [21] G. Dileo and A. M. Pastore, "Almost Kenmotsu manifolds with a condition of η -parallelism," *Differential Geometry and Its Applications*, vol. 27, no. 5, pp. 671–679, 2009.