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Almost α-Kenmotsu Pseudo-Riemannian Manifolds with CR-Integrable Structure

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Abstract: This article aims to study almost α-Kenmotsu pseudo-Riemannian structure. We first focus on the concept of almost α-Kenmotsu pseudo-Riemannian structure and its basic properties. Then, we shall prove some fundamental formulas and some classification results on such manifolds with CR-integrable structure. Finally, some illustrative examples of almost α-Kenmotsu pseudo-Riemannian manifold are given.

Keywords: almost α-Kenmotsu manifold; pseudo-Riemannian metric; Kaehler structure; contact distribution

MSC: 53C25; 53C35; 53D15

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1. Introduction

Contact metric manifolds with Riemannian metrics have been investigated by many authors. In particular, Blair obtained complete and detailed results on this topic [1]. Contact pseudo-Riemannian structures (η, g) are inherently generalizations of contact metric structures, where η is a contact one-form and g is a pseudo-Riemannian metric equipped with it. Takahashi first investigated contact metric structures with associated pseudo-Riemannian metric in Sasakian manifolds [2]. Calvaruso and Perrone introduced a systematic study of contact structures with associated pseudo-Riemannian metrics [3]. The relevance of the physics of contact pseudo-Riemannian structures was indicated in [4–7]. With the help of the contact pseudo-Riemannian structure mentioned in [7], it may provide more insight into the geometry of spacetime needed for physical problems in relativity. In recent years, some authors have studied almost contact pseudo-Riemannian manifolds [8–11]. We should remember that the primary source and the greatest motivation for researchers working on pseudo-Riemannian space is O'Neill's book [12].

The class of almost contact metric manifolds, Kenmotsu manifolds, was first introduced by Kenmotsu [13]. The Kenmotsu structure (ϕ, ξ, η, g) is normal and, in general, these structures are not Sasakian. Kenmotsu manifolds can be characterized through their Levi-Civita connection. Kenmotsu defined a structure closely related to the warped product which was characterized by tensor equations. A Kenmotsu manifold M of dimension $(2n+1)$ is identified with a warped product space $M = (-\varepsilon, +\varepsilon) \times_f {M_1}^{2n}$ such that $(-\varepsilon, +\varepsilon)$ is an open interval, ${M_1}^{2n}$ is a Kaehler manifold, c is a positive constant, and $f(t) = ce^t$. Given an almost Kenmotsu structure, an almost α-Kenmotsu structure can be obtained using the following homothetic deformation:

$$\eta^* = \left(\frac{1}{\alpha}\right)\eta,\ \xi^* = \alpha\xi,\ \phi^* = \phi,\ g^* = \left(\frac{1}{\alpha^2}\right)g$$

where α is a non-zero real constant. It is important to note that almost α-Kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures [14].

On the other hand, a systematic study of almost α-Kenmotsu pseudo-Riemannian manifolds has not been undertaken yet. In [11], Wang and Liu introduced the geometry of almost Kenmotsu pseudo-Riemannian manifolds. The authors emphasized the analogies and differences connected with the Riemannian metric tensor and obtained certain classification results related to local symmetry and nullity condition. Besides, the authors studied locally symmetric almost Kenmotsu manifolds of dimension $(2n+1)$ $(n>1)$ with CR-integrable structure [15]. These structures are locally isometric to either the hyperbolic space of constant sectional curvature -1 or the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.

Kenmotsu pseudo-Riemannian manifolds were investigated by Naik et al. [8]. In particular, the authors established necessary and sufficient conditions for Kenmotsu pseudo-Riemannian manifolds satisfying certain tensor conditions. Furthermore, Öztürk et al. studied α-Kenmotsu pseudo-Riemannian manifolds satisfying conformally flat conditions and the tensor conditions such as local symmetry, local ϕ-symmetry, global ϕ-symmetry, and semi-symmetry [16]. Then, the authors obtained some results related to the Einstein, η-Einstein manifolds, ξ-sectional, and ϕ-sectional curvatures on α-Kenmotsu pseudo-Riemannian structures [9]. After these studies, the η-parallelity of the tensor fields h and ϕh were investigated by the authors. They obtained some results with η-parallelity and η-cyclic parallelity of the torsion tensor τ, as well as the deformation of almost α-Kenmotsu pseudo-Riemannian structure [17].

The article is organized as follows. In Section 2, we focus on the concept of almost α-Kenmotsu pseudo-Riemannian manifold. We shall describe the basic formulas of almost α-Kenmotsu pseudo-Riemannian manifolds. In Section 3, we shall obtain some results on CR-integrable almost α-Kenmotsu pseudo-Riemannian manifolds. Finally, we give some illustrative examples of almost α-Kenmotsu pseudo-Riemannian manifold.

2. Materials and Methods

Let M be a $(2n+1)$-dimensional smooth manifold endowed with a triple (ϕ, ξ, η). Here, ϕ is a type of (1,1)-tensor field, ξ is a vector field, and η is a one-form on M which defines [1].

$$\eta(\xi) = 1,\ \phi^2 = -I + \eta \otimes \xi \tag{1}$$

$$\phi\xi = 0,\ \eta \circ \phi = 0,\ rank\phi = 2n \tag{2}$$

A pseudo-Riemannian metric g on M is said to be compatible with the almost contact structure (ϕ, ξ, η) if:

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y) \tag{3}$$

where $\eta(X) = \varepsilon g(X, \xi)$, $g(\xi, \xi) = \varepsilon$, and $\varepsilon = \pm 1$ [11].

On such a manifold, the fundamental 2-form Φ of M is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y on M. An almost contact pseudo-Riemannian manifold with structure (ϕ, ξ, η, g) is considered, such that:

$$d\eta = 0,\ d\Phi = 2\alpha(\eta \wedge \Phi) \tag{4}$$

is said to be an almost α-Kenmotsu pseudo-Riemannian manifold for $\alpha \neq 0$, $\alpha \in R$ [17]. The identically vanishing of the following tensor defined by:

$$N_\phi = [\phi, \phi] + 2d\eta \otimes \xi \tag{5}$$

which expresses the normality of almost contact metric structure, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [18]. We notice that an α-Kenmotsu manifold is a normal almost α-Kenmotsu pseudo-Riemannian manifold M with $\varepsilon = +1$, and the metric g is Riemannian.

When an almost α-Kenmotsu pseudo-Riemannian manifold M has a normal almost contact structure, we can say that M is an α-Kenmotsu pseudo-Riemannian manifold [11].

Throughout the paper, we shall denote by $\Gamma(TM)$, ∇, and D the Lie algebra of all tangent vector fields on M, the Levi Civita connection of pseudo-Riemannian metric g, and the distribution orthogonal to ξ called the contact distribution, that is,

$$D = Ker(\eta) = \{X : \eta(X) = 0\} \tag{6}$$

respectively. $d\eta = 0$, D is integrable and the $(2n)$-dimensional distribution is given by $\phi(D) = D$. Let N be a maximal integral submanifold of D. So, the vector field ξ restricted to integral submanifold N is the normal vector of N. Hence, there exists a Hermitian structure and ϕ induces an almost complex structure J $\left(J^2 = -I\right)$ on M by $J\widetilde{X} = \phi\widetilde{X}$ for any vector field $\widetilde{X}$ tangent to N. Let G be the pseudo-Riemannian metric induced on N defined by $G\left(\widetilde{X}, \widetilde{Y}\right) = g\left(\widetilde{X}, \widetilde{Y}\right)$. Then, (J, G) becomes an almost Hermitian structure on N such that $G\left(\widetilde{X}, \widetilde{Y}\right) = G\left(J\widetilde{X}, J\widetilde{Y}\right)$ for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ tangent to N. The fundamental 2-form Ω, $\Omega\left(\widetilde{X}, \widetilde{Y}\right) = G\left(\widetilde{X}, \widetilde{JY}\right)$ of (J, G) is induced on N. Additionally, we have $\left(\widetilde{X}, \widetilde{Y}\right) = \Phi(X, Y)$, i.e., Ω is the pull-back of the tensor field ϕ from M to N. Then, Ω is closed, i.e., $d\Omega = 0$. So, the pair (J, G) is an almost Kaehler structure on N of D. When the structure J is complex, (J, G) becomes a Kaehler structure on N. Suppose the structure (J, G) is Kaehler on every integral submanifold of the distribution D. In that case, this manifold is said to be an almost α-Kenmotsu pseudo-Riemannian manifold with Kaehler integral submanifold or a CR-integrable almost α-Kenmotsu pseudo-Riemannian manifold.

Proposition 1 [18]. *Let M be an almost contact metric manifold and ∇ be the Riemannian connection. Then, we have:*

$$(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \phi)Z) \tag{7}$$

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi) = (\nabla_X \Phi)(\xi, \phi Y) \tag{8}$$

$$(\nabla_X \Phi)(Y, Z) + (\nabla_X \Phi)(\phi Y, \phi Z) = \eta(Z)(\nabla_X \eta)\phi Y - \eta(Y)(\nabla_X \eta)\phi Z \tag{9}$$

$$2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X \tag{10}$$

$$3d\Phi(X, Y, Z) = \oplus_{X,Y,Z}(\nabla_X \Phi)(Y, Z) \tag{11}$$

Here, $\oplus_{X,Y,Z}$ denotes the cyclic sum over the vector fields X, Y and Z.

Lemma 1 [11]. *Let M be an almost contact pseudo-Riemannian manifold. Then, the following equation holds:*

$$\begin{aligned} 2g((\nabla_X \phi)Y, Z) = {} & 3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z) \\ & + g\left(N^{(0)}(Y, Z), \phi X\right) + \varepsilon N^{(1)}(Y, Z)\eta(X) \\ & + 2\varepsilon d\eta(\phi Y, X)\eta(Z) - 2\varepsilon d\eta(\phi Z, X)\eta(Y) \end{aligned} \tag{12}$$

for any $X, Y, Z \in \Gamma(TM)$, where $N^{(0)}, N^{(1)}$ is defined by:

$$N^{(0)}(X, Y) = N_\phi(X, Y) + 2d\eta(X, Y)\xi \tag{13}$$

and

$$N^{(1)}(X, Y) = \left(L_{\phi X}\eta\right)Y - \left(L_{\phi Y}\eta\right)X \tag{14}$$

respectively. Here, L_X denotes the Lie derivative in the direction of X.

Proposition 2 [17]. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold. Then, we have:*

$$h(X) = \frac{1}{2}(L_\xi \phi)X,\ h(\xi) = 0 \tag{15}$$

$$\nabla_X \xi = -\alpha\phi^2 X - \phi h X \tag{16}$$

$$\nabla_\xi \xi = 0,\ \nabla_\xi \phi = 0 \tag{17}$$

$$(\phi \circ h)X = -(h \circ \phi)X \tag{18}$$

$$(\nabla_X \eta)Y = \alpha[\varepsilon g(X,Y) - \eta(X)\eta(Y)] + \varepsilon g(\varphi Y, hX) \tag{19}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proposition 3. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold. For any* $X, Y, Z \in \Gamma(TM)$, *we have:*

$$\begin{aligned} 2g((\nabla_X \phi)Y, Z) = -2g\alpha(\varepsilon g(X, \phi Y)\xi + \eta(Y)\phi X, Z) \\ + g\left(N^{(0)}(Y,Z), \phi X\right) \end{aligned} \tag{20}$$

Proof. By the help of $N^{(1)}$, it follows that:

$$N^{(1)}(Y,Z) = 2d\eta(\phi Y, Z) - 2d\eta(\phi Z, Y) \tag{21}$$

From (12) and (21), it is easy to obtain:

$$2g((\nabla_X \phi)Y, Z) = 3d\Phi(X, \phi Y, \phi Z) - 3d\Phi(X, Y, Z) + g\left(N^{(0)}(Y,Z), \phi X\right) \tag{22}$$

Thus, make use of (22) and the definition of almost α-Kenmotsu pseudo-Riemannian manifold, then the proof follows from (7) and (8). □

Proposition 4 [17]. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold. Then, we have:*

$$R(X,Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\phi hY - \eta(Y)\phi hX] + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \tag{23}$$

for any $X, Y \in \Gamma(TM)$.

Proposition 5. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold. Then, the curvature conditions are as follows:*

$$R(X,\xi)\xi = \alpha^2\phi^2 X + 2\alpha\phi hX - h^2X + \phi(\nabla_\xi h)X \tag{24}$$

$$(\nabla_\xi h)X = -\phi R(X,\xi)\xi - \alpha^2\phi X - 2\alpha hX - \phi h^2 X \tag{25}$$

$$R(X,\xi)\xi - \phi R(\phi X, \xi)\xi = 2\left[\alpha^2\phi^2 X - h^2 X\right] \tag{26}$$

$$S(X,\xi) = -2n\alpha^2\eta(X) - (div(\phi h))X \tag{27}$$

$$S(\xi,\xi) = -\left[2n\alpha^2 + tr\left(h^2\right)\right] \tag{28}$$

$$div\xi = 2\alpha n,\ div\eta = -2\alpha n\varepsilon. \tag{29}$$

Proof. By the hypothesis, using (23) with $Y = \xi$ and considering the following equations,

$$(\nabla_\xi \phi h)X = \phi(\nabla_\xi h),\ (\nabla_X \phi h)\xi = h^2 X - \phi hX \tag{30}$$

we obtain (24). Applying ϕ to (24) and remarking that $g\big(\big(\nabla_\xi h\big)X,\xi\big)=0$, we obtain (25). Additionally, by the help of (24) for ϕX, we have:

$$R(\phi X,\xi)\xi=-\alpha^2\phi X-2\alpha\phi^2 hX-\phi h^2 X+\phi\big(\nabla_\xi h\big)(\phi X) \tag{31}$$

Then, we obtain:

$$R(X,\xi)\xi-\phi R(\phi X,\xi)\xi=2\alpha^2\phi^2 X-2h^2X+\phi\big(\nabla_\xi h\big)X+\phi^2\big(\nabla_\xi h\big)(\phi X) \tag{32}$$

which reduces to (26). We note that $\big(\nabla_\xi h\big)\circ\phi=-\phi\circ\big(\nabla_\xi h\big)$. Now, we may take a local orthonormal ϕ-basis $\{E_1,\cdots,E_{2n},\xi\}=\{e_1,\cdots,e_n,\phi e_1,\cdots,\phi e_n,\xi\}$. From (23) and using the definition of Ricci curvature tensor, we obtain

$$\begin{aligned} S(X,\xi)=&-2n\alpha 2\eta(X)+\alpha\eta(X)\big[\textstyle\sum_{i=1}^n\big[\varepsilon_i g(\phi he_i,e_i)-\varepsilon_{i+n}g\big(\phi^2 he_i,\phi e_i\big)\big]\big]\\ &+\textstyle\sum_{i=1}^n[\varepsilon_i g(\nabla_X\phi h)e_i,e_i)+\varepsilon_{i+n}g((\nabla_X\phi h)\phi e_i,\phi e_i)]\\ &-\sum_{i=1}^n\big[\varepsilon_i g((\nabla_{e_i}\phi h)X,e_i)+\varepsilon_{i+n}g\big(\big(\nabla_{\phi e_i}\phi h\big)X,\phi e_i\big)\big]\end{aligned}$$

Then, the above equation reduces to:

$$S(X,\xi)=-2n\alpha^2\eta(X)-\sum_{i=1}^{2n+1}\varepsilon_i g(R(E_i,X)\xi,E_i)$$

such that

$$div(\phi h)=\sum_{i=1}^{2n+1}\varepsilon_i g\big(\big(\nabla_{E_i}\phi h\big)X,E_i\big).$$

Since $tr(\phi h)=0$, we deduce:

$$0=\sum_{i=1}^n[\varepsilon_i g(\phi he_i,e_i)+\varepsilon_{i+n}g(\phi h\phi e_i,\phi e_i)].$$

Thus, the proof of (27) completes. Moreover, putting $X=\xi$ in (27), we have:

$$S(\xi,\xi)=-2n\alpha^2-(div(\phi h))\xi \tag{33}$$

where $(div(\phi h))\xi=tr\big(h^2\big)$. So (33) reduces to (28). It is well known that

$$div\eta=-tr(\nabla\eta)=-\sum_{i=1}^n\big\{(\nabla_{e_i}\eta)e_i+\big(\nabla_{\phi e_i}\eta\big)\phi e_i\big\}.$$

follows from the above equation, and so we deduce

$$div\xi=\alpha\Big(\textstyle\sum_{i=1}^n g(e_i-\phi he_i,e_i)+g\Big(\phi e_i+\phi^2 he_i,\phi e_i\Big)\Big)$$

and

$$div\eta=-tr(\nabla\eta)=-\varepsilon div\xi.$$

Thus, we complete the proof. □

3. Results

This section is devoted to study almost α-Kenmotsu pseudo-Riemannian manifolds whose integral submanifolds of D are Kaehler.

3.1. CR-Integrability

Proposition 6. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold. Then, we have*

$$(\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y = -\alpha[\eta(Y)\phi X - 2\varepsilon g(\phi X, Y)\xi] - \eta(Y)hX \tag{34}$$

$$\phi(\nabla_{\phi X}\phi)Y - (\nabla_X\phi)Y = 2\alpha\eta(Y)\phi X - \varepsilon(g(\alpha\phi X + hX, Y)\xi) \tag{35}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Using Koszul formula, we have:

$$\begin{aligned}2g((\nabla_X\phi)Y, Z) = 2\eta(X)\Phi(\phi Y, \phi Z) - 2[\eta(X)\Phi(Y, Z) + \eta(Y)\Phi(Z, X) + \eta(Z)\Phi(X, Y)] \\ +g\Big(N^{(0)}(Y, Z), \phi X\Big).\end{aligned} \tag{36}$$

Replacing X and Y with ϕX and ϕY in (36), respectively, so we obtain:

$$2g\big((\nabla_{\phi X}\phi)\phi Y, Z\big) = -2\eta(Z)\Phi(\phi X, \phi Y) + g\Big(N^{(0)}(\phi Y, Z), \phi^2 X\Big). \tag{37}$$

Next, from (36) and (37), it follows that:

$$\begin{aligned}g\big((\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y, Z\big) = 2\{\eta(X)\Phi(\phi Y, \phi Z) - \eta(X)\Phi(Y, Z) - \eta(Y)\Phi(Z, X) \\ -\eta(Z)\Phi(X, Y) - \eta(Z)\Phi(\phi X, \phi Y)\} - g\Big(N^{(0)}(\phi Y, Z) + \phi N^{(0)}(Y, Z), X\Big) \\ +(1/\varepsilon)\eta(X)\eta\Big(N^{(0)}(\phi Y, Z)\Big)\end{aligned} \tag{38}$$

Here, the sum of $N^{(0)}(\phi Y, Z) + \phi N^{(0)}(Y, Z)$ is given by:

$$\begin{aligned}N^{(0)}(\phi Y, Z) + \phi N^{(0)}(Y, Z) = \; & -\varepsilon[g(\phi Y, \nabla_Z\xi)\xi + g(Y, \nabla_{\varphi Z}\xi)\xi - g(\phi Y, \nabla_Z\xi)\xi \\ & -g(\phi Z, \nabla_Y\xi)\xi] + \eta(Y)[\phi\nabla_Z\xi - \nabla_{\phi Z}\xi].\end{aligned}$$

Simplifying the last equation, it reduces to:

$$N^{(0)}(\phi Y, Z) + \phi N^{(0)}(Y, Z) = 2\eta(Y)hZ. \tag{39}$$

Moreover, using (13) and (20), we have:

$$\eta\Big(N^{(0)}(\phi Y, Z)\Big) = \varepsilon\big[g(\phi Z, \nabla_Y\xi) - g\big(Y, \nabla_{\phi Z}\xi\big)\big]. \tag{40}$$

Following from (40), we obtain:

$$\eta\Big(N^{(0)}(\phi Y, Z)\Big) = -\alpha\Big[g\Big(\phi Z, \phi^2 Y\Big) + g(Y, \phi Z)\Big] + g(Y, hZ) + g\Big(Z, \phi^2 hY\Big). \tag{41}$$

In view of (39) and (41), it is easy to see that:

$$\eta\Big(N^{(0)}(\phi Y, Z)\Big) = 0. \tag{42}$$

Taking into account of (38), (39) and (42), we have

$$\begin{aligned}2g\big((\nabla_X\phi)Y + (\nabla_{\phi X}\phi)\phi Y, Z\big) = \; & -2\alpha\eta(Y)g(\phi X, Z) - 2\eta(Y)g(hX, Z) \\ & +4\varepsilon\alpha\eta(Z)g(\phi X, Y).\end{aligned} \tag{43}$$

Then, it follows from (43), and we lead to (34). Additionally, we consider the following formula

$$(\nabla_{\phi X}\phi)\phi Y = \nabla_{\phi X}\phi^2 Y - \phi(\nabla_{\phi X}\phi Y)$$

for any $X, Y \in \Gamma(TM)$ and then applying the covariant derivation $(\nabla_X \phi)Y$ in the above equation by the help of (34), we obtain:

$$(\nabla_X \phi)Y - \phi(\nabla_{\phi X} \phi)Y = -2\alpha\eta(Y)\phi X + \varepsilon g(\alpha\phi X, Y)\xi + \varepsilon g(hX, Y)\xi.$$

Thus, it completes the proof. □

Proposition 7. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold. Then, we have*

$$\begin{aligned} g(R_{\xi X}Y, Z) - g(R_{\xi X}\phi Y, \phi Z) + g(R_{\xi\phi X}Y, \phi Z) + g(R_{\xi\phi X}\phi Y, Z) \\ = 2(\nabla_{hX}\Phi)(Y, Z) + 2\alpha^2\eta(Y)g(X, Z) - 2\alpha^2\eta(Z)g(X, Y) \\ -2\alpha\eta(Y)g(\phi hX, Z) + 2\alpha\eta(Z)g(\phi hX, Y) \end{aligned} \tag{44}$$

for any $X, Y, Z \in \Gamma(TM)$. *Here,* $g(R_{\xi X}Y, Z) = R(\xi, X, Y, Z) = g(X, R_{YZ}\xi)$.

Proof. Making use of (23) and consider the definition of Riemannian curvature tensor, the left side of (44) takes the form

$$2\alpha^2\eta(Y)g(X, Z) - 2\alpha^2\eta(Z)g(X, Y) + G(X, Y, Z) - G(X, Z, Y), \tag{45}$$

such that G is defined by

$$\begin{aligned} G(X, Y, Z) &= g(X, -(\nabla_Y \phi h)Z + \phi(\nabla_Y \phi h)\phi Z)) \\ &+ g(X, (\nabla_{\phi Y}\phi h)\phi Z) - g(\phi X, (\nabla_{\phi Y}\phi h)Z). \end{aligned} \tag{46}$$

On the other hand, the following formulas can be written as

$$\begin{aligned} g(X, (\nabla_{\phi Y}\phi h)\phi Z + \phi(\nabla_{\phi Y}\phi h)Z) = g(\phi X, \phi(\nabla_{\phi Y}\phi h)\phi Z) \\ \eta(X)g(\xi, (\nabla_{\phi Y}\phi h)\phi Z) - (\nabla_{\phi Y}\phi h)Z \end{aligned} \tag{47}$$

$$\phi(\nabla_Y \phi h)\phi Z - (\nabla_Y \phi h)Z = h(\nabla_Y \phi)Z - (\nabla_Y \phi)hZ \tag{48}$$

and

$$g(\xi, (\nabla_{\phi Y}\phi h)\phi Z) = \varepsilon[g(hY, hZ) - g(hZ, \phi Y)]. \tag{49}$$

Taking into account of (34), (35), (47), (48) and (49) in (46), it follows that:

$$\begin{aligned} G(X, Y, Z) = -\alpha\eta(X)g(\phi Y, hZ) - \alpha\eta(X)g(hZ, hY) + 2g(hX, (\nabla_Y \phi)Z) \\ +2\alpha\eta(Z)g(\phi Y, hX) - \alpha\eta(X)\varepsilon g(hZ, \phi Y) + \alpha\eta(X)\varepsilon g(hZ, hY). \end{aligned} \tag{50}$$

Finally, by the help of (50), we arrange (45) with the following formula:

$$3d\Phi(Y, Z, hX) = (\nabla_Y \Phi)(Z, hX) + (\nabla_Z \Phi)(hX, Y) + (\nabla_{hX}\Phi)(Y, Z)$$

Thus, we obtain (44). This ends the proof. □

Proposition 8. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold. Then, we have*

$$A\phi + \phi A = -2\alpha\phi, \quad \eta \circ A = 0, \quad \eta \circ h = 0 \tag{51}$$

$$h = A \circ \phi + \alpha\phi, \quad hA + Ah = -2\alpha h \tag{52}$$

$$tr(A) = -\alpha \sum_{i=1}^{2n} \varepsilon_i, \quad tr(\phi A) = 0 \tag{53}$$

for any $X, Y \in \Gamma(TM)$. *Here, A is defined by* $A = -\nabla\xi$.

Proof. For any $X, Y \in \Gamma(TM)$, we have

$$g(X, AY) = g\big(X, \phi^2 Y + \phi h Y\big)$$
$$g(X, AY) = -g(X, Y) + \eta(Y)\eta(X) - g(\phi X, hY)$$

and

$$g(AX, Y) = g\big(\phi^2 X + \phi h X, Y\big)$$
$$g(AX, Y) = -g(X, Y) + \eta(X)\eta(Y) - g(\phi X, hY)$$

Here A is given by $A = \phi^2 + \phi h$. Also, using the definition of h, it yields

$$g\left(\frac{1}{2}(L_\xi \phi)X, Y\right) = g\left(X, \frac{1}{2}(L_\xi \phi)Y\right).$$

We also note that ϕ is a (1,1)-type tensor field. Otherwise, the above equality does not provide. So A and h are the symmetric operators. Then, taking the sum of $A\phi$ and ϕA, we deduce

$$A\phi + \phi A = 2\alpha\phi^3 + \phi h \phi + \phi^2 h$$

which ends the proof of the first side of (51). By considering the one-form and A, we get

$$(\eta \circ A)X = \eta(AX) = \varepsilon g(X, A\xi) = 0$$

and

$$(\eta \circ h)X = \eta(hX) = \varepsilon g(X, h\xi) = 0.$$

Hence, the rest of the proof of (51) is obvious. It is clear that $A\xi = h\xi = 0$. Furthermore, by substituting X for ϕX, we have

$$A\phi X = -\alpha\phi X + hX$$

and

$$hAX + AhX = \alpha\phi^2 hX - \phi h^2 X + \alpha\phi^2 hX + \phi h^2 X.$$

Thus, the proof of (52) ends. Finally, the trace of A can be written as

$$tr(A) = \textstyle\sum_{i=1}^{2n+1} \varepsilon_i g(AE_i, e_i) = \sum_{i=1}^{2n+1} \varepsilon_i g\big(\alpha\phi^2 E_i + \phi h E_i, E_i\big)$$
$$= -\alpha \textstyle\sum_{i=1}^{2n+1} \varepsilon_i g(\phi E_i, \phi E_i) + \varepsilon_i tr(\phi h)$$

where $tr(\phi h) = 0$. Analogously, the trace of ϕA is as follows:

$$tr(\phi A) = \textstyle\sum_{i=1}^{2n+1} \varepsilon_i g(-\alpha\phi E_i - hE_i, E_i).$$

Here, $tr(h) = 0$ and $\{E_1, \cdots, E_{2n}, \xi\}$ is a local orthonormal ϕ-basis. □

Theorem 1. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold and $h = 0$. Then, M is expressed by a locally warped product such that $M_\xi \times_g \widetilde{M}$ where M_ξ is an open interval with coordinate t, $\widetilde{M}$ is a $2n$-dimensional indefinite almost Kaehler manifold, and $g = \lambda e^{\alpha t}$ for $\lambda > 0$.*

Proof. First, we consider the contact distribution defined by $D = Ker\eta = Im\phi$. For $X \in D$, (16) takes the form $\nabla_X \xi = \alpha X$ when $h = 0$. Let $\widetilde{M}$ and $\widetilde{\nabla}$ be the integral submanifold of D and the Levi-Civita connection of $\widetilde{M}$, respectively. Then, the second fundamental form $B(X, Y)$ of pseudo-Riemannian immersion $\widetilde{M} \to M$ is defined by

$$\eta(B(X, Y)) = \eta\left(\nabla_X Y - \widetilde{\nabla}_X Y\right) = -g(\nabla_Y \xi, X) = -\alpha g(X, Y)$$

for any $X, Y \in D$. Thus, $\widetilde{M}$ is a totally umbilical submanifold of M. Additionally, the mean curvature vector field defined by $H = -\alpha\varepsilon\xi$. Following from (51), it is obvious that $\nabla_{\xi}\xi = 0$. Accordingly, M is locally a warped product space such that $M_{\xi} \times_{g} \widetilde{M}$, where M_{ξ} is an integral curve of ξ. It is well known that the mean vector field H is π related to $-(1/g)grad\, g$. Here, the projection $\pi : M_{\xi} \times_{g} \widetilde{M} \to M_{\xi}$ is a pseudo-Riemannian submersion. In other words, we have

$$\alpha\varepsilon g\xi = gradg. \tag{54}$$

Then, (54) shows that we can obtain $\xi = (\partial/(\partial t))$ with coordinate t in local sense. So, we can write $grad(f) = \alpha\varepsilon((\partial f)/(\partial t))\partial t$, where $g(\xi, \xi) = \varepsilon$. Thus, the general solution of this differential equation takes the form $g = \lambda e^{\alpha t}$, where λ is a positive constant. After all, we denote by J the restriction of ϕ on contact distribution, then we can see that $(\widetilde{M}, J)$ is an indefinite almost Kaehler manifold of dimension $2n$. □

Theorem 2. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold such that the integral manifolds of D are indefinite Kaehler. Then, M is an α-Kenmotsu pseudo- Riemannian manifold if and only if $\nabla\xi = -\alpha\phi^2$.*

Proof. With the help of similar technique in [19], we obtain that $N(X, \xi) = 2\alpha\phi(hX)$ for any $X \in \Gamma(TM)$. If the structure is normal, then $h(Y)$ vanishes for $Y \in D$. Since $h(\xi) = 0$, we have $h = 0$. Then from (16), it implies that $\nabla_X\xi = -\alpha\phi^2 X$. Conversely, if $\nabla_X\xi = -\alpha\phi^2 X$, then we obtain $h = 0$. So, we say that $N(X, \xi) = 0$ for any $X \in \Gamma(TM)$. Furthermore, it is clear that $N_J(X, Y) = N(X, Y) = 0$ for $X, Y \in D$. Thus, the integral manifolds of D are Kaehler manifolds. □

Theorem 3. *If M is an almost α-Kenmotsu pseudo-Riemannian manifold, then the integral manifolds of D are indefinite almost Kaehler manifolds given by the mean curvature vector field $H = -\alpha\varepsilon\xi$. Additionally, these integral manifolds are totally umbilical submanifolds if and only if $h = 0$.*

Proof. Let denote by $\widetilde{M}$ and $\widetilde{\nabla}$ the integral manifold of contact distribution D and the Levi-Civita connection of the integral manifold, respectively. Then, we take into account the pseudo-Riemannian immersion such that $\widetilde{M} \to M$, denoting by $B(X, Y)$ the second fundamental form for any $X, Y \in \Gamma(TM)$. In view of (16), we may write

$$g(B(X, Y), \xi) = g\left(\nabla_X Y - \widetilde{\nabla}_X Y, \xi\right) = -g(Y, \alpha X - \phi hX) \tag{55}$$

which reduces to

$$B(X, Y) = -\varepsilon g(Y, \alpha X - \phi hX)\xi. \tag{56}$$

Thus, it follows from (56) that we can say that $\widetilde{M}$ is totally umbilical submanifold of M if and only if $h = 0$. Therefore, for $h = 0$, we obtain $B(X, Y) = -\varepsilon\alpha g(Y, X)\xi$. By a straightforward calculation, the mean curvature vector field H takes the form $H = -\alpha\varepsilon\xi$. Thus, we complete the proof. □

3.2. (κ, μ)-Spaces

The notion of (κ, μ)-spaces was introduced by Blair as defined in the following equation

$$R(Y, Z)\xi = -\eta(Y)(\kappa I + \mu h)Z + \eta(Z)(\kappa I + \mu h)Y \tag{57}$$

for κ and μ constants [1]. Furthermore, Dileo and Pastore investigated $(\kappa, \mu)'$-spaces on almost Kenmotsu manifolds [19]. The characteristic vector field ξ belongs to the $(\kappa, \mu)'$-space if

$$R(Y, Z)\xi = -\eta(Y)\left(\kappa I + \mu h'\right)Z + \eta(Z)\left(\kappa I + \mu h'\right)Y \tag{58}$$

for any $X, Y, Z \in \Gamma(TM)$. Here, we remark that $h' = h \circ \phi$. Following this study, Öztürk generalized the nullity condition on almost α-cosymplectic manifolds

$$R(Y,Z)\xi = -\eta(Y)(\kappa I + \mu h + v\phi h)Z + \eta(Z)(\kappa I + \mu h + v\phi h)Y \tag{59}$$

for the smooth functions, such that $d\kappa \wedge \eta = d\mu \wedge \eta = dv \wedge \eta = 0$ [20].

Now, we obtain some results satisfying (57) and (58) on almost α-Kenmotsu pseudo-Riemannian (κ, μ)-spaces.

Theorem 4. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold whose ξ belongs to the (κ, μ)-space. Then, $\kappa = -\varepsilon\alpha^2$ and $h = 0$. Additionally, Theorem 1 holds.*

Proof. Letting $Y \in Ker\eta$ and $Z = \xi$. In view of (57), we have

$$R(Y,\xi)\xi = \varepsilon(\kappa Y + \mu hY). \tag{60}$$

Then, substituting (60) into (26) gives:

$$h^2 Y = -\left(\varepsilon\kappa + \alpha^2\right)Y. \tag{61}$$

In fact, by using (24) and (26), we obtain:

$$l\phi Y = \varepsilon(\kappa\phi Y + \mu h\phi Y), \; \phi l\phi Y = \varepsilon(-\kappa Y + \mu hY)$$

and

$$lY - \phi l\phi Y = 2\varepsilon\kappa Y = -2\alpha^2 Y - 2h^2 Y$$

where $l = R(.,\xi)\xi$ is the Jacobi operator with respect to the characteristic vector field ξ. Let Y be the eigenvector field of h with respect to the eigenvalue ρ defined by $hY = \rho Y \in Ker\eta$. Then, it follows from (23) that:

$$lY = -\alpha^2 Y + 2\alpha\phi hY - h^2 Y - \left(\nabla_\xi h\phi\right)Y. \tag{62}$$

Taking account of (60)–(62), we obtain:

$$\varepsilon\mu\rho Y - 2\alpha\rho\phi Y + \left(\nabla_\xi h\phi\right)Y = 0. \tag{63}$$

Next, taking scalar product with ϕY on both sides of (63), we have:

$$\rho = 0. \tag{64}$$

This implies that $h = 0$. Moreover, by the help of (61) and (64), we deduce:

$$\varepsilon\kappa + \alpha^2 = 0. \tag{65}$$

Thus, the proof of the rest of the theorem is obvious by using Theorem 1. □

Theorem 5. *Let M be an almost α-Kenmotsu pseudo-Riemannian manifold whose ξ belongs to the $(\kappa, \mu)'$-space and $h\phi \neq 0$. Then, the integral manifolds of D has indefinite Kaehler structure.*

Proof. Taking $Y, Z \in Ker\eta$ and from (58) we have $R(Y,Z)\xi = 0$. Again taking $Y, Z, W \in Ker\eta$, by the help of Proposition 7, we obtain:

$$(\nabla_{hY}\Phi)(Z,W) = 0. \tag{66}$$

On the other hand, from Proposition 3, we obtain

$$2(\nabla_{hY}\Phi)(Z,W) + g\left(N_\phi(Z,W), \phi hY\right) = 0 \tag{67}$$

such that $(\nabla_Y \Phi)(Z,W) = -g(W,(\nabla_Y \phi)Z)$. We note that $g\big(N_\phi(Z,W),\phi hY\big) = 0$ for $Y,Z,W \in Ker\eta$. By the hypothesis, using (58) in (26), the following is yielded:

$$(h\phi)^2 = -\left(\varepsilon\kappa + \alpha^2\right)Y. \tag{68}$$

Here, we have $h^2 = (h\phi)\circ(h\phi) = (h\phi)^2$. Since $\phi h \neq 0$ ($\varepsilon\kappa + \alpha^2 \neq 0$), then, we observe that $g\big(N_\phi(Z,W),Y\big) = 0$ for $Y,Z,W \in Ker\eta$. This implies that $N_\phi(Z,W) = 0$. Thus, it completes the proof. □

4. Examples

4.1. Example of an Arbitrary Dimensional Case

Let $M = M_1 \times I$, where M_1 is an open connected subset of R^{2n} and I is an open interval in R. Let $(u_1,\ldots,u_n,v_1,\ldots,v_n,z)$ be the Cartesian coordinates such that

$$M = \left\{(u_1,\ldots,u_n,,v_1,\ldots,v_n,z) \in R^{2n+1} : z \neq 0\right\}.$$

The global basis $\{U_1,\ldots,U_n,,V_1,\ldots,V_n,\xi\}$ on M defined by

$$U_i = 2z\left(\frac{\partial}{\partial u_i}\right), \quad V_i = -\left(\frac{2}{z^3}\right)\left(\frac{\partial}{\partial v_i}\right), \quad \xi = \left(\frac{\partial}{\partial z}\right)$$

for $i = 1,2,\ldots,n$. Now, we define the structure (ϕ,ξ,η,g) on M as follows:

$$\phi\left(\left(\tfrac{\partial}{\partial u_i}\right)\right) = -\left(\tfrac{1}{z^4}\right)\left(\tfrac{\partial}{\partial v_i}\right),\ \phi\left(\left(\tfrac{\partial}{\partial v_i}\right)\right) = z^4\left(\tfrac{\partial}{\partial u_i}\right),\ \phi\left(\tfrac{\partial}{\partial z}\right) = 0$$
$$\eta = dz,\ \xi = \tfrac{\partial}{\partial z}$$

and

$$g = (1/4)\textstyle\sum_{i=1}^n \left(1/z^2 du_i^2 + z^6 dv_i^2\right) + \varepsilon dz^2.$$

Here, $g(\xi,\xi) = \varepsilon = \mp 1$. This means that M is an almost contact pseudo-Riemannian structure with (ϕ,ξ,η,g). In order to check, whether it is almost α-Kenmotsu pseudo-Riemannian or not, we verify the condition $d\Phi = 2\alpha(\eta \wedge \Phi)$. On the other hand, all Φ_{ij}'s vanish except for

$$\Phi_{ii} = g\left(\left(\frac{\partial}{\partial u_i}\right),\phi\left(\frac{\partial}{\partial v_i}\right)\right) = z2/4.$$

Hence, we obtain

$$\Phi = \left(z^2/4\right)\textstyle\sum_{i=1}^n (du_i \wedge dv_i)$$

and

$$d\Phi = \left(\frac{z}{2}\right)(du \wedge dv \wedge dz) = \left(\frac{2}{z}\right)(\eta \wedge \Phi).$$

We remark that $N_\phi \neq 0$. Hence, M is an almost α-Kenmotsu pseudo-Riemannian manifold with the above (ϕ,ξ,η,g) structure. Since the structure is not normal, the tensor field h does not have to be zero. So, the integral submanifold of D is almost Kaehler manifold [20,21], and then Theorems 1–3 are held.

4.2. Example of a Three-Dimensional Case

Let us denote the Cartesian coordinates of $R^3(x,y,z)$ and consider three-dimensional manifold $M \subset R^3$ defined by

$$M = \left\{(x,y,z) \in R^3 \mid z \neq 0\right\}$$

where $M = M_1 \times I$, where M_1 is an open connected subset of R^2 and I is an open interval in R. The arbitrary vector fields are given by

$$E_1 = L_1(z)(\partial/\partial x) + L_2(z)(\partial/\partial y)$$
$$E_2 = -L_2(z)(\partial/\partial x) + L_1(z)(\partial/\partial y)$$
$$E_3 = (\partial/(\partial z))$$

such that

$$L_1(z) = c_2 e^{-\alpha z} cos\lambda z - c_1 e^{-\alpha z} sin\lambda z,$$
$$L_2(z) = c_1 e^{-\alpha z} cos\lambda z + c_2 e^{-\alpha z} sin\lambda z.$$

Here, it is noted that ${c_1}^2 + {c_2}^2$ for constants c_1, c_2, λ and $\alpha \neq 0$. We define the structure (ϕ, ξ, η, g) on M as follows:

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0$$
$$\boldsymbol{\eta = dz,\ \xi = \frac{\partial}{\partial z}}$$
$$g = \left({L_1}^2 + {L_2}^2\right)^{-1}\left(dx^2 + dy^2 + \varepsilon dz^2\right)$$

Let η be the one-form defined by

$$\eta(X) = \varepsilon g(X, e_3)$$

for any vector field X on M. Let h be the (1,1)-tensor field defined by

$$h(e_1) = -\lambda e_1, \quad h(e_2) = \lambda e_2, \quad h(e_3) = 0.$$

Then using linearity of g and ϕ, we obtain

$$\phi^2 X = -X + \eta(X)e_3,\ \eta(e_3) = g(\xi, \xi) = \varepsilon$$
$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$. Hence, M is an almost α-Kenmotsu pseudo-Riemannian manifold with the (ϕ, ξ, η, g) structure. However, it is sufficient to check that the only non-zero components of the second fundamental form Φ are

$$\Phi(\partial/\partial x, \partial/\partial y) = -\left({L_1}^2 + {L_2}^2\right)^{-1} = -e^{2\alpha z}\left({c_1}^2 + {c_2}^2\right)^{-1}.$$

So, the above equation takes the form

$$d\Phi = \left(-4\alpha e^{2\alpha z}\right)\left({c_1}^2 + {c_2}^2\right)^{-1}(dx \wedge dy \wedge dz)$$

which implies

$$d\Phi = 2\alpha(\eta \wedge \Phi)$$

on M. Additionally, the Nijenhuis torsion tensor of ϕ does not vanish. Because the structure is not normal, the tensor field h does not have to vanish. In three-dimensional case, for the D distribution on M to have Kaehler leaves, if and only if the following equation holds [20,21]:

$$(\nabla_X \phi)Y = \varepsilon g(\alpha\phi X + hX, Y)\xi - \eta(Y)(\alpha\phi X + hX).$$

By the help of the above equation, Theorems 1–3 are verified. In the case of dimension 3, the integral submanifolds of the D distribution are almost Kaehler regarding dimension 2.

5. Discussion

This article deals with almost α-Kenmotsu pseudo-Riemannian manifolds whose integral submanifolds are Kaehler. In other words, the main object of this article is to give some results about CR-Integrable α-Kenmotsu pseudo-Riemannian manifolds. It is well known that an almost CR-structure is said to be a CR-structure if it is integrable. In particular, we have planned our future works using some tensor conditions thanks to the studies on these subjects, which are the sources of our motivation [3,8,10,15,22].

The theory of solitons on manifolds is currently quite popular. Exciting results continue to be obtained in this topic. Ricci solitons have been studied extensively in various frameworks and from different perspectives. In particular, the physical applications of these subjects are interesting. The Ricci and gradient Ricci solitons play a crucial role in developing mathematics and physics. For this reason, in our further studies, we will study almost α-Kenmotsu manifolds endowed with different metric connections admitting some solitons. Moreover, all curvature and tensor products on (κ, μ, ν)-spaces will be investigated by using the soliton theory.

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