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Alessandro Ferraro, Leandro Aolita, Daniel Cavalcanti, Fernando M. Cucchietti ...+1 more authors

Institutions: Catalan Institution for Research and Advanced Studies

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Almost all quantum states have nonclassical correlations

A. Ferraro,[1] L. Aolita,[1] D. Cavalcanti,[1] F. M. Cucchietti,[1] and A. Acín[1,2]

[1]*Institut de Ciencies Fotoniques, Mediterranean Technology Park, E-08860 Castelldefels (Barcelona), Spain*

[2]*Institució Catalana de Recerca i Estudis Avançats, Lluis Companys 23, E-08010 Barcelona, Spain*

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Quantum discord quantifies nonclassical correlations in a quantum system including those not captured by entanglement. Thus, only states with zero discord exhibit strictly classical correlations. We prove that these states are negligible in the whole Hilbert space: typically a state picked out at random has positive discord and, given a state with zero discord, a generic arbitrarily small perturbation drives it to a positive-discord state. These results hold for any Hilbert-space dimension and have direct implications for quantum computation and for the foundations of the theory of open systems. In addition, we provide a simple necessary criterion for zero quantum discord. Finally, we show that, for almost all positive-discord states, an arbitrary Markovian evolution cannot lead to a sudden, permanent vanishing of discord.

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I. INTRODUCTION

The emergence of quantum-information science motivated a major effort toward the characterization of entangled states, generally believed to be an essential resource for quantum-information tasks that outperform their classical counterparts. In particular, the geometry of the sets of entangled or nonentangled states received much attention [1], starting from the fundamental result that the set of separable (nonentangled) states has nonzero volume in a finite-dimensional Hilbert space [2]. In other words, separable states are not at all negligible, which has direct implications for some implementations of quantum computing [3] and on the definition of entanglement quantifiers [4].

Apart from entanglement, quantum states display other correlations [5–7] not present in classical systems (meaning, here, systems where all observables commute). Aiming at capturing such correlations, Ollivier and Zurek introduced quantum discord [5]. They showed that only in the absence of discord does there exist a local measurement protocol that enables distant observers to extract all the information about a bipartite system without perturbing it. This completeness of local measurements is featured by any classical state, but not by quantum states, even some separable ones. Thus, zero discord is a necessary condition for classical-only correlations.

Very recently, quantum discord has received increasing attention [8–15]. A prevailing observation in all results obtained so far is that the absence or presence of discord is directly associated to nontrivial properties of states. Thus, it is natural to ask how typical positive-discord states are. Here we prove that a particular subset of states that contains the set of zero-discord states has measure zero and is nowhere dense. That is, it is topologically negligible: typically, every state picked out at random has positive discord and, given a state with zero discord, a generic (arbitrarily small) perturbation will take it to a state of strictly positive discord. Remarkably, these results hold true for any Hilbert-space dimension and are thus in contrast with expectations based on the structure of entangled states [2]: while the set of separable states has positive volume, the set of only classically correlated states does not. In addition, we provide a necessary condition, of very simple evaluation, for zero quantum discord. With this tool we suggest a schematic geometrical representation of the set of zero discord, and we study the open-system dynamics of discord. We find that, for almost all states of positive discord, the interaction with any (non-necessarily local) Markovian bath can *never* lead either to a sudden, permanent vanishing of discord or to one lasting a finite time interval. In strong contrast with entanglement—which typically vanishes suddenly and permanently at a finite time [16]—discord can only permanently vanish in the asymptotic infinite-time limit (i.e., at the steady state).

Our results have wide-ranging implications. First, from a fundamental perspective, they imply that classical-only correlated states are extremely rare in the space of all quantum states. Second, it was recently discovered that an arbitrary unitary evolution for any system and bath is described (upon tracing the bath out) as a completely positive map on the system if, and only if, system and bath are initially in a zero-discord state [11]. In view of the rarity of zero-discord states, the fundamental recipe "unitary evolution + partial trace" is now in conflict with complete positivity—one of the most basic and fundamental requirements that physical evolution is demanded to fulfill [17]—for almost all quantum states. Another interesting fact is that quantum discord is present in typical instances of a mixed-state quantum computation [18], even when entanglement is absent [9,10,14]. This led Datta *et al.* [9,10] to suggest that discord might be the resource responsible for the quantum speedup in this computational model. If the mere presence of discord was by itself responsible for some speedup, then our results would imply that almost all quantum states are useful resources. Furthermore, Piani *et al.* [12] introduced a new task—local broadcasting—to operationally distinguish among different varieties of states with zero quantum discord. They showed that only some zero-discord states can be locally broadcasted, which, according to us, now means hardly any quantum state. Also, our general results on the Markovian dynamics of discord complement and generalize the specific results reported in Refs. [13]. There, for particular cases of local channels and two-qubit systems, discord was never observed to vanish permanently at a finite time. As said, we prove the generality of this behavior. Finally, our results also apply to quantifiers of quantum correlations other than discord.

II. QUANTUM DISCORD

Consider a bipartite system in a composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, of dimension $d = d_A \times d_B$, with $d_A = \dim(\mathcal{H}_A)$ and $d_B = \dim(\mathcal{H}_B)$, respectively. Given a quantum state $\rho \in \mathcal{B}(H)$ [where $\mathcal{B}(H)$ denotes the set of bounded, positive-semidefinite operators on $\mathcal{H}$ with unit trace], the von Neumann mutual information I_{AB} between A and B is defined as

$$I_{AB}(\rho) \doteq S(\rho_A) + S(\rho_B) - S(\rho), \tag{1}$$

where $S(\rho) = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy and $\rho_{A,B} = \text{Tr}_{B,A}[\rho]$. Mutual information (1) quantifies the total amount of correlations in quantum states [6].

A classically equivalent definition of mutual information is $S(\rho_B) - S(\rho_{B|A})$, where $\rho_{B|A}$ is the state of B given a measurement in A. Thus, classical mutual information quantifies the decrease in ignorance (gain of information) about subsystem B upon local measurement on A. Let us now consider a measurement consisting of (non-necessarily orthogonal) one-dimensional measurement elements $\{M_j\}$ on $\mathcal{H}_A$. We can write the state of system B conditioned on the outcome j for A as $\rho_{B|j} = \text{Tr}_A[M_j\rho]/p'_j$, where the probability of outcome j is given by $p'_j = \text{Tr}[\rho M_j]$. By optimizing over the measurement set $\{M_j\}$, one can define

$$J_{AB}(\rho) \doteq S(\rho_B) - \min_{\{M_j\}} \sum_j p'_j S(\rho_{B|j}), \tag{2}$$

which quantifies the classical correlations in ρ [6].

Despite both definitions for the mutual information being equivalent for classical systems, the quantum generalizations I_{AB} and J_{AB} in general do not coincide. Their discrepancy defines the discord:

$$D_{AB}(\rho) \doteq I_{AB}(\rho) - J_{AB}(\rho). \tag{3}$$

Notice that quantum discord is always non-negative and it is asymmetric with respect to A and B [5].

III. NULL-DISCORD STATES

Let us denote by Ω_0 the set composed of all states with zero discord:

$$\Omega_0 \doteq \{\rho \in \mathcal{B}(H) \;\text{ such that }\; D_{AB}(\rho) = 0\}. \tag{4}$$

The members of this set are characterized [5,10] by being invariant under von Neumann measurements on A in some orthonormal basis $\{\Pi_j\}$; that is,

$$\rho \in \Omega_0 \Longleftrightarrow \exists \;\{\Pi_j\}\; \text{ such that }\; \rho = \sum_{j=1}^{d_A} \Pi_j \rho \Pi_j. \tag{5}$$

This implies that the set $\{\Pi_j\}$ defines a basis of $\mathcal{H}_A$ with respect to which ρ is block diagonal [10]:

$$\rho \in \Omega_0 \Longleftrightarrow \exists \;\{\Pi_j\}\; \text{ such that }\; \rho = \sum_{j=1}^{d_A} p_j \Pi_j \otimes \sigma_j, \tag{6}$$

where σ_j are quantum states in $\mathcal{B}(\mathcal{H}_B)$ and $\{p_j\}$ defines a probability distribution.

The characterization of Ω_0 presented just above is not practical in the sense that one has to check for the existence of a measurement basis for which conditions (5) and (6) are satisfied. With this motivation, we derive a sufficient condition for positive quantum discord that is *basis independent*. From condition (6), and denoting the commutator by square brackets, the following is applied.

Proposition 1. If $\rho \in \Omega_0$ then

$$[\rho, \rho_A \otimes \mathbb{1}_B] = 0, \tag{7}$$

where $\mathbb{1}_B$ is the identity operator on $\mathcal{H}_B$. Hence, $[\rho, \rho_A \otimes \mathbb{1}_B] \neq 0$ implies that $D_{AB}(\rho) > 0$. The converse, however, is not true: there are some states with positive discord that commute with their reduced ones. States of interest such as all pure maximally entangled states are an example. Let us introduce the auxiliary set C_0 of all states satisfying Eq. (7):

$$C_0 \doteq \{\rho \in \mathcal{B}(H) \;\text{ such that }\; [\rho, \rho_A \otimes \mathbb{1}_B] = 0\}. \tag{8}$$

One has that $\Omega_0 \subset C_0$. We prove next that C_0 has measure zero and is nowhere dense, thereby implying the same properties for Ω_0 [19].

A. C_0 has measure zero

The key observation here is that Eq. (7) imposes a nontrivial constraint on ρ that confines it to a lower-dimensional subspace of $\mathcal{B}(H)$. This already suggests that the volume of C_0 in $\mathcal{B}(H)$ is zero, a proof of which we sketch next (a detailed proof is given in Appendix A). Consider a generic state $\rho \in \mathcal{B}(H)$ expressed, for example, in an orthogonal basis given by the tensor product between the traceless generators of the group SU(d_A) and those of SU(d_B). In this basis, the calculation of commutator (7) is straightforward and gives a set of implicit constraints on a state to belong to C_0. These constraints can be inverted to obtain an explicit differentiable parametrization of the set C_0 which uses strictly fewer independent real parameters than the ones needed to parametrize $\mathcal{B}(H)$. Since a differentiable parametrization of a set measure zero is also measure zero, C_0 has measure zero in $\mathcal{B}(H)$.

B. C_0 is nowhere dense

The set C_0, apart from being of zero measure, is also nowhere dense, two concepts that are *a priori* independent. A set $\mathcal{A}$ is called nowhere dense (in $\mathcal{X}$) if there is no neighborhood in $\mathcal{X}$ on which $\mathcal{A}$ is dense. Equivalently, $\mathcal{A}$ is said to be nowhere dense if its closure has an empty interior. In particular, this implies that within an arbitrarily small vicinity of any state that belongs to C_0 (Ω_0) there are always states out of C_0 (Ω_0). Let us now observe that C_0 is closed, which implies that the closure of C_0 is itself. This follows from the fact that the function $f(\rho) = [\rho, \rho_A \otimes \mathbb{1}_B]$ is a continuous map and the zeros of a continuous map form a closed set. Since any set whose closure has measure zero is nowhere dense, it suffices to conclude that C_0 is nowhere dense (see Appendix B). Being both closed and nowhere dense implies in particular that a generic perturbation of a state inside the set will drive it not just to a state outside but also to an entire region (open set) outside of it.

IV. GEOMETRY OF THE SET OF ZERO QUANTUM DISCORD

First, let us observe that Ω_0 is not a convex set. In fact, an arbitrary convex mixture between two states ρ_1 and ρ_2 that are block diagonal in incompatible local bases is typically not block diagonal. On the other hand, if one mixes states that are block diagonal in the same local basis, then the resulting state is necessarily block diagonal (in the same basis) and therefore belongs to Ω_0. In particular, every state of zero discord is connected to the maximally mixed state $\mathbb{1}/d$, as the latter is trivially block diagonal in any local basis. This already shows us that the set Ω_0 is connected. From a geometrical viewpoint, this means that when moving rectilinearly from every state in Ω_0 toward $\mathbb{1}/d$, only states in Ω_0 are encountered. Accordingly, the segment from every state out of Ω_0 to $\mathbb{1}/d$ is exclusively composed of states out of Ω_0 (the two-qubit Werner state is an instructive and simple example of this [5]). All in all, this leaves us with some sort of starlike hyperstructure for Ω_0 (with $\mathbb{1}/d$ at the center), represented in Fig. 1.

Some details of the set have been sacrificed in the figure for the sake of clarity. For example, the tips of the star rays are pure separable states, always at the border of $\mathcal{B}(H)$, even though some of them are shown in its interior. Also, all the rays are connected not only through $\mathbb{1}/d$, but also by (nonconvex) continuous trajectories induced by local unitaries. Nevertheless, as we already know, this lies fully in a lower-dimensional subspace without volume and is not represented in Fig. 1. The picture should thus not be taken as rigorous but just as a pictorial representation to illustrate the main features of Ω_0.

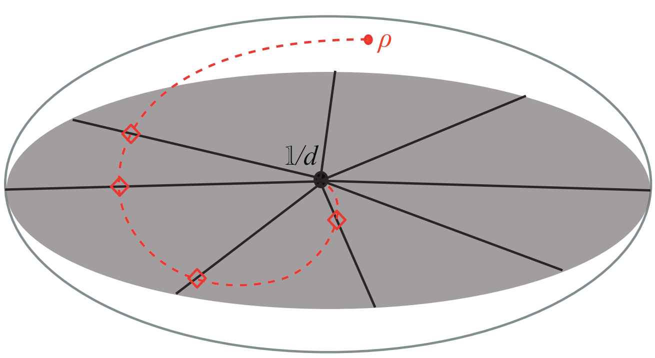

FIG. 1. (Color online) Schematic two-dimensional representation of the set Ω_0 of states with zero discord (black lines). The set of all possible states $\mathcal{B}(H)$ (enclosing ellipse) contains the set of separable ones, depicted in gray, with the maximally mixed state $\mathbb{1}/d$ in its center. All block-diagonal states, including pure separable states at the border of $\mathcal{B}(H)$, compose Ω_0 and can be connected to $\mathbb{1}/d$ through states in Ω_0. Arbitrary states in Ω_0, however, cannot in general be combined to form a state in Ω_0. The whole of Ω_0 lives in a lower-dimensional subspace of $\mathcal{B}(H)$. The dynamical trajectory of an arbitrary state ρ caused by a Markovian bath is represented by the dashed (red) line. In this example the trajectory leads toward $\mathbb{1}/d$. During its evolution, the evolved state can only cross Ω_0 a finite number of times, and permanent vanishing of discord cannot happen before the infinite-time limit, at the stationary state.

The geometrical notion of moving rectilinearly toward $\mathbb{1}/d$ corresponds to the dynamical process of global depolarization (global white noise). From the above considerations, it is clear now that global depolarization can never induce finite-time vanishing of positive discord. It only induces the disappearance of discord in the asymptotic infinite-time limit, when $\mathbb{1}/d$ is actually reached. In fact, given the singular geometry of Ω_0 suggested here, it seems highly unlikely that a noisy dynamical evolution inducing a smooth trajectory in $\mathcal{B}(H)$ is able to take a state outside of Ω_0 into its interior and to keep it there permanently. This notion is discussed next.

V. OPEN-SYSTEM DYNAMICS OF DISCORD

We now show for any state $\rho \notin C_0$, that is, for almost all (positive-discord) states, that the interaction with any (not necessarily local) Markovian bath can never lead to a sudden permanent vanishing of discord. Unless the asymptotic infinite-time limit is reached, a Markovian map can take ρ through the singular set C_0 (and therefore also through Ω_0) at most a finite number of times, equal to $\tilde{d}_\lambda(\tilde{d}_\lambda - 1)/2 - 1$, where $\tilde{d}_\lambda$ is the number of different eigenvalues of the map.

Consider the system interacts with a generic (not necessarily local) bath during an arbitrary time τ. We describe the evolution of the system with a completely positive, trace-preserving map $\Lambda_\tau : \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$. In what follows we use the notation of Ref. [20]. The map Λ_τ can be written in its (diagonal) spectral decomposition, $\Lambda_\tau = \sum \lambda_i |\mu_i)(\nu_i|$, where λ_i, $|\nu_i)$, and $|\mu_i)$ are, respectively, the eigenvalues and left and right eigenoperators of the map, $\Lambda_\tau|\mu_i) \equiv \lambda_i|\mu_i)$ and $(\nu_i|\Lambda_\tau \equiv \lambda_i(\nu_i|$. For a general map, $|\nu_i)$ and $|\mu_i)$ span two nonorthogonal complete bases of $\mathcal{B}(H)$ and satisfy the conditions $(\nu_i|\mu_j) \equiv \delta_{ij}$ and $(\nu_i|\nu_j) \neq \delta_{ij} \neq (\mu_i|\mu_j)$, where δ_{ij} is the Kronecker delta and where $(X|Y)$ is nothing but the Hilbert-Schmidt inner product: $(X|Y) \equiv \mathrm{Tr}[X^\dagger.Y]$. In addition, these maps are always contractive; that is, $|\lambda_i| \leqslant 1\, \forall\, i$ and $|\lambda_i| = 1$ for at least one i. For the specific case of normal maps (those commuting with their adjoints), the left and right eigenoperators coincide and the basis they span becomes orthonormal. Also, since we are interested in maps that describe some decoherence process, we assume that $|\lambda_i| < 1$ for at least one i, because the case $|\lambda_i| = 1\ \forall\ i$ corresponds to the case of unitary evolution of the composite system.

We now consider all maps Λ_t that can be expressed as the successive composition of n times Λ_τ: $\Lambda_t = \sum \lambda_i^n |\mu_i)(\nu_i|$, with $t = n\tau$. All Markovian maps fall into this category. From a strictly mathematical viewpoint, it is possible that some of the eigenvalues of Λ_τ are null. Nevertheless, since the initial condition $\Lambda_{\tau=0} \equiv \mathbb{1}$ must be satisfied, because of continuity there is always a sufficiently small τ for which all eigenvalues are non-null. With this physically motivated observation in mind, we restrict our discussion to all maps such that $\lambda_i \neq 0\ \forall\ i$. The initial state ρ is expanded in the basis $\{|\mu_i)\}$ as $\rho = \sum \rho_i|\mu_i)$, with $\rho_i \equiv (\nu_i|\rho)$, and after time t it evolves to $\rho_t \equiv \Lambda_t(\rho) = \sum \rho_i \lambda_i^n |\mu_i)$. Now we can show that for a generic (positive-discord) initial state ρ such that $[\rho, \rho_A \otimes \mathbb{1}_B] \neq 0$, there exists no $t_s \in [0,\infty)$ such that $\rho_t \in C_0$ (and in particular such that $\rho_t \in \Omega_0$) for all $t > t_s$. We do it by *reductio ad absurdum*. Assume then that the opposite is true. This means that there exists a state ρ_t that satisfies $[\rho_t, \rho_{A_t} \otimes$

$\mathbb{1}_B] = 0$, with $\rho_{A_t} \equiv \mathrm{Tr}_B[\rho_t]$, for all $t > t_s$. This, however, defines an *infinite* set of linearly independent equations (as many as $n > n_s \equiv t_s/\tau$), which can never be satisfied. An analogous contradiction is also obtained if it is assumed that ρ_t satisfies $[\rho_t, \rho_{A_t} \otimes \mathbb{1}_B] = 0$ only during the finite-time interval $(t_s, t_s + \Delta t]$, with any $\Delta t > 0$. Furthermore, we prove that ρ_t can enter C_0 (and in consequence also Ω_0) a maximum of $\tilde{d}_\lambda(\tilde{d}_\lambda - 1)/2 - 1$ times, where $\tilde{d}_\lambda$ is the number of different eigenvalues λ_i (see Appendix C).

VI. DISCUSSION

We have shown here that a random quantum state possesses in general strictly positive discord and that a generic arbitrarily small perturbation of a state with zero discord will generate discord. These results imply that classical-only correlated quantum states are extremely rare. An interesting analogy can now be established: almost all states possess discord just as almost all *pure* states possess entanglement. This means that the mere presence of positive quantum discord lacks *per se* informative content (for example, as a computational resource), because it is a common feature of almost all quantum states. Of course, this by no means excludes the possibility that a more quantitative characterization of the discord gives valuable assessment of a state's usefulness for some task. In addition, in a future perspective, our results call for a better understanding of the conflict between the standard approach to open quantum systems and complete positivity of maps.

A final comment about the experimental implications is in place. We have shown that states with zero discord are (densely) surrounded by states with positive discord. As a consequence, ruling out the presence of quantum discord is, strictly speaking, experimentally impossible, unless further assumptions are made. Clearly, if further suppositions are made, then quantum discord can trustworthily be taken to be null. The clearest and most trivial example of this is given by two quantum systems independently prepared in distant laboratories. In such a situation, the most natural and reasonable assumption is that the systems do not share any correlations at all, and therefore their joint state is a product state, which has trivially no discord. However, we stress that any finite-precision measurement on a fully unknown state is compatible with a positive amount of discord. This is in striking contrast to what happens for entanglement, whose presence can instead be strictly ruled out in experiments.

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APPENDIX A: THE SET C_0 HAS MEASURE ZERO

We express $\rho \in \mathcal{B}(H)$ in the basis given by the traceless, orthogonal generators $\gamma_i^A \otimes \gamma_j^B$ of the product group $\mathrm{SU}(d_A) \otimes \mathrm{SU}(d_B)$ (we use the same notation as Eq. (5.2) of Ref. [21]):

$$\rho = \frac{1}{d_A d_B}\left[\mathbb{1}_{AB} + \sum_{i=1}^{d_A-1} \tau_i^A \gamma_i^A \otimes \mathbb{1}_B + \sum_{j=1}^{d_B-1} \tau_i^B \mathbb{1}_A \otimes \gamma_i^B + \sum_{h=1}^{d_A-1}\sum_{k=1}^{d_B-1} \beta_{hk}\gamma_h^A \otimes \gamma_k^B\right]. \tag{A1}$$

The expression above maps the Hilbert space $\mathcal{H}$ to $\mathbb{R}^{d_0}$ ($d_0 = d_A^2 d_B^2 - 1$) via the parameters τ_i^A, τ_i^B, and β_{hk}. The partial trace of ρ over $\mathcal{H}_B$ gives

$$\rho_A = \frac{1}{d_A}\left[\mathbb{1}_A + \sum_{i=1}^{d_A-1} \tau_i^A \gamma_i^A\right]. \tag{A2}$$

The generators form a closed set with respect to commutation: $[\gamma_i^A, \gamma_j^A] = 2i \sum_k f_{ijk}\gamma_k^A$, where f_{ijk} is a rank 3 antisymmetric tensor called the structure constant of the group $\mathrm{SU}(d_A)$ (see, for example, Ref. [22]). The calculation of commutator (7) is straightforward in this representation:

$$[\rho, \rho_A \otimes \mathbb{1}_B] = 2i \sum_{h,l,m=1}^{d_A-1}\sum_{k=1}^{d_B-1} \beta_{hk}\tau_l^A f_{hlm}\gamma_m^A \otimes \gamma_k^B. \tag{A3}$$

Since matrices γ_m^A, γ_k^B are orthogonal, imposing $[\rho, \rho_A \otimes \mathbb{1}_B] = 0$ accounts for constraining parameters β_{hk} and τ_l^A in the following way:

$$\sum_{h,l=1}^{d_A-1} \beta_{hk}\tau_l^A f_{hlm} = 0 \tag{A4}$$

for all k and m. These equations can be inverted. In particular, even the inversion of only one of them is sufficient for our purposes. By doing this, one obtains an explicit differentiable parametrization of the set C_0 with strictly fewer real independent parameters than d_0, that is, the ones required to parametrize $\mathcal{B}(H)$. Thus, C_0 has Lebesgue measure zero in $\mathcal{B}(H)$. ■

APPENDIX B: THE SET C_0 IS NOWHERE DENSE

Let us first show that C_0 is closed. Since the partial trace is a contractive map—meaning that the (trace) distance between any two operators is larger than, or equal to, that between the operators resulting from the application of the map—the map $f : \mathcal{B}(H) \longrightarrow f(\mathcal{B}(H))$ is continuous. The operator zero (the operator whose matrix representation is composed only of zero elements) in turn forms a closed subset of the set image of f, $f(\mathcal{B}(H))$. By the topological definition of a continuous map, the preimage of a closed set is also closed. Thus, C_0 is closed, being the preimage of the closed set "operator zero."

To complete the proof, recall that the closure of a closed set is, by definition, the set itself. Then a closed set of measure zero is nowhere dense because the fact that it is measure zero implies that it has no interior point. We show the latter with our example of interest, C_0: Suppose that there exists an interior point in the closed, zero-measure set C_0. By the definition of an interior point, this would mean that there exists a state $\rho \in C_0$ surrounded by an open ball of positive radius entirely contained in C_0. (The metric used to define the ball is not relevant, since we are considering finite dimensions.) Nevertheless, since open balls have positive Lebesgue measure in $\mathbb{R}^n$ for any n, this would contradict the fact that C_0 has measure zero. Then, there exists no interior point of C_0, implying that the set is nowhere dense. ∎

APPENDIX C: NO FINITE-TIME ACCORDING

For any initial state ρ such that $[\rho,\rho_A \otimes \mathbb{1}_B] \neq 0$, we prove here that there exists no finite time t_s after which the evolved state $\Lambda_t(\rho)$ belongs to C_0, either for all $t \geqslant t_s$ or for $t \in (t_s,t_s+\Delta t]$, with any $\Delta t > 0$. Following the notation from the text above, we write the condition $[\rho_t,\rho_{A_t} \otimes \mathbb{1}_B] = 0\ \forall\, t > t_s$ explicitly as a system of equations for the purpose of seeing their linear independence:

$$\sum_{i,j=1}^{d^2} \rho_i \rho_j (\lambda_i \lambda_j)^n [|\mu_i),|\mu_{A\,j}) \otimes \mathbb{1}_B] = 0\ \forall\, n > n_s \in \mathbb{N}, \tag{C1}$$

where $|\mu_{A\,j}) = \mathrm{Tr}_B[|\mu_j)]$ and the expansion $\rho = \sum \rho_i |\mu_i)$ has been used. Let us relabel the pair of indices (i,j) using a single index $k = 1,\ldots,d^2 \times d^2$ and define $R_k = \rho_i\rho_j$, $L_k = \lambda_i\lambda_j$, and $D_k = [|\mu_i),|\mu_{A\,j}) \otimes \mathbb{1}_B]$. Then Eqs. (C1) above can be recast in the form of linear equations in R_k's:

$$\begin{aligned} &\sum_k R_k L_k^{n_s+1} D_k = 0,\\ &\qquad\vdots \\ &\sum_k R_k L_k^{n_s+m} D_k = 0, \end{aligned} \tag{C2}$$

for any $m \in \mathbb{N}$, which have to be satisfied conditioned on the initial condition $[\rho,\rho_A \otimes \mathbb{1}_B] \neq 0$,

$$\sum_k R_k D_k \neq 0. \tag{C3}$$

We can already intuit that operator equations (C2) compose a set of m linearly independent equations from the fact that coefficients L_k (with $0 < |L_k| \leqslant 1$) all appear in a geometric progression. We demonstrate this formally by writing Eqs. (C2) and (C3) in a matrix representation and, thus, recasting them as a set of linearly independent equations for complex numbers.

In Eqs. (C2) and (C3), we keep only the $\bar{d}$ terms such that D_k and R_k are both different from zero. Thus,

$$\sum_{k=1}^{\bar{d}} R_k L_k^n D_k = 0, \qquad \sum_{k=1}^{\bar{d}} R_k D_k \neq 0, \tag{C4}$$

for $n_s < n \leqslant n_s + m$. Let us now express the operators D_k in an arbitrary matrix representation and focus on their matrix elements $[D_k]_{p,q}$. The initial condition, Eq. (C4), implies that there exists at least a couple $(p_{\bar{0}},q_{\bar{0}})$ such that $\sum_{k=1}^{d'} R_k [D_k]_{p_{\bar{0}},q_{\bar{0}}} \neq 0$, for some $d' \leqslant \bar{d}$. Focusing on such a couple $(p_{\bar{0}},q_{\bar{0}})$, and denoting $d_k \equiv [D_k]_{p_{\bar{0}},q_{\bar{0}}} \neq 0$, we have that Eqs. (C4) reduce to ordinary equations with complex coefficients d_k and L_k:

$$\sum_{k=1}^{d'} R_k L_k^n d_k = 0, \tag{C5}$$

$$\sum_{k=1}^{d'} R_k d_k \neq 0. \tag{C6}$$

We now change variables to the non-null coefficients $r_k \doteq R_k d_k$:

$$\sum_{k=1}^{d'} r_k L_k^n = 0, \tag{C7}$$

$$\sum_{k=1}^{d'} r_k \neq 0, \tag{C8}$$

for all $n_s < n \leqslant n_s + m$. If the coefficients L_k are degenerate, one can define another set of variables by grouping together all the r_k's that correspond to the same degenerate L_k. Namely, we introduce $s_h = \sum r_k$, where the sum extends to the r_k's corresponding to the same L_h. Denoting by $\tilde{d}$ the number of different L_h's we have that Eqs. (C7) and (C8) are equivalent to,

$$\begin{aligned} &\sum_{h=1}^{\tilde{d}} s_h L_h^{n_s+1} = 0\\ &\qquad\vdots \\ &\sum_{h=1}^{\tilde{d}} s_h L_h^{n_s+m} = 0, \end{aligned} \tag{C9}$$

$$\sum_{h=1}^{\tilde{d}} s_h \neq 0, \tag{C10}$$

with $L_h \neq L_{h'}$ if $h \neq h'$ and $n_s < n \leqslant n_s + m$. Equations (C9) are linear in s_h with complex, non-null, nondegenerate coefficients in geometric progression. From the properties of eigenvalues λ_i mentioned in the text, we see that the coefficients L_h necessarily satisfy $|L_h| \leqslant 1\ \ \forall\ h$, with $|L_h| = 1$ for some h and $|L_h| < 1$ for all other h's. Thus, Eqs. (C9) yield a homogenous system of m independent linear equations for $\tilde{d}$ unknowns s_h. For $m < \tilde{d}$ there are m nontrivial solutions that are also compatible with Eq. (C10). For $m \geqslant \tilde{d}$ though, Eqs. (C9) become a uniquely determined homogenous system, whose unique solution is the trivial one $s_h = 0$ for all $h = 1,\ldots,\tilde{d}$. This solution, however, is not acceptable, since it contradicts the initial condition of Eq. (C10). ∎

As said, trajectories that cross C_0 at most $\tilde{d} - 1$ times might in principle give acceptable solutions to Eqs. (C9). If there are $\tilde{d}_\lambda$ different λ_i eigenvalues, it is straightforward to count that there are $\tilde{d} = \tilde{d}_\lambda(\tilde{d}_\lambda - 1)/2$ different L_h's. As an example, we can now easily calculate an upper bound to the number of times C_0 can be crossed by usual maps, such as a local depolarizing or dephasing channels (three different eigenvalues, two times), or the global depolarizing channel (two different eigenvalues, never).

On the other hand, also from Eqs. (C9), one can see that $\rho_t \in C_0$, for $t \to \infty$ if, and only if, the steady state of the map is itself a state inside C_0. This is clear when one considers the limit $n_s \to \infty$ in Eqs. (C9), where all powers of L_h, from $L_h^{n_s+1}$ to $L_h^{n_s+m}$, are exactly equal to zero for $|L_h| < 1$, and equal to 1 for the single L_h equal to 1. Equations (C9) simply converge to the single condition $s_H = 0$, where H is the one h for which $L_H = 1$. This condition is in turn not in conflict with Eq. (C10) and therefore provides an acceptable solution. The coefficient s_H is associated with the projection of the initial state onto the map's steady state. So it simply gives the trivial fact that the final state will end up in C_0 if and only if the steady state of the map is itself in C_0.

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[19] Notice that these properties automatically extend also to "two-side measures" of quantumness [8] based on the disturbance caused by local measurements on both subsystems [7,12]. This is because discord yields a general lower bound to such measures [8], because the disturbance when measuring both particles is never less than that caused when only one particle is subject to measurements.

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