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Almost analytic forms with respect to a quadratic endomorphism and their cohomology

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Abstract: The goal of this paper is to consider the notion of almost analytic form in a unifying setting for both almost complex and almost paracomplex geometries. We use a global formalism, which yields, in addition to generalizations of the main results of the previously known almost complex case, a relationship with the Frölicher–Nijenhuis theory. A cohomology of almost analytic forms is also introduced and studied as well as deformations of almost analytic forms with pairs of almost analytic functions.

Key words: Quadratic endomorphism, almost F-analytic form, F-symmetric form, almost (para)complex structure, cohomology

1. Introduction

The notion of almost analytic form was introduced a long time ago in the almost complex geometry and hence it was treated in local coordinates, especially by Japanese geometers [15, 16, 17, 18]. A global approach appeared in [14], unfortunately only in Romanian. Some of these global techniques were used in [9] and [13]; for example, in the former paper a differential is introduced in the algebra of pairs of almost analytic forms and a corresponding Poincaré type lemma is proved.

The present work aims to consider almost analytic forms in a unifying setting, which adds the almost paracomplex geometry. This type of even dimensional geometry is now in the mainstream of research as the surveys [1] and [4] and their several citations prove. In this way, we reveal the common parts of these geometries with respect to differential forms and present the techniques of [14] to a larger audience. An important feature of the global approach is that it yields a relationship with the Frölicher-Nijenhuis theory, widely used now for several important topics. Namely, we prove that for an almost F-analytic form its closeness with respect to the Frölicher–Nijenhuis derivative d_F is characterized by the usual (i.e. exterior derivative) closeness.

The content of the paper is as follows. In the first subsection of Section 2 we consider only 1-forms in order to offer a detailed picture of the techniques used herein. In the next subsection we consider the general case of r-forms with r less than or equal to n = half of the dimension of the underlying manifold. A d_F-cohomology of almost analytic forms is introduced and studied and also some deformations of almost analytic forms with pairs of almost analytic functions are considered. In Section 3 we restrict ourselves to the Hermitian and para-Norden framework and reobtain the characterization of almost analyticity for n-forms in terms of

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harmonicity. Considering again the case of 1-forms, a local computation in the case of integrability of given endomorphism F gives an usual characterization of coefficients in terms of (para)Cauchy–Riemann equations.

2. Almost analytic forms with respect to a quadratic endomorphism

2.1. Almost analytic 1-forms

Fix a triple (M, F, ω) with M a smooth m-dimensional manifold, F a tensor field of $(1,1)$-type on M, and ω a differentiable 1-form, i.e. $\omega \in \Omega^1(M)$.

Definition 2.1 *i) F is a quadratic endomorphism if there exists $\varepsilon \in \mathbb{R}^*$ such that:*

$$F^2 = \varepsilon I. \tag{2.1}$$

ii) The F-conjugate of ω is the 1-form:

$$\overline{\omega} = \omega_F := \omega \circ F^{-1} = \frac{1}{\varepsilon}\omega \circ F. \tag{2.2}$$

It follows that:

$$\overline{\overline{\omega}} = \frac{1}{\varepsilon}\overline{\omega} \circ F = \frac{1}{\varepsilon}\omega. \tag{2.3}$$

To the pair (F, ω) we associate a 2-form defined by:

$$\Omega_{F,\omega}(X,Y) := d\omega(FX, Y) - \varepsilon d\overline{\omega}(X,Y), \tag{2.4}$$

which yields the main notion of this subsection:

Definition 2.2 *The 1-form ω is called almost F-analytic if $\Omega_{F,\omega} = 0$. Let $\Omega^1(M, F)$ be the set of almost F-analytic 1-forms.*

In the following we use the identity:

$$2d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \tag{2.5}$$

A first result shows that this property is invariant under F-conjugation:

Proposition 2.1 *The 1-form ω is almost F-analytic if and only if its F-conjugate $\overline{\omega}$ is almost F-analytic. If ω is almost F-analytic then ω is closed if and only if $\overline{\omega}$ is closed.*

Proof Using (2.1), (2.3), and (2.4) we get:

$$\Omega_{F,\overline{\omega}}(X,Y) = -\frac{1}{\varepsilon}\Omega_{F,\omega}(FX,Y)\,,\ \Omega_{F,\omega}(X,Y) = -\Omega_{F,\overline{\omega}}(FX,Y) \tag{2.6}$$

and the conclusion follows directly from (2.6). □

Recall now the Nijenhuis tensor field of F:

$$N_F(X,Y) := [FX, FY] - F[FX,Y] - F[X,FY] + F^2[X,Y], \tag{2.7}$$

which for our case (2.1) becomes $N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + \varepsilon[X,Y]$. We have the following skew-symmetries:

$$N_F(FX,Y) = -FN_F(X,Y) = N_F(X,FY), \quad N_F(FX,FY) = \varepsilon N_F(X,Y) \tag{2.8}$$

which yields a second property of almost F-analytic forms:

Proposition 2.2 *If ω is almost F-analytic then:*

$$\omega \circ N_F = \overline{\omega} \circ N_F = 0. \tag{2.9}$$

Proof Let ω be almost F-analytic. Using (2.5), $\omega \circ F = \varepsilon\overline{\omega}$, and $\overline{\omega} \circ F = \omega$, from $\Omega_{F,\omega}(X,Y) = d\omega(FX,Y) - \varepsilon d\overline{\omega}(X,Y) = 0$ we easily obtain:

$$\omega(F[X,Y]) = \varepsilon X\overline{\omega}(Y) - FX\omega(Y) + \omega([FX,Y]). \tag{2.10}$$

Putting $X \mapsto FX$ and $Y \mapsto FY$ in (2.10), by direct calculus we obtain:

$$\begin{aligned}(\omega \circ N_F)(X,Y) &= \omega([FX,FY]) - \omega(F[FX,Y]) - \omega(F[X,FY]) + \varepsilon\omega([X,Y]) \\ &= \omega([FX,FY]) - \varepsilon FX(\overline{\omega}(Y)) + \varepsilon X\omega(Y) - \varepsilon\omega([X,Y]) \\ &\quad -\varepsilon X\omega(Y) + \varepsilon FX\overline{\omega}(Y) - \omega([FX,FY]) + \varepsilon\omega([X,Y]) = 0.\end{aligned}$$

By Proposition 2.1 $\overline{\omega}$ is also almost F-analytic and the relation $(\overline{\omega} \circ N_F)(X,Y) = 0$ follows in a similar manner starting from $\Omega_{F,\overline{\omega}}(X,Y) = d\overline{\omega}(FX,Y) - d\omega(X,Y) = 0$. □

Another tool in our study is provided by the Obata operators associated to F, namely the maps $O_F, O_F^* : \Omega^2(M) \to \Omega^2(M)$:

$$\begin{cases} O_F(\rho)(X,Y) := \frac{1}{2}[\rho(X,Y) - \rho(FX,FY)] \\ O_F^*(\rho)(X,Y) := \frac{1}{2}[\rho(X,Y) + \rho(FX,FY)], \end{cases} \tag{2.11}$$

which give a classification of 2-forms with respect to F:

Definition 2.3 *The 2-form ρ is called F-pure if $O_F^*(\rho) = 0$ and respectively F-hybrid if $O_F(\rho) = 0$.*

Proposition 2.3 *i) If F is an almost complex structure ($\varepsilon = -1$) and ω is almost F-analytic form then the 2-forms $d\omega$, $d\overline{\omega}$ are F-pure.*

ii) If F is an almost product structure ($\varepsilon = 1$) and ω is almost F-analytic form then the 2-forms $d\omega$, $d\overline{\omega}$ are F-hybrid.

Proof i) Let $\varepsilon = -1$. From the characterization of almost F-analyticity, setting $X \mapsto FX$ in (2.10) we have:

$$X(\omega(Y)) + FX(\omega(FY)) = \omega([X,Y]) + \omega(F[FX,Y]), \tag{2.12}$$

and now $X \to Y$ in (2.12):

$$Y(\omega(X)) + FY(\omega(FX)) = -\omega([X,Y]) - \omega(F[X,FY]). \tag{2.13}$$

From (2.13) minus (2.12) we get:

$$2d\omega + \omega([X,Y]) + 2d\overline{\omega}(FX,FY) + \omega([FX,FY]) = 2\omega([X,Y]) + \omega\circ F([FX,Y]+[X,FY]),$$

which means:

$$4O^*_F(d\omega) = -\omega\circ N_F = 0.$$

By analogy:

$$4O^*_F(d\overline{\omega}) = -\overline{\omega}\circ N_F = 0.$$

ii) Let $\varepsilon = 1$. Again, with $X \to FX$ in relation (2.10) we have:

$$X(\omega(Y)) - FX(\omega(FY)) = \omega([X,Y]) - \omega(F[FX,Y]) \tag{2.14}$$

and $X \leftrightarrow Y$ in this equality gives:

$$Y(\omega(X)) - FY(\omega(FX)) = -\omega([X,Y]) + \omega(F[X,FY]). \tag{2.15}$$

With (2.14) minus (2.15) we obtain:

$$2d\omega(X,Y) + \omega([X,Y]) - 2d\overline{\omega}([FX,FY]) - \omega([FX,FY]) = 2\omega([X,Y]) - \omega\circ F([FX,Y]+[X,FY]),$$

which means: $4O_F(d\omega) = \omega\circ N_F = 0$. Also: $4O_F(d\overline{\omega}) = \overline{\omega}\circ N_F = 0$ and the assertion is proved. □

An important consequence of this result is the following:

Corollary 2.1 *If $\varepsilon \in \{-1,+1\}$ then definition (2.4) and hence the definition of almost F-analyticity do not depend on the place of F.*

Proof From Proposition 2.3 we have that the almost F-analyticity implies:

$$d\omega(X,Y) = \varepsilon d\omega(FX,FY), \tag{2.16}$$

and then the right-hand side of (2.4) is:

$$d\omega(FX,Y) - \varepsilon d\overline{\omega}(X,Y) = \varepsilon d\omega(F^2X,FY) - \varepsilon d\overline{\omega}(X,Y) = \varepsilon^2 d\omega(X,FY) - \varepsilon d\overline{\omega}(X,Y)$$

and since $\varepsilon^2 = 1$ we get the conclusion. □

We finish this subsection with a relationship of this formalism with the Frölicher–Nijenhuis theory. Recall that given a tensor field F of $(1,1)$-type it defines the following:

i) an interior product i_F; for an r-form ω we have that $i_F\omega$ is again an r-form given by:

$$i_F\omega(X_1,\dots,X_r) := \sum_{i=1}^{r}\omega(X_1\dots,FX_i,\dots,X_r),\ r\geq 1 \text{ and } i_Ff=0,\ \forall f\in C^\infty(M); \tag{2.17}$$

ii) an exterior F-derivative d_F with:

$$d_F := i_F\circ d - d\circ i_F. \tag{2.18}$$

Proposition 2.4 *If $\varepsilon = \pm 1$ and ω is almost F-analytic then the exterior F-derivatives of ω and $\overline{\omega}$ are:*

$$d_F\omega = \frac{1}{2} i_F \circ d\omega = \varepsilon d\overline{\omega}, \quad d_F\overline{\omega} = d\omega. \tag{2.19}$$

Proof For $r = 1$ we have:

$$i_F\omega = \varepsilon\overline{\omega} \tag{2.20}$$

and then:

$$\begin{aligned}(d_F\omega)(X,Y) &= i_F(d\omega)(X,Y) - d(\varepsilon\overline{\omega})(X,Y) \\ &= d\omega(FX,Y) + d\omega(X,FY) - \varepsilon d\overline{\omega}(X,Y) = \Omega_{F,\omega}(X,Y) + d\omega(X,FY),\end{aligned}$$

which means that $d_F\omega(\cdot,\cdot) = d\omega(\cdot, F\cdot)$. We apply the previous Corollary 2.1 to get the first part of (2.19). The second part of the required formula follows by duality. □

Similarly to [6, 10, 16], a smooth function f on M is called *almost F-analytic* if there exists a smooth function g on M such that:

$$df \circ F = dg, \tag{2.21}$$

and in this case g is called *the corresponding function* of f. In this case g is also almost F-analytic with corresponding function εf. Let us denote by $C^\infty(M,F)$ the set of all almost F-analytic functions on M. If $f \in C^\infty(M,F)$, then by (2.20) we have:

$$d_F f = i_F \circ df = \varepsilon\overline{df} = df \circ F = dg. \tag{2.22}$$

Proposition 2.5 *If $f \in C^\infty(M,F)$ then df and $d_F f$ are both almost F-analytic.*

Proof Let $f \in C^\infty(M,F)$. Then:

$$\Omega_{F,df}(X,Y) = (d(df))(FX,Y) - \varepsilon(d(\overline{df}))(X,Y) = -(d(dg))(X,Y) = 0,$$

which says that df is almost F-analytic. The second assertion follows by setting $X \mapsto FX$ in the above relation. □

2.2. Almost F-analytic r-forms and d_F-cohomology

In this subsection we give a generalization of previous results to r-forms for $r \geq 2$ with ε restricted to $\{-1,+1\}$ and we study the d_F-cohomology of almost analytic r-forms.

Firstly, inspired by Proposition 2.3, we introduce a class of r-forms adapted to F:

Definition 2.4 *The r-form ω is called F-symmetric if for all vector fields $X_1,\dots,X_r$:*

$$\omega(FX_1,\dots,X_r) = \omega(X_1,\dots,FX_i,\dots,X_r), \quad 2 \leq i \leq r. \tag{2.23}$$

Example 2.1 *i) If $\theta \in \Omega^1(M,F)$ then the 2-forms $\omega = d\theta$ and $\overline{\omega} = d\overline{\theta}$ are F-symmetric. Indeed, equation (2.16) means:*

$$d\theta(X,Y) = \varepsilon d\theta(FX,FY), \quad d\overline{\theta}(X,Y) = \varepsilon d\overline{\theta}(FX,FY)$$

and with $X \to FX$ we get the conclusion.

ii) More generally than i) if $\varepsilon = +1$ then a F-hybrid 2-form is F-symmetric and for $\varepsilon = -1$ an F-pure 2-form is F-symmetric. □

Secondly, we associate a conjugate form and an $(r+1)$-form:

Definition 2.5 *If $\omega \in \Omega^r(M)$ is F-symmetric then its F-conjugate is $\overline{\omega} = \omega_F \in \Omega^r(M)$ given by:*

$$\overline{\omega}(X_1, \dots, X_r) := \frac{1}{\varepsilon}\omega(FX_1, \dots, X_r). \tag{2.24}$$

We associate $\Omega_{F,\omega} \in \Omega^{r+1}(M)$ given by:

$$\Omega_{F,\omega}(X_1, \dots, X_{r+1}) := d\omega(FX_1, \dots, X_{r+1}) - \varepsilon d\overline{\omega}(X_1, \dots, X_{r+1}). \tag{2.25}$$

Thirdly, we define the natural generalization of the previous subsection:

Definition 2.6 *The F-symmetric form $\omega \in \Omega^r(M)$ is called almost F-analytic if:*

$$\Omega_{F,\omega} = 0. \tag{2.26}$$

In order to unify the property that says when an F-symmetric r-form is almost F-analytic for both almost complex and paracomplex cases, we present:

Proposition 2.6 *An F-symmetric r-form ω ($r \geq 1$) is almost F-analytic iff*

$$\begin{aligned} &FX_1(\omega(X_2, \dots, X_{r+1})) - X_1(\omega(FX_2, \dots, X_{r+1})) = \\ &= \sum_{j=2}^{r+1}(-1)^{1+j}\omega(F[X_1, X_j] - [FX_1, X_j], X_2, \dots, \widehat{X_j}, \dots, X_{r+1}). \end{aligned} \tag{2.27}$$

Proof It follows by a direct calculation involving the definition of the exterior derivative. □

Remark 2.1 *In a more general case of $(0,r)$-tensor fields we can consider the operator $\Phi_F : \mathcal{T}^0_r(M) \to \mathcal{T}^0_{r+1}(M)$; see [18]:*

$$\begin{aligned} \Phi_F\omega(X, Y_1, \dots, Y_r) &= FX(\omega(Y_1, \dots, Y_r)) - X(\omega(FY_1, Y_2, \dots, Y_r)) \\ &+\omega((L_{Y_1}F)X, Y_2, \dots, Y_r) + \dots + \omega(Y_1, Y_2, \dots, (L_{Y_r}F)X), \end{aligned} \tag{2.28}$$

for every vector field $X, Y_1, \dots, Y_r$, where L_X denotes the Lie derivative with respect to X. Then, similarly to [5, 6, 7, 10, 12, 15, 18], the tensor field ω is called almost F-analytic if $\Phi_F\omega = 0$ and for r-forms this condition is equivalent to (2.26).

Let $\Omega^r(M, F)$ be the set of almost F-analytic r-forms.

The following result is a motivation for this notion and also a generalization of the first remark above:

Proposition 2.7 *If $\omega \in \Omega^r(M, F)$ then its differential is F-symmetric and its exterior F-differential of ω is:*

$$d_F\omega = \frac{1}{r+1}i_F \circ d\omega = \varepsilon d\overline{\omega}. \tag{2.29}$$

Proof The first part follows directly from the skew-symmetry of $d\overline{\omega}$ and the relation:

$$d\omega(FX_1,\ldots,X_{r+1})=\varepsilon d\overline{\omega}(X_1,\ldots,X_{r+1}), \tag{2.30}$$

provided by the definition. For the second part we get that $i_F\omega=\varepsilon r\overline{\omega}$ and with a similar calculus as in Proposition 2.4 we derive:

$$d_F\omega(X_1,\ldots,X_{r+1})=d\omega(FX_1,\ldots,X_{r+1}),$$

by using the first part. Equation (2.29) follows then directly. □

Proposition 2.8 *The F-symmetric r-form ω is almost F-analytic if and only if $\overline{\omega}$ is almost F-analytic. If ω is almost F-analytic then ω is closed if and only if $\overline{\omega}$ is closed, and equivalently ω and $\overline{\omega}$ are d_F-closed.*

Proof It is sufficient to prove the implication that ω is almost F-analytic $\Rightarrow$ $\overline{\omega}$ is almost F-analytic since:

$$\overline{\overline{\omega}}(X_1,\ldots,X_r)=\frac{1}{\varepsilon}\overline{\omega}(FX_1,\ldots,X_r)=\omega(F^2X_1,\ldots,X_r)=\varepsilon\omega(X_1,\ldots,X_r)=\frac{1}{\varepsilon}\omega(X_1,\ldots,X_r) \tag{2.31}$$

and remark that almost F-analyticity is invariant with respect to scalings $\omega\to\lambda\omega$.

Firstly we must prove that $\overline{\omega}$ is F-symmetric. We have:

$$\overline{\omega}(FX_1,\ldots,X_r)=\frac{1}{\varepsilon}\omega(F^2X_1,\ldots,X_r)=\omega(X_1,\ldots,X_r). \tag{2.32}$$

Also:

$$\begin{aligned}\bar{\omega}(X_1,\ldots,FX_i,\ldots,X_r)&=\frac{1}{\varepsilon}\omega(FX_1,\ldots,FX_i,\ldots,X_r)\\&=\frac{1}{\varepsilon}\omega(X_1,\ldots,F^2X_i,\ldots,X_r)=\omega(X_1,\ldots,X_r),\end{aligned}$$

which is what we claim.

Secondly, we must verify Definition 2.6. A straightforward calculation gives the generalization of (2.6):

$$\Omega_{F,\overline{\omega}}(X_1,\ldots,X_{r+1})=-\frac{1}{\varepsilon}\Omega_{F,\omega}(FX_1,\ldots,X_{r+1}) \tag{2.33}$$

and the conclusion follows. □

Proposition 2.9 *If $\omega\in\Omega^r(M,F)$ then:*

$$\omega(N_F(X_1,X_2),\ldots,X_{r+1})=\overline{\omega}(N_F(X_1,X_2),\ldots,X_{r+1})=0. \tag{2.34}$$

Proof Using the characterization of almost F-analyticity of ω from (2.27) but with $X_2\mapsto FX_2$, we have

$$\begin{aligned}&FX_1(\omega(FX_2,\ldots,X_{r+1}))-\varepsilon X_1(\omega(X_2,\ldots,X_{r+1}))=\\&=-\omega(F[X_1,FX_2]-[FX_1,FX_2],X_3,\ldots,X_{r+1})+\\&+\sum_{j=3}^{r+1}(-1)^{1+j}\omega(F[X_1,X_j]-[FX_1,X_j],FX_2,X_3,\ldots,\widehat{X_j},\ldots,X_{r+1}).\end{aligned} \tag{2.35}$$

On the other hand, $\varpi \in \Omega^r(M,F)$, too, and using again (2.27) for ϖ, we have

$$\begin{aligned} &FX_1(\omega(FX_2,\dots,X_{r+1})) - \varepsilon X_1(\omega(X_2,\dots,X_{r+1})) = \\ &= -\omega(\varepsilon[X_1,X_2] - F[FX_1,X_2],X_3,\dots,X_{r+1}) + \\ &+ \sum_{j=3}^{r+1}(-1)^{1+j}\omega(F[X_1,X_j] - [FX_1,X_j],FX_2,X_3,\dots,\widehat{X_j},\dots,X_{r+1}). \end{aligned} \tag{2.36}$$

Now, by (2.35) and (2.36), the first equality follows easily. The second equality follows in a similar manner. □

Inspired by the $\varepsilon = -1$ case, we suppose now that $m = 2n$ and for the $\varepsilon = +1$ we suppose that F is an almost paracomplex structure, i.e. the dimensions of $(+1)$-eigenspace and (-1)-eigenspaces are both equal to n. It follows for both cases of ε the existence of local basis of vector fields of type $B = \{e_1,\dots,e_n,Fe_1,\dots,Fe_n\}$, where for the case $\varepsilon = 1$ we must have $F \neq \mathrm{Id}$, and then there exist nontrivial F-symmetric r forms only for $r \leq n$. An important result for this choice of dimension is:

Proposition 2.10 *An F-symmetric n-form ω is almost F-analytic if and only if ω and ϖ are both closed.*

Proof Suppose firstly that ω is almost F-analytic. When its differential is applied on data $\{FX_1,X_1,\dots,X_n\}$ of elements of B we have $d\omega(FX_1,X_1,\dots,X_n) = \varepsilon d\varpi(X_1,X_1,\dots,X_n) = 0$ and deduce that ω (and consequently ϖ) is closed. The proof of the converse part is directly from Definition 2.25 □

We introduce now an exterior product adapted to our setting:

Definition 2.7 *The exterior F-product is the map $\wedge_F : \Omega^r(M)\times\Omega^s(M) \to \Omega^{r+s}(M)$ given by:*

$$\theta \wedge_F \omega := \theta\wedge\omega + \varepsilon\overline{\theta}\wedge\bar{\omega} \tag{2.37}$$

where $\wedge$ is the usual exterior product of M.

A long but straightforward computation in the basis B gives:

Proposition 2.11 *Let θ and ω be F-symmetric forms.*

i) The $(r+s)$-form $\theta\wedge_F\omega$ is also F-symmetric.

ii) The F-conjugate of the $(r+s)$-form above is:

$$(\theta\wedge_F\omega)_F = \theta\wedge\overline{\omega} + \overline{\theta}\wedge\omega. \tag{2.38}$$

As a consequence, if θ and ω are almost F-analytic forms then $\theta\wedge_F\omega$ is also an almost F-analytic form.

Proposition 2.12 *Let $\omega \in \Omega^r(M,F)$ and $\theta \in \Omega^s(M,F)$, $r,s \geq 0$, where $\Omega^0(M,F) = C^\infty(M,F)$. Then:*

i) $d_F\omega \in \Omega^{r+1}(M,F)$;

ii) $d_F^2\omega = 0$;

iii) $d_F(\omega\wedge_F\theta) = d_F\omega\wedge_F\theta + (-1)^r\omega\wedge_F d_F\theta$.

Proof i) If $\omega \in \Omega^r(M,F)$ then by (2.24) and (2.30) we have:

$$d\overline{\omega} = \overline{d\omega}. \tag{2.39}$$

Now, using (2.29) and (2.39), we have:

$$\Omega_{F,d_F\omega}(X_1,\ldots,X_{r+2}) = (d(\varepsilon d\overline{\omega}))(FX_1,\ldots,X_{r+2}) - \varepsilon(d(d\omega))(X_1,\ldots,X_{r+2}) = 0,$$

which says that $d_F\omega \in \Omega^{r+1}(M,F)$.
ii) Using (2.29), (2.31), and (2.39) we have:

$$d_F(d_F\omega) = d_F(\varepsilon d\overline{\omega}) = \varepsilon^2 d(\overline{d\overline{\omega}}) = \varepsilon^2 d(d\overline{\overline{\omega}}) = \varepsilon d(d\omega) = 0.$$

iii) Follows using (2.29), (2.37), and (2.38). □

We notice that $(\Omega^r(M,F),\wedge_F)$ is a graded $C^\infty(M,F)$-algebra. Also, by ii) Proposition 2.12 we have the differential complex $(\Omega^\bullet(M,F),d_F)$ and its cohomology $H^\bullet(M,F)$ is called the *d_F-cohomology of almost F-analytic forms* on M.

Another important property of the operator d_F is the following Poincaré type lemma:

Theorem 2.1 *Let $\omega \in \Omega^r(U,F)$, $r \geq 1$, where $U \subset M$ such that $d_F\omega = 0$ on U. Then there exists $\theta \in \Omega^{r-1}(U',F)$ where $U' \subset U$ such that $\omega = d_F\theta$ on U'.*

Proof Let ω as in the hypothesis. Taking into account that $d_F\omega = 0$ is equivalent with $d\overline{\omega} = 0$ and by applying the classical Poincaré lemma for the operator d it follows that there exists $\theta \in \Omega^r(U')$ where $U' \subset U$ and such that $\overline{\omega} = d\theta$ on U'. From $0 = \overline{d\overline{\omega}} = d\overline{\overline{\omega}} = \frac{1}{\varepsilon}d\omega$ it follows also by Poincaré lemma for the operator d that there exists $\theta_1 \in \Omega^r(U')$ where $U' \subset U$ and such that $\omega = d\theta_1$ on U'.
Similar arguments as in the proof of Theorem 1 from [9] show that both θ and θ_1 are almost F-analytic and $\theta = \overline{\theta_1}$. Now, the proof follows easily since $\omega = d\theta_1 = \varepsilon d\overline{\theta} = d_F\theta$. □

We notice that $\ker\{d_F : \Omega^0(U,F) \to \Omega^1(U,F)\} \cong \widetilde{\mathbb{R}}$ where $\widetilde{\mathbb{R}}$ is the sheaf of germs associated to the constant pre-sheaf $\mathbb{R}$. Also, consider $\Phi^r(M,F)$ the sheaf of germs of almost F-analytic r-forms on M and $i : \widetilde{\mathbb{R}} \to \Phi^0(M,F)$ the natural inclusion. The sheaves $\Phi^r(M,F)$ are fine and taking into account Theorem 2.1 it follows that the following sequence of sheaves:

$$0 \longrightarrow \widetilde{\mathbb{R}} \xrightarrow{i} \Phi^0(M,F) \xrightarrow{d_F} \Phi^1(M,F) \xrightarrow{d_F} \ldots \xrightarrow{d_F} \Phi^n(M,F) \xrightarrow{d_F} 0$$

is a fine resolution of $\widetilde{\mathbb{R}}$ and we denote by $H^r(M,F;\widetilde{\mathbb{R}})$ the cohomology groups of M with coefficients in the sheaf $\widetilde{\mathbb{R}}$. Thus, we obtain a de Rham theorem for the d_F-cohomology of almost F-analytic forms, namely:

Theorem 2.2 *Then d_F-cohomology groups of almost F-analytic forms on M are given by:*

i) $H^0(M,F;\widetilde{\mathbb{R}}) \cong \mathbb{R}$,

ii) $H^r(M,F;\widetilde{\mathbb{R}}) \cong H^r(M,F)$, $1 \leq r \leq n-1$,

iii) $H^n(M,F;\widetilde{\mathbb{R}}) \cong \Omega^n(M,F)/d_F(\Omega^{n-1}(M,F))$, *iv)* $H^r(M,F;\widetilde{\mathbb{R}})=0$, $n+1\leq r\leq 2n$.

We consider now a deformation of almost F-analytic forms with pairs of almost F-analytic functions:

Definition 2.8 *Fix $\omega\in\Omega^r(M)$ an almost F-analytic form and $\alpha,\beta\in C^\infty(M)$. The (α,β)-deformation of ω is the r-form:*

$$\omega_{\alpha,\beta}:=\alpha\omega+\beta\bar{\omega}. \tag{2.40}$$

Since $\omega_{\alpha,\beta}$ is an F-symmetric form it is natural to ask in what conditions regarding these functions the new r-form is also an almost F-analytic one:

Proposition 2.13 *The F-symmetric r-form $\omega_{\alpha,\beta}$ is almost F-analytic if and only if α is almost F-analytic with corresponding function β.*

Proof We have:

$$(\omega_{\alpha,\beta})_F=\alpha\overline{\omega}+\frac{\beta}{\varepsilon}\omega=(\omega_F)_{\alpha,\beta}. \tag{2.41}$$

The proof is easy to see in the case $r=1$ where the almost F-analyticity of $\omega_{\alpha,\beta}$ means:

$$d\alpha(FX)\omega(Y)+\frac{1}{\varepsilon}d\beta(FX)\omega(FY)=d\alpha(X)\omega(FY)+d\beta(X)\omega(Y) \tag{2.42}$$

for all vector fields X,Y. A detailed proof for $r\geq 2$ can be found in [3] for the case of almost (para) complex Lie algebroids. □

This results yields the introduction of the set:

$$\widetilde{C}^\infty(M,F)=\{(\alpha,\beta)\in C^\infty(M,F)\times C^\infty(M,F);\quad d\beta=d\alpha\circ F\}. \tag{2.43}$$

A straightforward computation gives that $\widetilde{C}^\infty(M,F)$ is a commutative algebra with respect to the product:

$$(\alpha_1,\beta_1)\cdot(\alpha_2,\beta_2):=(\alpha_1\alpha_2+\varepsilon\beta_1\beta_2,\alpha_1\beta_2+\alpha_2\beta_1), \tag{2.44}$$

having as a unit the pair of the constant functions $(1,0)\in\widetilde{C}^\infty(M,F)$. The inverse of the element $(\alpha,\beta)\in\widetilde{C}^\infty(M,F)$ different from $(0,0)$ is the pair $\left(\frac{\alpha}{\alpha^2-\varepsilon\beta^2},\frac{-\beta}{\alpha^2-\varepsilon\beta^2}\right)$; for the case $\varepsilon=+1$ we also exclude the cases $(\alpha,\pm\alpha)$.

Let us introduce the set of pairs of forms:

$$\widetilde{\Omega}^r(M,F)=\{(\omega,\bar{\omega});\omega\in\Omega^r(M,F)\}. \tag{2.45}$$

Proposition 2.13 says that $\widetilde{\Omega}^r(M,F)$ is a $\widetilde{C}^\infty(M,F)$-module for all $1\leq r\leq n$ and hence the set

$$\widetilde{\Omega}(M,F)=\sum_{r=1}^{n}\widetilde{\Omega}^r(M,F)$$

is a graded $\widetilde{C}^\infty(M,F)$-algebra. We consider the wedge product

$$(\omega,\overline{\omega})\widetilde{\wedge}(\theta,\overline{\theta})=(\omega\wedge_F\theta,(\omega\wedge_F\theta)_F) \tag{2.46}$$

and the operator

$$D : \widetilde{\Omega}^r(M,F) \to \widetilde{\Omega}^{r+1}(M,F), \quad D(\omega,\overline{\omega}) = (d\omega, d\overline{\omega}). \tag{2.47}$$

It follows that:

i) D is a local operator and $\mathbb{R}$-linear;

ii) for every $(\omega,\overline{\omega}) \in \widetilde{\Omega}^r(M,F)$ and $(\theta,\overline{\theta}) \in \widetilde{\Omega}^s(M,F)$ we have

$$D\left[(\omega,\overline{\omega})\widetilde{\wedge}(\theta,\overline{\theta})\right] = D(\omega,\overline{\omega})\widetilde{\wedge}(\theta,\overline{\theta}) + (-1)^r(\omega,\overline{\omega})\widetilde{\wedge}D(\theta,\overline{\theta});$$

iii) $D^2 = (0,0)$;

and an associated cohomology of the differential complex $(\widetilde{\Omega}(M,F), D)$ can be considered exactly as in [9].

3. Almost analytic forms on almost para-Norden manifolds and examples

We continue with the setting of Subsection 2.2, namely $\varepsilon = \pm 1$, but we add a Riemannian metric g to our framework, which satisfies

$$g(FX,Y) = \varepsilon g(X,FY). \tag{3.1}$$

Then:

a) for $\varepsilon = -1$ the triple (M,F,g) is an usual almost Hermitian manifold,

b) for $\varepsilon = +1$ the triple (M,F,g) is an almost para-Norden manifold; see, for instance, [11].

In order to unify these cases we get the following formula:

$$g(FX,FY) = g(X,Y), \forall X,Y \in \mathcal{X}(M). \tag{3.2}$$

The fundamental 2-form of an almost Hermitian manifold is $\omega(X,Y) := g(X,FY)$, which is not F-symmetric, since $\omega(FX,Y) = -\omega(X,FY)$, while the symmetric bilinear form $\omega(X,Y) := g(X,FY)$ associated to an almost para-Norden manifold is F-symmetric.

The characterization of almost analyticity of differential forms on almost Hermitian manifolds in terms of their harmonicity was studied in [13]. In order to unify these results for both cases presented above, in this section we extend some similar results for the case of almost para-Norden manifolds.

The metric g yields the Hodge star operator $\star$ and the orthonormal basis B of the type discussed above. Hence, similar to the almost Hermitian case, see Proposition 2.3 in [13, p. 77], a direct computation yields:

Proposition 3.1 *If the n-form ω is F-symmetric on the almost para-Norden manifold (M^{2n},F,g) then $\star\omega$ is also F-symmetric.*

The important consequence of this result is:

Proposition 3.2 *If ω is an almost F-analytic n-form on the almost para-Norden manifold (M^{2n},F,g) then $\star\omega$ is also almost F-analytic.*

We arrive now to the main result of this section, which provides a large class of almost F-analytic forms:

Proposition 3.3 *An F-symmetric n-form on the almost para-Norden manifold (M, g, F) is almost F-analytic if and only if ω and $\overline{\omega}$ are both harmonic.*

Proof It is a direct consequence of $d\omega = d(\star\omega) = 0$. □

Suppose $n = 2$ and $\varepsilon = -1$. By using the corollary 18 of [8, p. 208] it results that on a compact, oriented surface M^2 with positive Ricci (equivalently Gaussian, if M is embedded in $\mathbb{R}^3$) curvature at one point we have $\Omega^1(M, F) = 0$.

We end this section with some examples of (almost) F-analytic forms. In order to find large classes of almost F-analytic forms we suppose now that F is integrable. Then we call *F-analytic forms* the differential forms studied until now.

The integrability of F yields the local coordinates $\{x^i, y^i; 1 \leq i \leq n\}$ such that the expression of F is:

$$F\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad F\left(\frac{\partial}{\partial y^i}\right) = \varepsilon\frac{\partial}{\partial x^i}. \tag{3.3}$$

Let $\omega = a_i dx^i + b_i dy^i$ be a 1-form on M; hence, $\bar{\omega} = \varepsilon b_i dx^i + a_i dy^i$. The F-analyticity of ω means:

$$FX(\omega(Y)) - \omega([FX, Y]) = X(\omega(FY)) - \omega(F[X, Y]), \tag{3.4}$$

and the choice of X, Y in the basis $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}; 1 \leq i \leq n\}$ gives the following characterization:

Theorem 3.1 *The 1-form ω is an F-analytic form if and only if its coefficients satisfy the ε-Cauchy–Riemann equations:*

$$\frac{\partial a_j}{\partial y^i} = \frac{\partial b_j}{\partial x^i}, \frac{\partial a_j}{\partial x^i} = \varepsilon\frac{\partial b_j}{\partial y^i}. \tag{3.5}$$

Similarly, the pair of smooth functions (α, β) belongs to $\widetilde{C}^\infty(M, F)$ if and only if α and β satisfies the ε-Cauchy–Riemann equations (3.5).

A natural framework where quadratic endomorphisms are involved is provided by *ε-contact structures*, namely triples (φ, ξ, η) consisting of an endomorphism, a vector field, and a 1-form on M^{2n+1} satisfying:

$$\varphi^2 = \varepsilon(I_M - \eta \otimes \xi),\ \eta(\xi) = 1. \tag{3.6}$$

For $\varepsilon = -1$ we get the almost contact geometry [2], while for $\varepsilon = +1$ we have the almost paracontact geometry [19]. On the product manifold $M \times \mathbb{R}$ we consider:

$$J(X, a\frac{d}{dt}) = (\varphi X + \varepsilon a\xi, \eta(X)\frac{d}{dt}), \tag{3.7}$$

and a straightforward computation yields that $J^2 = \varepsilon I_{M\times\mathbb{R}}$. For the 1-form $\omega_b = \eta + bdt$ with $b \in \mathbb{R}$, its conjugate with respect to J is:

$$(\omega_b)_J = \varepsilon b\eta + dt, \tag{3.8}$$

and then ω_b is almost J-analytic form if and only if:

$$d\eta(\varphi X, Y) = b\varepsilon d\eta(X, Y) \tag{3.9}$$

for all vector fields X, Y. In particular, if (M, φ, ξ, η) is *ε-cosymplectic*, i.e. η is closed, then all ω_b are almost J-analytic.

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