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Almost Complex Structures of Tangent Bundles and Finsler Metrics

By

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The notion of complex Finsler spaces was first introduced by P. Finsler himself. However, it is not until recently that a number of papers in this field have been published. It seems, so fas as we know, that one of the most interesting papers is that of E. Heil [5][1], to which we will refer in the last section of this paper. On the other hand, many authors consider the theory of Finsler spaces as a special geometry of tangent bundles.

Now the main purpose of the present paper is to study the geometry of tangent bundles and of generalized almost complex structures, based on non-linear connections in tangent bundles, and thus to develope a theory of almost complex Finsler spaces.

We first introduce the notions of φ-connections and quasi tensor fields, which enable us to define the lifts of tensor fields on manifolds to their tangent bundles.

In section 2, almost complex structures are naturally introduced in tangent bundles by φ-connections.

In section 3, the affine connections of tangent bundles are studied. Then we introduce a special class of connections, named connections of Finsler type, which have close relations to several connections of Finsler spaces.

Quasi almost complex structures are defined in section 4, and we investigate the properties of f-structures and almost complex structures in tangent bundles which are obtained as the lifts of quasi almost complex structures.

1) Numbers in brackets refer to the references at the end of the paper.

The last two sections are devoted to consideration on structures constructed by generalized metrics together with quasi almost complex structures. Finsler metrics are considered as a special case of generalized metrics.

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§ 1. φ-connections and φ-spaces

Let M be an n-dimensional differentiable C^∞ manifold and $T(M)$ be its tangent bundle, the canonical projection being denoted by $\tau: T(M)\to M$. All the zero vectors of M are left out from $T(M)$, that is to say, we consider $T(M)-M$, and denote it, from now on, by the same notation $T(M)$, and call it, for brevity, the tangent bundle over M.

The tangent vector space of $T(M)$ at a point z is denoted by T_z. It is well known that the vertical distribution Φ^v is correspondence $z\in T(M)\to\Phi_z{}^v$, where $\Phi_z{}^v$ is the subspace of T_z tangent to the fibre over $x=\tau(z)$.

Definition. The *φ-connection of* $T(M)$ is an assignment of a horizontal C^∞ distribution Φ^h such that, at each point of $T(M)$,

$$T_z = \Phi_z{}^v+\Phi_z{}^h \text{ (direct sum)}.$$

The φ-connection is called *homogeneous* when and only when, in each point z of $T(M)$ and for any positive number λ, the condition

$$d\lambda^*\Phi^h{}_{(x,y)} = \Phi^h{}_{(x,\lambda y)} \tag{1.1}$$

is satisfied, where $x=\tau(z)$ and $y\in M_x$, and $d\lambda^*$ is the differential of the mapping $\lambda^*: T(M)\to T(M)$ $((x, y)\to(x, \lambda y))$.

The φ-connection is nothing but a generalization of *non-linear connections* ([11], [29]). To see this, we refer to the canonical coordinates $(z^A)=(x^i, y^i)$ of $z=(x, y)\in T(M)$, where $x=(x^i)$ and $y=y^i(\partial/\partial x^i)_x$. It is well known that the matrix of transformation of canonical coordinates is

$$\left(\frac{\partial z^{A'}}{\partial z^A}\right) = \begin{pmatrix} \dfrac{\partial x^{i'}}{\partial x^i}, & 0 \\ \dfrac{\partial^2 x^{i'}}{\partial x^i\partial x^l}y^l, & \dfrac{\partial x^{i'}}{\partial x^i} \end{pmatrix} \tag{1.2}$$

We denote by the capital letter V a tangent vector field on $T(M)$, by p^h and p^v the projection operators corresponding to the distributions Φ^h and Φ^v respectively. Since a vector field V has the form $V=u^i\dfrac{\partial}{\partial x^i}+v^i\dfrac{\partial}{\partial y^i}$, operators p^h and p^v are written as

$$(1.3)\qquad p^h=\begin{pmatrix} E_n, & 0 \\ -\varphi, & 0\end{pmatrix}, \qquad p^v=\begin{pmatrix} 0, & 0 \\ \varphi, & E_n\end{pmatrix},$$

where E_n is a unit n-matrix and φ is an n-matrix determined by the φ-connection. It follows from (1.2) that the condition for p^h and p^v to be tensors is written as

$$(1.4)\qquad \frac{\partial x^{i'}}{\partial x^i}\varphi^i{}_l-\varphi^{i'}{}_{j'}\frac{\partial x^{j'}}{\partial x^l}=\frac{\partial^2 x^{i'}}{\partial x^l\partial x^p}y^p,$$

which is the law of transformation of non-linear connections.

Conversely, if $\varphi=(\varphi^i{}_j)$ satisfies the law (1.4), the two projection tensor p^h and p^v are determined by (1.3), and then the φ-connection is assigned.

Moreover we have "*A necessary and sufficient condition for φ-connection to be homogeneous is that components $\varphi^i{}_j$ of φ in terms of canonical coordinates are positively homogeneous of degree* 1 *with respect to y^i.*" Indeed, in terms of a canonical coordinate system, the condition (1.1) is expressed as

$$(1.5)\qquad \varphi^i{}_j(x,\lambda y)=\lambda\varphi^i{}_j(x,y). \qquad (\lambda>0)$$

Now we introduce the *horizontal lift* and *vertical lift* of vector field v on M to $T(M)$ by means of the φ-connection (Dombrowski [4], Yano-Davies [28]), which are denoted by v^h and v^v respectively. If v is $v_x=v^i(x)\left(\dfrac{\partial}{\partial x^i}\right)_x$, v^v and v^h are written in the forms

$$(1.6)\quad v_z{}^v=v^i(x)\left(\frac{\partial}{\partial x^i}\right)_z,\quad v_z{}^h=v^i(x)\left(\frac{\partial}{\partial x^i}\right)_z-\varphi^i{}_m(x,y)v^m(x)\left(\frac{\partial}{\partial y^i}\right)_z.$$

We consider a tangent vector field V on $T(M)$. Since components of the projection $d\tau\cdot V$ depend upon both x and y, $d\tau\cdot V$ is not always a vector field on M. However, in the geometry of Finsler spaces and non-linear connections, vector fields with line elements have been treated. Hence, if we denote by $\mathfrak{X}(T(M))$ the set of tangent vector fields on $T(M)$, we call elements of $\tilde{\mathfrak{X}}(M)=d\tau\cdot\mathfrak{X}(T(M))$ *quasi vector fields on M.* Hereafter the composite concept of $(M, T(M), \tau, \varphi, \tilde{\mathfrak{X}}(M))$ is called the *φ-space.* In the following, φ-space are mainly treated.

Now the *vertical* and *horizontal lifts of a quasi vector field v* $\left(v_x=v^i(x, y)\left(\frac{\partial}{\partial x^i}\right)_x\right)$ are defined respectively by

$$(1.6') \qquad \begin{cases} v_z{}^v = v^i(x, y)\left(\dfrac{\partial}{\partial y^i}\right)_z, \\ v_z{}^h = v^i(x, y)\left(\dfrac{\partial}{\partial x^i}\right)_z - \varphi^i{}_m(x, y)v^m(x, y)\left(\dfrac{\partial}{\partial y^i}\right)_z. \end{cases}$$

Then we have

Proposition 1.1. *Any vector field V on $T(M)$ is expressed as*

$$V = v_1{}^h + v_2{}^v$$

where v_1 and v_2 are quasi vector fields on M, uniquely determined by V.

Proof. The vector field V is written as $V=p^hV+p^vV$. If we put $v_1=d\tau\cdot p^hV$, then $v_1\in\tilde{\mathfrak{X}}(M)$ and $v_1{}^h=p^hV$. On the other hand p^vV has a form $v_2{}^i(x, y)\left(\frac{\partial}{\partial y^i}\right)_z$, and $v_2{}^i(x, y)\left(\frac{\partial}{\partial x^i}\right)_x$ form a quasi vector field v_2 on M. The p^vV coincides with $v_2{}^v$.

Next, let us consider tensor fields on $T(M)$. If K is, for example, a tensor field of type (1, 2), then K is a bilinear mapping $\mathfrak{X}(T(M))\times\mathfrak{X}(T(M))\to\mathfrak{X}(T(M))$, i.e., for any vector fields U and V, $K(U, V)$ is another vector field on $T(M)$ and is bilinear with respect to U and V. Now we shall treat a bilinear mapping $\tilde{\mathfrak{X}}(M)\times\tilde{\mathfrak{X}}(M)\to\tilde{\mathfrak{X}}(M)$ and call it a quasi tensor field of type (1, 2). Quasi tensor fields of several types are similarly defined. Of course, tensor fields on M and tensor fields on Finsler spaces are also kinds of quasi tensor fields. Since the vector fields U and V on $T(M)$ are decomposed uniquely, by means of Prop. 1.1, as $U=u_1{}^h+u_2{}^v$ and $V=v_1{}^h+v_2{}^v$, it follows that, for a tensor field K of type (1, 2) on $T(M)$,

$$K(U,\ V) = h(K(u_1^h,\ v_1^h))+h(K(u_1^h,\ v_2^v))+h(K(u_2^v,\ v_1^h))+h(K(u_2^v,\ v_2^v))$$
$$+v(K(u_1^h,\ v_1^h))+v(K(u_1^h,\ v_2^v))+v(K(u_2^v,\ v_1^h))+v(K(u_2^v,\ v_2^v))\,,$$

where $h(V)$ and $v(V)$ denote the horizontal and vertical components of the vector V respectively. Now we are concerned with the first term of the right-hand side $h(K(u_1^h,\ v_1^h))$ which is a vector field on $T(M)$. Then, by virtue of Prop. 1. 1, there exists a quasi vector w_1 whose horizontal lift coincides with $h(K(u_1^h,\ v_1^h))$. We then observe that the tensor field K induces a quasi tensor field a of type (1, 2), satisfying $a(u_1,\ v_1)=w_1$. Thus we can define a tensor field $l^{-}_{--}(a)$ on $T(M)$

$$l^{-}_{--}(a)(U,\ V) = h(K(h(U),\ h(V)) = (a(u_1,\ v_1))^h\,.$$

The other seven components of $K(U,\ V)$ also induce respectively quasi tensor fields b, c, d, e, f, g and h. Then K is expressed uniquely by

$$\text{(1. 7)}\qquad K = l^{-}_{--}(a)+l^{-}_{-+}(e)+l^{-}_{+-}(c)+l^{-}_{++}(g)$$
$$+l^{+}_{--}(b)+l^{+}_{-+}(f)+l^{+}_{+-}(d)+l^{+}_{++}(h)\,,$$

where the sign $-$ implies a horizontal component and $+$ a vertical one. These eight operators l's are called lift operators of tensor. If k is a quasi tensor field of type (1, 2), then, for example, $K=l^{+}_{-+}(k)$ is a tensor field on $T(M)$ which is expressed as

$$K(U,\ V) = l^{+}_{-+}(k)(U,\ V) = (k(u_1,\ v_2))^v,$$

where we put $U=u_1^h+u_2^v$ and $V=u_1^h+v_2^v$.

Remark 1. In terms of canonical coordinates, components of the tensor K defined by (1. 7) are given by

$$\text{(1. 8)}\begin{cases} K^i{}_{kj} = a^i{}_{kj}+c^i{}_{lj}\varphi^l{}_k+e^i{}_{kl}\varphi^l{}_j+g^i{}_{lm}\varphi^l{}_k\varphi^m{}_j\,, \\ K^i{}_{\bar{k}j} = c^i{}_{kj}+g^i{}_{kl}\varphi^l{}_j\,,\quad K^i{}_{k\bar{j}} = e^i{}_{kj}+g^i{}_{lj}\varphi^l{}_k\,,\quad K^i{}_{\bar{k}\bar{j}} = g^i{}_{kj}\,, \\ K^{\bar{i}}{}_{kj} = -\varphi^i{}_l a^l{}_{kj}+b^i{}_{kj}-\varphi^i{}_l c^l{}_{mj}\varphi^m{}_k+d^i{}_{lj}\varphi^l{}_k-\varphi^i{}_l e^l{}_{km}\varphi^m{}_j \\ \qquad\qquad +f^i{}_{kl}\varphi^l{}_j-\varphi^i{}_r g^r{}_{lm}\varphi^l{}_k\varphi^m{}_j+h^i{}_{lm}\varphi^l{}_k\varphi^m{}_j\,, \\ K^{\bar{i}}{}_{\bar{k}j} = -\varphi^i{}_l c^l{}_{kj}+d^i{}_{kj}-\varphi^i{}_r g^r{}_{kl}\varphi^l{}_j+h^i{}_{kl}\varphi^l{}_j\,, \\ K^{\bar{i}}{}_{k\bar{j}} = -\varphi^i{}_l e^l{}_{kj}+f^i{}_{kj}-\varphi^i{}_r g^r{}_{lj}\varphi^l{}_k+h^i{}_{lj}\varphi^l{}_k\,, \\ K^{\bar{i}}{}_{\bar{k}\bar{j}} = -\varphi^i{}_r g^r{}_{kj}+h^i{}_{kj}\,. \end{cases}$$

In these equations components of quasi tensor fields $a, \cdots$ and h are denoted by $a^i{}_{kj}, \cdots$ and $h^i{}_{kj}$ respectively, which are functions of x and y, and their transformation law is the one satisfied by ordinary tensors. At the same time, it follows, from (1. 8), that we have the components of the lift operators in canonical coordinates, for example, we have the components of the tensor $l^{-}_{--}(a)$ as a case where $b=c=d=e=f=g=h=0$ in (1. 8).

Remark 2. Since tensors on $T(M)$ of type (1, 2) are of the form (1. 8), several kinds of lifts of a tensor field on M, i.e. the extension (or the complete lift) and the horizontal lift e.t.c.$\cdots$ are given as special ones in our case. For example, the complete lift of a tensor field of type (1, 1) defined by Yano-Kobayashi [30] (or the extension defined by Sasaki [22]) is given as follows. The general form of a tensor field K of type (1, 1) is $K=l^{-}_{-}(a)+l^{+}_{-}(b)+l^{-}_{+}(c)+l^{+}_{+}(d)$. If we take $d=a$, $c=0$, and $b=(b^i{}_j)=(y^r\partial_r a^i{}_j-y^r\varphi^p{}_r\dot{\partial}_p a^i{}_j+\varphi^i{}_l a^l{}_j-a^i{}_l\varphi^l{}_j)$, K is expressed by

$$K=(K^A{}_B)=\begin{pmatrix} a^i{}_k, & 0 \\ y^r(\partial_r a^i{}_k-\varphi^p{}_r\partial_p a^i{}_k), & a^i{}_k \end{pmatrix},$$

where we put $\partial_r=\partial/\partial x^r$, $\dot{\partial}_r=\partial/\partial y^r$. If a is a tensar field on M, i.e. a satisfies $\dot{\partial}_r a^i{}_k=0$, then the above tensor K coincides with the complete lift of a tensor a.

Proposition 1. 2. *On putting*

$$[u^h, v^h]=([u, v]^{\#})^h+(R_\varphi(u, v))^v,$$
$$[u^h, v^v]=(-\dot{\nabla}_v{}^{\#}\cdot u)^h+(\nabla_u{}^{\#}\cdot v)^v,$$

for any quasi vector fields u and v, it follows that

$$[u^v, v^v]=(\dot{\nabla}_u{}^{\#}\cdot v-\dot{\nabla}_v{}^{\#}\cdot u)^v,$$

and $R_\varphi(u, v)$ and $T_\varphi(u, v)=\nabla_u{}^{\#}\cdot v-\nabla_v{}^{\#}\cdot u-[u, v]^{\#}$ are quasi tensor fields of type (1, 2).

Proof. It is enough to express each vector by the canonical coordinates. By direct calculation we obtain

$$
(1.9)\quad \begin{cases}
[u^v, v^v] = (u^j \partial_j v^i - v^j \partial_j u^i)\dfrac{\partial}{\partial y^i}, \\
[u, v]^\# = \{u^l(\partial_l v^i - \varphi^m{}_l \dot{\partial}_m v^i) - v^l(\partial_l u^i - \varphi^m{}_l \dot{\partial}_m u^i)\}\dfrac{\partial}{\partial x^i}, \\
\nabla_u{}^\# \cdot v = (\partial_l v^i - \varphi^m{}_l \dot{\partial}_m v^i + \dot{\partial}_m \varphi^i{}_l v^m) u^l \dfrac{\partial}{\partial x^i}, \\
\dot{\nabla}_u{}^\# \cdot v = u^l \dot{\partial}_l v^i \dfrac{\partial}{\partial x^i}, \\
R_\varphi(u, v) = R^i{}_{kj} u^k v^j \dfrac{\partial}{\partial x^i}, \\
R^i{}_{kj} = \partial_j \varphi^i{}_k - \partial_k \varphi^i{}_j - \varphi^m{}_j \dot{\partial}_m \varphi^i{}_k + \varphi^m{}_k \dot{\partial}_m \varphi^i{}_j, \\
T_\varphi(u, u) = (\dot{\partial}_m \varphi^i{}_l - \dot{\partial}_l \varphi^i{}_m) u^l v^m \dfrac{\partial}{\partial x^i},
\end{cases}
$$

which prove our assertion.

The quasi tensor field R_φ and T_φ are called a *curvature tensor* and *torsion tensor with respect to the connection* φ respectively. Now we shall define the covariant differential $\nabla^\# k$ and $\dot{\nabla}^\# k$ of any quasi tensor field k of type $(1, r)$ or $(0, r)$ with respect to a φ-connection. If k is of type $(1, r)$, then we set

$$
(1.10)\quad \begin{cases}
\nabla_v{}^\# k \cdot (u^1, \cdots, u^r) = \nabla_v{}^\# \cdot k(u^1, \cdots, u^r) - \sum_{j=1}^{r} k(u^1, \cdots, \nabla_v{}^\# \cdot u^j, \cdots, u^r), \\
\dot{\nabla}_v{}^\# k \cdot (u^1, \cdots, u^r) = \dot{\nabla}_v{}^\# \cdot k(u^1, \cdots, u^r) - \sum_{j=1}^{r} k(u^1, \cdots, \dot{\nabla}_v{}^\# \cdot u^j, \cdots, u^r).
\end{cases}
$$

In the case of type $(0, r)$ we set also

$$
(1.10')\quad \begin{cases}
\nabla_v{}^\# k \cdot (u^1, \cdots, u^r) = v^h \cdot k(u^1, \cdots, u^r) - \sum_{j=1}^{r} k(u^1, \cdots, \nabla_v{}^\# \cdot u^j, \cdots, u^r), \\
\dot{\nabla}_v{}^\# k \cdot (u^1, \cdots, u^r) = v^v \cdot k(u^1, \cdots, u^r) - \sum_{j=1}^{r} k(u^1, \cdots, \dot{\nabla}_v{}^\# \cdot u^j, \cdots, u^r).
\end{cases}
$$

In these definitions, if we put $\nabla_v{}^\# k \cdot (u^1, \cdots, u^r) = \nabla^\# k \cdot (u^1, \cdots, u^r, v)$ and $\dot{\nabla}_v{}^\# k \cdot (u^1, \cdots, u^r) = \dot{\nabla}^\# k \cdot (u^1, \cdots, u^r, v)$, $\nabla^\# k$ and $\dot{\nabla}^\# k$ are quasi tensor fields of type $(1, r+1)$ or $(0, r+1)$. For later applications, we calculate the components of these quasi tensors in the case where k is of type $(1, 1)$:

$$
(1.11)\quad \begin{cases}
\nabla_k{}^\# k^i{}_j = \partial_k k^i{}_j - \varphi^m{}_k \dot{\partial}_m k^i{}_j + \dot{\partial}_m \varphi^i{}_k k^m{}_j - k^i{}_m \dot{\partial}_j \varphi^m{}_k, \\
\dot{\nabla}_k{}^\# k^i{}_j = \dot{\partial}_k k^i{}_j,
\end{cases}
$$

where we put

$$\nabla^{\#}k\cdot(u, v) = (\nabla_k^{\#}k^i{}_j)v^k u^j \frac{\partial}{\partial x^i}, \quad \dot{\nabla}^{\#}k\cdot(u, v) = (\dot{\nabla}_k^{\#}k^i{}_j)v^k u^j \frac{\partial}{\partial x^i}.$$

Now we shall give a simple example of a φ-space. In a connected manifold M, if the second countability axiom holds good, then M always admits a positive definite non-Riemannian Finsler metric. (Kashiwabara [10]) Hence we take the connction of Cartan with respect to the Finsler metric, and denote it by $\gamma^{*i}{}_{kj}$. By putting $\varphi^i{}_j=\gamma^{*i}{}_{jl}y^l$, $\varphi^i{}_j$ satisfies (1. 4) and

$$\varphi^i{}_j = \frac{1}{2}\dot{\partial}_j(\gamma^{*i}{}_{lm}y^l y^m) = G^i{}_{jl}y^l\,, \tag{1. 12}$$

where $G^i{}_{jl}$ is the connection introduced by Berwald. This φ-connection is called a *φ-connection derived from a Finsler metric.* In this case the covariant differential $\nabla^{\#}$ coincides with the one defined by Berwald, and the following relations hold:

$$R^i{}_{kj} = R_{jkr}{}^i y^r\,, \qquad T_\varphi = 0\,, \tag{1. 13}$$

where $R_{jkr}{}^i$ is a curvature tensor expressed by

$$\begin{aligned} R_{jkr}{}^i = {} & (\partial_j\gamma^{*i}{}_{rk} - \varphi^t{}_j\dot{\partial}_t\gamma^{*i}{}_{rk}) - (\partial_k\gamma^{*i}{}_{rj} - \varphi^t{}_k\dot{\partial}_t\gamma^{*i}{}_{rj}) \\ & + \gamma^{*i}{}_{tj}\gamma^{*t}{}_{rk} - \gamma^{*i}{}_{tk}\gamma^{*t}{}_{rj}\,. \end{aligned}$$

§ 2. Natural almost complex structure

We shall concern with almost complex structures in the tangent bundle over a manifold M, which are more general ones than the structures studied by Nagano [17], Dombrowski [2], Yano-Davies [28], Akbar-Zadeh [1], and Matsumoto [14].

Theorem 2. 1. *In the tangent bundle $T(M)$ over a φ-space M there exists an almost complex structure J defined by the following:*

$$\begin{cases} J(u^h) = \alpha u^h - \dfrac{1+\alpha^2}{\rho}u^v\,, \\ J(u^v) = \rho u^h - \alpha u^v\,, \end{cases} \tag{2. 1}$$

where α and ρ are scalar fields on $T(M)$ satisfying $\rho \neq 0$ and u is any quasi vector field.

Proof. For any scalar fields α, β, ρ and σ on $T(M)$, if we put $J=\alpha l^{-}_{-}(1)+\beta l^{+}_{+}(1)+\rho l^{-}_{+}(1)+\sigma l^{+}_{-}(1)$, then J is a tensor field on $T(M)$, where 1 means $(\delta^i{}_j)$. The condition $J^2=-E^{2n}$ gives the relations $\beta=-\alpha$ and $\sigma=-(1+\alpha^2)/\rho$. Thus $\alpha l^{-}_{-}(1)-\alpha l^{+}_{+}(1)-\dfrac{1+\alpha^2}{\rho} l^{+}_{-}(1)+\rho l^{-}_{+}(1)$ is an almost complex structure of $T(M)$, which complete the proof of the Theorem.

The above proof gives that J given by (2.1) is, in $T(M)$, the most general type of the almost complex structure in the ones induced from 1 and φ. In a case where $\alpha=0$ and $\rho=-1$, J is denoted hereafter by J^*, and J^* satisfies for any quasi vector field u,

$$J^*(u^h)=u^v, \quad J^*(u^v)=-u^h, \tag{2.2}$$

which implies that J^* is the *natural almost complex structure* named by Matsumoto [14]. Especially, if the manifold M admits an affine connection $\gamma^i{}_{jk}(x)$, then M admits a φ-conection defined by $\varphi^i{}_j=\gamma^i{}_{jl}y^l$, and J^* coincides with an almost complex structure treated by Nagano [17] and Dombrowski [4].

As the structure J depends upon the choice of φ, if necessary, J is denoted by $J_\varphi(\rho, \alpha)$. In canonical coordinates, the components of $J_\varphi(\rho, \alpha)$ are written in the form

$$J_\varphi(\rho, \alpha)=\begin{pmatrix} \alpha E_n+\rho\varphi\,, & \rho E_n \\ -2\alpha\varphi-\rho\varphi^2-\dfrac{1+\alpha^2}{\rho}E_n\,, & -\rho\varphi-\alpha E_n \end{pmatrix}. \tag{2.3}$$

On the other hand, if we put

$$\varphi'=\varphi+\frac{\alpha}{\rho}E_n\,, \tag{2.4}$$

then φ' is also a φ-connection. From these, it follows that

$$J_\varphi(\rho, \alpha)=J_{\varphi'}(\rho, 0)=\rho \underset{\varphi'}{l^{-}_{+}}(1)-\frac{1}{\rho}\underset{\varphi'}{l^{+}_{-}}(1)\,. \tag{2.5}$$

If $T(M)$ admits a certain structure K of type $(1, 1)$, then the Nijenhuis tensor $N_{(K)}$ of K is defined by

$$N_{(K)}(U, V)=-K^2[U, V]+K[KU, V]+K[U, KV]-[KU, KV]\,,$$

for any vector fields U and V in $T(M)$. On the other hand, as K

is of type (1,1), K is written as $K=l^-_-(a)+l^+_-(b)+l^-_+(c)+l^+_+(d)$, where a, b, c and d are quasi tensor fields of type (1, 1). Hence $N_{(K)}$ should be expressed by a, b, c, d and φ. Thus

Proposition 2.1. *In the tangent bundle $T(M)$, a structure tensor given by $K=l^-_-(a)+l^+_-(b)+l^-_+(c)+l^+_+(d)$ of type* (1, 1) *is integrable if and only if any quasi vector fields u and v on M satisfy the following*:

$$
\left\{
\begin{aligned}
& a[au, v]^\sharp+a[u, av]^\sharp-[au, av]^\sharp-a^2[u, v]^\sharp-(ac+cd)R_\varphi(u, v) \\
&\qquad +cR_\varphi(au, v)+cR_\varphi(u, av)-\dot{\nabla}^\sharp a\cdot(v, bu)+\dot{\nabla}^\sharp a\cdot(u, bv) \\
&\qquad +c\nabla^\sharp b\cdot(v, u)-c\nabla^\sharp b\cdot(u,v)+cbT_\varphi(u, v) = 0\,, \\
& dR_\varphi(au, v)+dR_\varphi(u, av)-R_\varphi(au, av)-(bc+d^2)R_\varphi(u, v) \\
&\qquad +(ba+db)T_\varphi(u, v)-bT_\varphi(au, v)-bT_\varphi(u, av)+\nabla^\sharp b\cdot(u, av) \\
&\qquad -\nabla^\sharp b\cdot(v, au)+\dot{\nabla}^\sharp b\cdot(u, bv)-\dot{\nabla}^\sharp b\cdot(v, bu) +b\nabla^\sharp a\cdot(v, u) \\
&\qquad -b\nabla^\sharp a\cdot(u, v)+d\nabla^\sharp b\cdot(v, u)-d\nabla^\sharp b\cdot(u, v) = 0\,, \\
& T_\varphi(cu, cv)+\nabla^\sharp c\cdot(u, cv)-\nabla^\sharp c\cdot(v, cu)+\dot{\nabla}^\sharp c\cdot(u, dv)-\dot{\nabla}^\sharp c\cdot(v, du) \\
&\qquad -a\dot{\nabla}^\sharp c\cdot(u, v)+a\dot{\nabla}^\sharp c\cdot(v, u)+c\dot{\nabla}^\sharp d\cdot(v, u)-c\dot{\nabla}^\sharp d\cdot(u, v) = 0\,, \\
& \nabla^\sharp d\cdot(u, cv)-\nabla^\sharp d\cdot(v, cu)+d\dot{\nabla}^\sharp d\cdot(v, u)-d\dot{\nabla}^\sharp d\cdot(u, v)+\dot{\nabla}^\sharp d\cdot(u, dv) \\
&\qquad -\dot{\nabla}^\sharp d\cdot(v, du)+b\dot{\nabla}^\sharp c\cdot(v, u)-b\dot{\nabla}^\sharp c\cdot(u, v)-R_\varphi(cu, cv) = 0\,, \\
& cR_\varphi(u, cv)-aT_\varphi(u, cv)-T_\varphi(au, cv)+\nabla^\sharp a\cdot(u, cv)+a\nabla^\sharp c\cdot(v, u) \\
&\qquad +c\nabla^\sharp d\cdot(v, u)-a\dot{\nabla}^\sharp a\cdot(u, v)+\dot{\nabla}^\sharp a\cdot(u, dv)-c\dot{\nabla}^\sharp b\cdot(u, v) \\
&\qquad -\dot{\nabla}^\sharp c\cdot(v, bu)-\dot{\nabla}^\sharp c\cdot(v, au) = 0\,, \\
& dR_\varphi(u, cv)-R_\varphi(au, cv)-bT_\varphi(u, cv)+b\nabla^\sharp c\cdot(v, u)-\nabla^\sharp d\cdot(v, au) \\
&\qquad +d\nabla^\sharp d\cdot(v, u)+\nabla^\sharp b\cdot(u, cv)-b\dot{\nabla}^\sharp a\cdot(u, v)+\dot{\nabla}^\sharp b\cdot(u, dv) \\
&\qquad -d\dot{\nabla}^\sharp b\cdot(u, v)-\dot{\nabla}^\sharp d\cdot(v, bu) = 0\,.
\end{aligned}
\right.
$$

Proof. If $U=u^h$ and $V=v^h$, it follows from Prop. 1.1 that

$$
\begin{aligned}
h(N_K(u^h, v^h)) = & -(a^2+cb)[u, v]^\sharp+a[au, v]^\sharp+a[u, av]^\sharp-[au, av]^\sharp \\
& -(ac+cd)R_\varphi(u, v)+cR_\varphi(u, av)+cR_\varphi(au, u) \\
& +a\dot{\nabla}_{bu}{}^\sharp v-c\nabla_v{}^\sharp bu-a\dot{\nabla}_{bv}{}^\sharp u+c\nabla_u{}^\sharp bv+\dot{\nabla}_{bv}{}^\sharp au-\dot{\nabla}_{bu}{}^\sharp av \\
= & \; a[au, v]^\sharp+a[u, av]^\sharp-[au, av]^\sharp-a^2[u, v]^\sharp \\
& -(ac+cd)R_\varphi(u, v) +cR_\varphi(au, v)+cR_\varphi(u, av) \\
& -\dot{\nabla}^\sharp a\cdot(v, bu)+\dot{\nabla}^\sharp a\cdot(u, bv)+c\nabla^\sharp b\cdot(v, u) \\
& -c\nabla^\sharp b\cdot(u, v)+cbT_\varphi(u, v)\,.
\end{aligned}
$$

Thus $N_{(K)}=0$ gives us the first equation of Prop. 2.1. Obviously the other equations are led by the similar method.

Theorem 2.2. *In the tangent bundle over φ-space M, the almost complex structure J introduced in Theorem 2.1 is integrable if and only if the following conditions are satisfied, by any quasi vector fields u and v,*

$$\text{(2.6)} \qquad \begin{cases} \rho T_{\varphi'}(u, v) - (\underset{\varphi'}{\nabla}_u{}^\sharp \rho) v + (\underset{\varphi'}{\nabla}_v{}^\sharp \rho) u = 0 , \\ \rho^3 R_{\varphi'}(u, v) - (\dot{\nabla}_u{}^\sharp \rho) v + (\dot{\nabla}_v{}^\sharp \rho) u = 0 , \end{cases}$$

where $\varphi' = \varphi + \dfrac{\alpha}{\rho} E_n$.

Proof. The integrability condition for $J(\rho, 0)$ leads us to

$$\rho T_\varphi(u, v) - (\underset{\varphi}{\nabla}_u{}^\sharp \rho) v + (\underset{\varphi}{\nabla}_v{}^\sharp \rho) u = 0 ,$$

$$\rho^3 R_\varphi(u, v) - (\dot{\nabla}_u{}^\sharp \rho) v + (\dot{\nabla}_v{}^\sharp \rho) u = 0 ,$$

by virtue of Prop, 2.1 and relations $\nabla^\sharp E_n = 0$ and $\dot{\nabla}^\sharp E_n = 0$. These results and the fact $J_\varphi(\rho, \alpha) = J_{\varphi'}(\rho, 0)$ establish the Theorem.

Corollary. *In $T(M)$ over φ-space, the natural almost complex structure J^* is integrable if and only if $R_\varphi = 0$ and $T_\varphi = 0$.* (Dombrowski [4], Matsumoto [4])

Proof. For the structure J^*, we have $J^* = J_\varphi(-1, 0)$. Hence Theorem 2.2 gives us our result.

§ 3. Linear connections of Finsler type

Let $P(T(M)) = P(T(M), \pi, G)$ be the bundle of frames over the $T(M)$, where G is a structure group, and π is the projection $P \to T(M)$ which maps a frame p at a point $z \in T(M)$ into z. It is well known [18] that if a linear connection Γ in P is assigned, then we obtain a lift L of vectors on $T(M)$ to P. Since we are now considering φ-space, T_z is written as $T_z = \Phi_z{}^h + \Phi_z{}^v$. Therefore $\Gamma_p{}^h = L(\Phi_z{}^h)$ and $\Gamma_p{}^v = L(\Phi_z{}^v)$ construct distributions on P and the relation $\Gamma_p = \Gamma_p{}^h + \Gamma_p{}^v$ holds good where Γ_p is a distribution at p defined by the linear connection Γ. Hence we find

$$\text{(3.1)} \qquad P_p = P_p{}^v + \Gamma_p{}^h + \Gamma_p{}^v$$

where P_p^v is a vertical subspace of P_p, and any vector $X^* \in P_p$ is decomposed uniquely as

$$X^* = v(X^*) + h^h(X^*) + h^v(X^*)$$

where $v(X^*) \in P_p^v$, $h^h(X^*) \in \Gamma_p^h$ and $h^v(X^*) \in \Gamma_p^v$. Now let $\Gamma^A{}_{BC}$ be parameters of the connection Γ in terms of the canonical coordinates and (p_A) be a frame at a point z, then p_A is represented as $p_A = p^B{}_A \left(\frac{\partial}{\partial z^B}\right)_z$. If we put

$$(3.2) \qquad \begin{cases} z^*{}_A{}^Q = \dfrac{\partial}{\partial p^A{}_Q}, \\ X_i^* = \dfrac{\partial}{\partial x^i} - \varphi^k{}_i \dfrac{\partial}{\partial y^k} - p^C{}_Q(\Gamma^B{}_{Ci} - \varphi^k{}_i \Gamma^B{}_{C\bar{k}}) \dfrac{\partial}{\partial p^B{}_Q}, \\ Y_i^* = \dfrac{\partial}{\partial y^i} - p^C{}_Q \Gamma^B{}_{C\bar{i}} \dfrac{\partial}{\partial p^B{}_Q}, \end{cases}$$

these constitute basis of P^v, Γ^h and Γ^v respectively.

Proposition 3.1. *In the frame bundle P over $T(M)$, the distribution $P^v + \Gamma^v$ is integrable. While the distribution $P^v + \Gamma^h$ is integrable if and only if the horizontal distribution Φ^h in $T(M)$ is integrable, i.e., $R_\varphi = 0$.*

Proof. For the distribution $P^v + \Gamma^v$, it is sufficient to verify $[U^*, V^*] \in P^v + \Gamma^v$ for any vector fields $U^*, V^* \in P^v + \Gamma^v$. So, if $U^*, V^* \in P^v$, it is clear $[U^*, V^*] \in P^v$. If $U^* \in P^v$, $V^* \in \Gamma^v$, then there exist functions f^i such that $V^* = f^i Y_i^*$, from which it follows

$$[U^*, V^*] = \left[U^A{}_Q \frac{\partial}{\partial p^A{}_Q}, \ f^i \frac{\partial}{\partial y^i} - f^i p^C{}_R \Gamma^B{}_{C\bar{i}} \frac{\partial}{\partial p^B{}_R} \right] \in P^v + \Gamma^v.$$

Similar calculation shows us that if $U^*, V^* \in \Gamma^v$, then $[U^*, V^*] \in P^v + \Gamma^v$.

For the distribution $P^v + \Gamma^h$, if $U^*, V^* \in P^v$, it is clear $[U^*, V^*] \in P^v + \Gamma^h$. If $U^* \in P^v$, $V^* \in \Gamma^h$, then there exist functions f^i and $U^A{}_Q$ such that $V^* = f^i X_i^*$, $U^* = U^A{}_Q \dfrac{\partial}{\partial p^A{}_Q}$, from which it follows

$$[U^*, X_i^*] = -U^A{}_Q(\Gamma^B{}_{Ai} - \varphi^l{}_i \Gamma^B{}_{A\bar{l}}) \frac{\partial}{\partial p^B{}_Q} - X_i^*(U^A{}_Q) \frac{\partial}{\partial p^A{}_Q} \in P^v.$$

Hence we have $[U^*, V^*]=f^i[U^*, X_i^*]+U^*(f^i)X_i^* \in P^v+\Gamma^h$. Finally if $U^*, V^* \in \Gamma^h$ then they have the form $U^*=f^iX_i^*$, $V^*=g^iX_i^*$ and the relation

$$[U^*, V^*]=f^ig^j[X_i^*, X_j^*]+f^iX_i^*(g^j)X_j^*-g^jX_j^*(f^i)X_i^*$$

holds good. Thus $P^v+\Gamma^h$ is integrable if and only if $[X_i^*X_j^*] \in \Gamma^h+P^v$. On the other hand $X_i^*=L(X_i)$, so X_i^* and X_i are π-related. Hence the condition $[X_i^*, X_j^*] \in \Gamma^h+P^v$ gives us $[X_i, X_j] \in \Phi^h$, i.e., $R_\varphi=0$. Q.E.D.

If there is assigned a linear connection Γ in P, we can consider in $T(M)$ a covariant derivative ∇ with respect to the Γ.

Definition. *A linear connection of horizontal Finsler type*, or simply *of h-f-type*, is a linear connection satisfying $v(\nabla_U v^h)=0$ for any vector field U on $T(M)$ and quasi vector field v on M. *A linear connection of vertical Finsler type*, or simply *of v-f-type*, is the one satisfying $h(\nabla_U v^v)=0$. Moreover a linear connection of *h-f-* and at the same time *v-f*-type is called *of quasi Finsler type*.

If Γ is of *h-f*-type, then we may put

$$\nabla_{u^h} v^h = (\nabla_u^{(h)} v)^h, \quad \nabla_{u^v} v^h = (\dot{\nabla}_u^{(h)} v)^h. \tag{3.3}$$

Similary, if Γ is of *v-f*-type, we may put

$$\nabla_{u^h} v^v = (\nabla_u^{(v)} v)^v, \quad \nabla_{u^v} v^v = (\dot{\nabla}_u^{(v)} v)^v. \tag{3.4}$$

Apparently, the quasi vector fields $\nabla_u^{(h)}v$, $\dot{\nabla}_u^{(h)}v$, $\nabla_u^{(v)}v$ and $\dot{\nabla}_u^{(v)}v$, which are defined in this way, are regarded as covariant derivatives of quasi vector field v with respect to u. For quasi tensor fields, e.g., a quasi tensor field f of type (1, 1) and a linear connection Γ of *h-f*-type, we may define

$$\begin{cases} \nabla_u^{(h)} f\cdot(v) = \nabla^{(h)} f\cdot(v, u) = \nabla_u^{(h)}\cdot f(v) - f(\nabla_u^{(h)} v), \\ \dot{\nabla}_u^{(h)} f\cdot(v) = \dot{\nabla}^{(h)} f\cdot(v, u) = \dot{\nabla}_u^{(h)}\cdot f(v) - f(\dot{\nabla}_u^{(h)} v). \end{cases} \tag{3.5}$$

If the Γ is of quasi Finsler type, then we may put

$$\nabla_{u^h} V = (\nabla_u^{(h)} v_1)^h + (\nabla_u^{(v)} v_2)^v, \ \nabla_{u^v} V = (\dot{\nabla}_u^{(h)} v_1)^h + (\dot{\nabla}_u^{(v)} v_2)^v, \tag{3.6}$$

where we put $V=v_1^h+v_2^v$. From the definition itself, it follows that a linear connection of quasi Finsler type is a linear connection with respect to which the distributions Φ^h and Φ^v are respectively parallel. It is well known that there exists a linear connection Γ admitting the above properties (e.g., see Walker [25]). However we shall here decide the connection Γ explicitly.

Let $\overset{0}{\Gamma}$ be a linear connection of $T(M)$ and denote by $\overset{0}{\nabla}$ covaciant differentiation with respect to $\overset{0}{\Gamma}$. Now, for any vector fields U and V, if $U=u^h$ and $V=v^h$, we define K by $K(U,V)=-v(\overset{0}{\nabla}_{u^h}v^h)$: otherwise we define K by $K(U,V)=0$. Then K is a (1, 2)-tensor field of type l^{+}_{--} on $T(M)$. Indeed, it is clear from the definition that $K(U,V)$ is a vector field on $T(M)$ and is linear with respect to U. The linearity with respect to V is shown as follows;

$$\begin{aligned} K(U, \rho v_1^h+\sigma v_2^h) &= -v[\rho\overset{0}{\nabla}_U v_1^h+\sigma\overset{0}{\nabla}_U v_2^h+U(\rho)v_1^h+U(\sigma)v_2^h] \\ &= \rho K(U, v_1^h)+\sigma K(U, v_2^h)\,. \end{aligned}$$

Hence, by the definition, K is a (1, 2)-tensor field of type l^{+}_{--}.

Similarly we define $H, \bar{K}$ and $\bar{H}$ as follows:

$$\begin{aligned} &\text{if } U=u^v \text{ and } V=v^h,\ H(U,V)=-v(\overset{0}{\nabla}_{u^v}v^h):\ \text{otherwise } H(U,V)=0\,, \\ &\text{if } U=u^h \text{ and } V=v^v,\ \bar{K}(U,V)=-h(\overset{0}{\nabla}_{u^h}v^v):\ \text{otherwise } \bar{K}(U,V)=0\,, \\ &\text{if } U=u^v \text{ and } V=v^v,\ \bar{H}(U,V)=-h(\overset{0}{\nabla}_{u^v}v^v):\ \text{otherwise } \bar{H}(U,V)=0\,. \end{aligned}$$

Then H, $\bar{K}$ and $\bar{H}$ are (2, 1)-tensor fields of type l^{+}_{+-}, l^{-}_{-+} and l^{-}_{++} respectively.

Now let $\overset{0}{\Gamma}$ be a linear connection of $T(M)$ over φ-space M, and K, H, $\bar{K}$ and $\bar{H}$ be tensor fields defined above with respect to $\overset{0}{\Gamma}$. If we put $U=u_1^h+u_2^v$ $\Gamma=\overset{0}{\Gamma}+K+H+\bar{K}+\bar{H}$, and denote by ∇ the covariant derivative with respect to Γ, then we get

$$\begin{aligned} \nabla_U v^h &= \overset{0}{\nabla}_{u_1^h}v^h+K(u_1^h, v^h)+\overset{0}{\nabla}_{u_2^v}v^h+H(u_2^v, v^h) \\ &= h(\overset{0}{\nabla}_{u_1^h}v^h+\overset{0}{\nabla}_{u_2^v}v^h)\,, \\ \nabla_U v^v &= \overset{0}{\nabla}_{u_1^h}v^v+\bar{K}(u_1^h, v^v)+\overset{0}{\nabla}_{u_2^v}v^v+\bar{H}(u_1^v, v^v) \\ &= v(\overset{0}{\nabla}_{u_1^h}v^v+\overset{0}{\nabla}_{u_2^v}v^v)\,. \end{aligned}$$

Hence Γ is of the quasi Finsler type. The above consideration shows us that a linear connection Γ of $T(M)$ is of the quasi Finsler type if and only if Γ is written in the form

$$\Gamma = \overset{0}{\Gamma}+K+H+\bar{K}+\bar{H}+l^{-}_{--}(a)+l^{-}_{+-}(c)+l^{+}_{-+}(f)+l^{+}_{++}(h)\,, \tag{3.7}$$

where a, c, f and h are arbitrary quary quasi tensor fields of type $(1, 1)$. Besides, it follows that Γ is of h–f–type if and only if Γ is written, for any quasi tensor fields a, c, e, f, g and h, in the form

$$\Gamma = \overset{0}{\Gamma} + K + H + l^{-}_{--}(a) + l^{-}_{+-}(c) + l^{-}_{-+}(e) + l^{+}_{-+}(f) + l^{-}_{++}(g) + l^{+}_{++}(h), \tag{3.8}$$

and at the same time it follows that Γ is of v–f–type if and only if Γ is written, for any quasi tensor fields a, b, c, d, f and h, in the form

$$\Gamma = \overset{0}{\Gamma} + \bar{K} + \bar{H} + l^{-}_{--}(a) + l^{+}_{--}(b) + l^{-}_{+-}(c) + l^{+}_{+-}(d) + l^{+}_{-+}(f) + l^{+}_{++}(h). \tag{3.9}$$

Theorem 3.1. *In the tangent bundle $T(M)$ over a φ–space M, there exists always a symmetric linear connection of v–f–type, and there exists a symmetric linear connection of h–f–type (or of the quasi Finsler type) if and only if $R_\varphi = 0$.*

Proof. A symmetric linear connection Γ is of v–f–type if and only if the distribution Φ^v is parallel with respect to Γ. A necessary and sufficient condition that there exists a symmetric linear connection with respect to which the given distribution is parallel is that the distribution is integrable (Walker [25], Willmore [26]). Hence, as the distribution Φ^v is integrable, there exists always a symmetric linear connection of v–f–type. Similar discussion gives us that the condition of the existence of a symmetric linear connection of h–f–type is that the horizontal distribution Φ^h is integrable, i.e., the equation $R_\varphi = 0$ holds good. A symmetric linear connection of quasi Finsler type is a symmetric connection with respect to which the distribution Φ^h and Φ^v are respectively parallel. The condition of the existence of such a connection is that the distribution Φ^h, Φ^v and $\Phi^h + \Phi^v$ are all integrable. (Walker [25]) Hence our condition is also given by $R_\varphi = 0$.

Definition. A connection of a quasi Finsler type satisfying

$$\nabla_u{}^{(h)} v = \nabla_u{}^{(v)} v, \quad \dot{\nabla}_u{}^{(h)} v = \dot{\nabla}_u{}^{(v)} v, \tag{3.10}$$

for any quasi vector fields u and v is called *a connection of Finsler type.*

The reason why the above connection is called the *connection of Finsler type* will be stated at the end of this section.

For the covariant differentiations with respect to the connection of Finsler type, there is no difference between (h) and (v), therefore we shall denote them simply by the notation ∇ and $\dot{\nabla}$ in this case.

Theorem 3.2. *With respect to the connection of Finsler type, the covariant differentiations commute with the lift operators, i.e.,*

$$l\nabla_u = \nabla_{u^h}\cdot l\,, \quad l\dot{\nabla}_u = \dot{\nabla}_{u^v}\cdot l\,. \tag{3.11}$$

Proof. We shall verify (3.11), as an example, in the case where f is a quasi tensor field of type (1, 2) and l is of type l^{+}_{+-}. For any vector fields $V=v_1{}^h+v_2{}^v$ and $W=w_1{}^h+w_2{}^v$, we find

$$\begin{aligned} l^{+}_{+-}(\nabla_u f)\cdot(V,\, W) &= (\nabla_u f\cdot(v_2,\, w_1))^v \\ &= (\nabla_u\cdot f(v_2,\, w_1)-f(\nabla_u v_2,\, w_1)-f(v_2,\, \nabla_u w_1))^v \\ &= \nabla_{u^h}\cdot(l^{+}_{+-}f\cdot(V,\, W))-l^{+}_{+-}f\cdot(\nabla_{u^h}V,\, W))-l^{+}_{+-}f\cdot(V,\, \nabla_{u^h}W) \\ &= \nabla_{u^h}\cdot(l^{+}_{+-}f)\cdot(V, W)\,. \end{aligned}$$

The other cases are also directly verified by similar method.

Now we are in a position to show the components of the linear connections of the above mentioned several types in terms of canonical coordinates. If $\overset{0}{\Gamma}$ is a linear connection of $T(M)$, for any linear connection Γ, there exists a tensor K satisfying $\Gamma=\overset{0}{\Gamma}+K$. The components of K in canonical coordinates are given by (1.8). In order to determine the components of Γ, we consider the quantity $\gamma^i{}_{kj}$ whose transformation law is the one satisfied by affine connections (the present $\gamma^i{}_{kj}$ is, however not always a function of x only, but may be a function of x and y). Then Γ whose components are given by

$$\begin{cases} \overset{0}{\Gamma}{}^i{}_{kj} = \gamma^i{}_{kj}, \quad \overset{0}{\Gamma}{}^i{}_{k\bar{j}} = 0, \quad \overset{0}{\Gamma}{}^i{}_{\bar{k}j} = 0, \quad \overset{0}{\Gamma}{}^i{}_{\bar{k}\bar{j}} = 0, \quad \overset{0}{\Gamma}{}^{\bar{i}}{}_{\bar{k}\bar{j}} = 0, \\ \overset{0}{\Gamma}{}^{\bar{i}}{}_{kj} = y^l(\partial_l\gamma^i{}_{kj}-\varphi^p{}_l\dot{\partial}_p\gamma^i{}_{kj}), \quad \overset{0}{\Gamma}{}^{\bar{i}}{}_{k\bar{j}} = \gamma^i{}_{kj}, \quad \overset{0}{\Gamma}{}^{\bar{i}}{}_{\bar{k}j} = \gamma^i{}_{kj}, \end{cases} \tag{3.12}$$

is a linear connection of $T(M)$ by virtue of (1.2). Combining the above $\overset{0}{\Gamma}$ with (1.7) and (1.8) we have components of the most general form of linear connections $\Gamma=\overset{0}{\Gamma}+K$ of $T(M)$.

Now, the direct calculation shows us that the tensors K, H, $\bar{K}$ and $\bar{H}$, which are defined above with respect to the $\overset{0}{\Gamma}$, are given by the formulae $K=l^{+}_{--}(b)$, $H=l^{+}_{+-}(d)$, $\bar{K}=l^{-}_{-+}(0)$ and $\bar{H}=l^{-}_{++}(0)$, where

$$\begin{aligned} b^{i}{}_{kj} &= -y^{l}\partial_{l}\gamma^{i}{}_{kj}+y^{l}\varphi^{m}{}_{l}\dot{\partial}_{m}\gamma^{i}{}_{kj}+\partial_{k}\varphi^{i}{}_{j}-\varphi^{m}{}_{k}\dot{\partial}_{m}\varphi^{i}{}_{j}-\varphi^{i}{}_{m}\gamma^{m}{}_{kj} \\ &\quad +\gamma^{i}{}_{km}\varphi^{m}{}_{j}+\gamma^{i}{}_{mj}\varphi^{m}{}_{k}\,, \\ d^{i}{}_{kj} &= \dot{\partial}_{k}\varphi^{i}{}_{j}-\gamma^{i}{}_{kj}\,. \end{aligned}$$

Hence the formula (3.8) gives us that a linear connection Γ of h-f-type is given by

$$(3.13)\quad \begin{cases} \Gamma^{i}{}_{kj} = \gamma^{i}{}_{kj}+a^{i}{}_{kj}+c^{i}{}_{mj}\varphi^{m}{}_{k}+e^{i}{}_{km}\varphi^{m}{}_{j}+g^{i}{}_{lm}\varphi^{l}{}_{k}\varphi^{m}{}_{j}\,, \\ \Gamma^{i}{}_{\bar{k}j} = c^{i}{}_{kj}+g^{i}{}_{km}\varphi^{m}{}_{j},\ \Gamma^{i}{}_{k\bar{j}} = e^{i}{}_{kj}+g^{i}{}_{mj}\varphi^{m}{}_{k},\ \Gamma^{i}{}_{\bar{k}\bar{j}} = g^{i}{}_{kj}, \\ \Gamma^{\bar{i}}{}_{kj} = \partial_{k}\varphi^{i}{}_{j}-\varphi^{i}{}_{m}\gamma^{m}{}_{kj}+\gamma^{i}{}_{km}\varphi^{m}{}_{j}-\varphi^{i}{}_{m}a^{m}{}_{kj}-\varphi^{i}{}_{m}c^{m}{}_{lj}\varphi^{l}{}_{k} \\ \qquad +f^{i}{}_{km}\varphi^{m}{}_{j}+h^{i}{}_{lm}\varphi^{l}{}_{k}\varphi^{m}{}_{j}-\varphi^{i}{}_{l}e^{l}{}_{km}\varphi^{m}{}_{j}-\varphi^{i}{}_{r}g^{r}{}_{lm}\varphi^{l}{}_{k}\varphi^{m}{}_{j}, \\ \Gamma^{\bar{i}}{}_{\bar{k}j} = \dot{\partial}_{k}\varphi^{i}{}_{j}-\varphi^{i}{}_{l}c^{l}{}_{kj}+h^{i}{}_{km}\varphi^{m}{}_{j}-\varphi^{i}{}_{r}g^{r}{}_{kl}\varphi^{l}{}_{j}\,, \\ \Gamma^{\bar{i}}{}_{k\bar{j}} = \gamma^{i}{}_{kj}+f^{i}{}_{kj}+h^{i}{}_{mj}\varphi^{m}{}_{k}-\varphi^{i}{}_{l}e^{l}{}_{kj}-\varphi^{i}{}_{r}g^{r}{}_{lj}\varphi^{l}{}_{k}\,, \\ \Gamma^{\bar{i}}{}_{\bar{k}\bar{j}} = h^{i}{}_{kj}-\varphi^{i}{}_{r}g^{r}{}_{kj}\,, \end{cases}$$

where a, c, e, f, g and h are arbitrary quasi tensor fields of type (1, 2). Similarly we get the components of quasi Finsler type, i.e., for any quasi tensor fields a, c, f and h of type (1, 2) as follows:

$$(3.14)\quad \begin{cases} \Gamma^{i}{}_{kj} = \gamma^{i}{}_{kj}+a^{i}{}_{kj}+c^{i}{}_{mj}\varphi^{m}{}_{k}\,,\quad \Gamma^{i}{}_{\bar{k}j} = c^{i}{}_{kj}\,, \\ \Gamma^{i}{}_{k\bar{j}} = 0\,,\quad \Gamma^{i}{}_{\bar{k}\bar{j}} = 0\,, \\ \Gamma^{\bar{i}}{}_{kj} = \partial_{k}\varphi^{i}{}_{j}-\varphi^{i}{}_{m}\gamma^{m}{}_{kj}+\gamma^{i}{}_{km}\varphi^{m}{}_{j}-\varphi^{i}{}_{m}a^{m}{}_{kj}-\varphi^{i}{}_{m}c^{m}{}_{lj}\varphi^{l}{}_{k} \\ \qquad +f^{i}{}_{km}\varphi^{m}{}_{j}+h^{i}{}_{lm}\varphi^{l}{}_{k}\varphi^{m}{}_{j}\,, \\ \Gamma^{\bar{i}}{}_{\bar{k}j} = \dot{\partial}_{k}\varphi^{i}{}_{j}-\varphi^{i}{}_{l}c^{l}{}_{kj}+h^{i}{}_{kl}\varphi^{l}{}_{j},\quad \Gamma^{\bar{i}}{}_{k\bar{j}} = \gamma^{i}{}_{kj}+f^{i}{}_{kj}+h^{i}{}_{mj}\varphi^{m}{}_{k}\,, \\ \Gamma^{\bar{i}}{}_{\bar{k}\bar{j}} = h^{i}{}_{kj}\,. \end{cases}$$

This is a general form of components of a linear connection of quasi Finsler type.

The covariant differentiations with respect to a connection of h-f-type are given by

$$
(3.15)\quad \begin{cases} \nabla_u^{(h)} v = u^k(\partial_k v^i - \dot{\partial}_p v^i \varphi^p{}_k + (\gamma^i{}_{kj} + a^i{}_{kj})v^j)\dfrac{\partial}{\partial x^i}\,, \\ \dot{\nabla}_u^{(h)} v = u^k(\dot{\partial}_k v^i + c^i{}_{kj} v^j)\dfrac{\partial}{\partial x^i}\,. \end{cases}
$$

Similarly the covariant differentials with respect to a connection of v-f-type are also given by

$$
(3.16)\quad \begin{cases} \nabla_u^{(v)} v = u^k(\partial_k v^i - \dot{\partial}_p v^i \varphi^p{}_k + (\gamma^i{}_{kj} + f^i{}_{kj})v^j)\dfrac{\partial}{\partial x^i}\,, \\ \nabla_u^{(v)} v = u^k(\dot{\partial}_k v^i + h^i{}_{kj} v^j)\dfrac{\partial}{\partial x^i}\,. \end{cases}
$$

These two formulae give us that a connection Γ is of Finsler type if and only if Γ is a connction of quasi Finsler type satisfying $f=a$ and $h=c$. Hence the components of an arbitrary connection of Finsler type are written as

$$
(3.17)\quad \begin{cases} \Gamma^i{}_{kj} = \gamma^i{}_{kj} + a^i{}_{kj} + c^i{}_{mj}\varphi^m{}_k\,, \quad \Gamma^i{}_{\bar{k}j} = c^i{}_{kj}\,, \\ \Gamma^i{}_{k\bar{j}} = 0\,, \quad \Gamma^i{}_{\bar{k}\bar{j}} = 0\,, \\ \Gamma^{\bar{i}}{}_{kj} = \partial_k \varphi^i{}_j - \varphi^i{}_m \gamma^m{}_{kj} + \gamma^i{}_{km}\varphi^m{}_j - \varphi^i{}_m a^m{}_{kj} - \varphi^i{}_m c^m{}_{lj}\varphi^l{}_k \\ \qquad\qquad + a^i{}_{km}\varphi^m{}_j + c^i{}_{lm}\varphi^l{}_k \varphi^m{}_j\,, \\ \Gamma^{\bar{i}}{}_{\bar{k}j} = \dot{\partial}_k \varphi^i{}_j - \varphi^i{}_l c^l{}_{kj} + c^i{}_{kl}\varphi^l{}_j\,, \\ \Gamma^{\bar{i}}{}_{k\bar{j}} = \gamma^i{}_{kj} + a^i{}_{kj} + c^i{}_{lj}\varphi^l{}_k\,, \quad \Gamma^{\bar{i}}{}_{\bar{k}\bar{j}} = c^i{}_{kj}\,, \end{cases}
$$

where a and c are arbitrary quasi tensor fields on M.

Remark. Let there be given a Finsler metric, φ be the φ-connection derived from the given Finsler metric, and γ^* be the Cartan's connection. If a linear connection Γ is of h-f-type and satisfies $\gamma + a = \gamma^*$ and $c^i{}_{kj} = \frac{1}{2} g^{im}\dot{\partial}_m g_{kj}$, then the covariant differential $\nabla^{(h)}$ coincides with the covariant differential defined by Cartan [2]. Moreover if Γ is assumed to be of Finsler type satisfying the above conditions, then Γ is uniquely deterimined. If Γ is of h-f-type and satisfies $\gamma + a = \dot{\partial}\varphi$, the $\nabla^{(h)}$ coincides with Berwald's. The connections of Rund and others are also derived from our connenctions of h-f-type, quasi Finsler type or Finsler type.

§4. Generalized almost complex structures

If a manifold admits an r-dimensional complex distribution Π^r satisfying the relation $\Pi^r \cap \bar{\Pi}^r = \{0\}$ at each point of the manifold where $\bar{\Pi}^r$ means a complex conjugate distribution of Π^r, then the manifold is called *a manifold with a Π_r-structure* (Ichijyô [7]). Hereafter we treat a manifold M of φ-space whose dimension is even $(=2m)$.

If the tangent bundle $T(M)$ over a φ-space M^{2m} admits a horizontal Π_m-structure (a Π_m-structure whose distribution Π^m is horizontal), then $T(M)$ admits an f_m-structure F_1 satisfying

$$(4.1) \qquad F_1^3+F_1 = 0\,, \text{ rank of } (F_1) = 2m\,,\ F_1 = l\,\Xi(f)\,,\ f^2 = -1\,.$$

Because, the complexification $(\Phi^h)^c$ of the horizontal distribution Φ^h is written in the form $(\Phi^h)^c_z = \Pi^m_z + \bar{\Pi}^m_z$ for any point $z \in T(M)$ where Π^m is the given horizontal complex distribution. Hence $(T)^c_z = \Pi_z^m + \bar{\Pi}_z^m + (\Phi^v)^c_z$ where T^c and $(\Phi^v)^c$ are complexifications of $T(M)$ and Φ^v respectively. Now, if we denote by Ψ the projection operator of Π^m, then $\bar{\Psi}$ is the projection operator of $\bar{\Pi}^m$ and the relations

$$\Psi^2 = \Psi\,, \quad \Psi\bar{\Psi} = 0\,, \quad \bar{\Psi}^2 = \bar{\Psi}\,, \quad \bar{\Psi}\Psi = 0$$

hold good. Here on putting

$$(4.2) \qquad F_1 = -\sqrt{-1}(\Psi - \bar{\Psi})$$

it follows that F_1 is a real tensor field of type (1, 1) on $T(M)$ and satisfies $F_1^3+F_1=0$ and the rank of $(F_1)=2m$. At the same time (4.2) gives us $F_1 = l\,\Xi(f)$ and $(F_1^3+F_1)u^h = ((f^3+f)u)^h$ for any quasi vector field u, from which we have $f^2=-1$.

Moreover, since $-\sqrt{-1}$ is the eigen value of F_1, the distribution, which is constructed by eigen vector space corresponding to $-\sqrt{-1}$ at each point, coincides with Π^m by virtue of (4.2). Hence, thus defined f_m-structure F_1 is called *an f_m-structure associated with horizontal Π_m-structure,* and conversely, the distribution Π^m is called *a complex distribution associated with F_1*. We call also hereafter the φ-space M which admits a quasi tensor field f satsfying $f^2=-1$ *a generalized almost complex space* (or simply *g.a.c.s.*) and call f *a quasi almost complex structure* (or simply *g.a.c.str.*).

Conversely, if M admits a quasi tensor field f of type $(1, 1)$ satisfying $f^2=-1$, then $F_1=l_-^-(f)$ forms an f_m-structure of $T(M)$ and the complex distribution associated with F_1 forms apparently a horizontal Π_m-structure. Thus the above argument leads us to

Proposition 4. 1. *In order that a φ-space M^{2m} admits g.a.c. str. f, it is necessary and sufficient that the tangent bundle $T(M)$ over M admits a horizontal Π_m-structure.*

Proposition 4. 2. *In order that a φ-space M^{2m} admits g.a.c. str. it is necessary and sufficient that the tangent bundle $T(M)$ over M^{2m} admits a vertical Π_m-structure.*

Proof. In *g.a.c.s.*, $l_-^-(f)=F_1$ gives a horizontal Π_m structure. If we put

$$F_2 = -J^*F_1J^*, \tag{4. 3}$$

where J^* is the natural almost complex structure, then we have

$$\text{rank of } (F_2) = 2m, \quad F_2^3+F_2 = 0\,, \quad F_2(u^v) = (fu)^v, \quad F_2(u^h) = 0\,,$$

from which we have $F_2=l_+^+(f)$. Hence the same argument as Prop. 4. 1 gives our result.

In the tangent bundle over *g.a.c.s.*, let the horizontal complex distribution Π^m associated with F_1 be parallel with respect to a linear connection Γ, then for any vector field U the equation $\nabla_U F_1 \cdot F_1=0$ holds good, where ∇ means the covariant derivative with respect to the Γ. Because, a necessary and sufficient condition for the complex distribution associated with an f-structure F_1 to be parallel with respect to a given connection Γ is that the equation $\nabla_U F_1 \cdot F_1=0$ holds good for any vector field U. (Ichijyo [7]) The converse is also true.

Theorem 4. 1. *In the tangent bundle over g.a.c.s., a necessary and sufficient condition for the horizontal distribution Π^m associated with F_1 to be parallel with respect to the given connection Γ is that the connection Γ is of h-f-type and satisfies $\nabla^{(h)}f=0$ and $\dot{\nabla}^{(h)}f=0$.*

Proof. Since the connection Γ is real and $\Phi^h = \mathfrak{Re}[\Pi^m + \bar{\Pi}^m]$, if the distribution Π^m is parallel with respect to the Γ, so are $\bar{\Pi}^m$ and Φ^h. Hence Γ is of h-f-type. Now, we have

$$\begin{aligned} \nabla_{u^h} F_1 \cdot (v^h) &= \nabla_{u^h} \cdot F_1(v^h) - F_1(\nabla_{u^h} v^h) \\ &= (\nabla_u^{(h)} \cdot (fv) - f(\nabla_u^{(h)} v))^h \\ &= (\nabla_u^{(h)} f \cdot (v))^h . \end{aligned}$$

Similarly we have $\nabla_{u^v} F_1 \cdot (v^h) = (\dot{\nabla}_u^{(h)} f \cdot (v))^h$. These results lead us to $\nabla_{u^h} F_1 \cdot (F_1 v^h) = (\nabla_u^{(h)} f \cdot f(v))^h$, $\nabla_{u^v} F_1 \cdot (F_1 v^h) = (\dot{\nabla}_u^{(h)} f \cdot (fv))^h$. Hence, from the preceding consideration, we have $\nabla^{(h)} f \cdot f = 0$ and $\dot{\nabla}^{(h)} f \cdot f = 0$. Since the rank of $(f) = 2m$, we obtain $\nabla^{(h)} f = 0$ and $\dot{\nabla}^{(h)} f = 0$. The converse also follows from above consideration.

The present author proved in his paper [8] that in a manifold with a Π_m-structure a necessary and sufficient condition that there exists a symmetric linear connection Γ with respect to which the distribution Π^m is parallel is that the distributions Π^m and $\mathfrak{Re}[\Pi^m + \bar{\Pi}^m]$ are both integrable. Hence, in the tangent bundle over *g.a.c.s.*, there exists a symmetric linear connection Γ with respect to which the horizontal complex distribution Π^m associated with F_1 is parallel if and only if the distribution Π^m is integrable and the equation $R_\varphi = 0$ holds good.

Now applying the Ishihara-Yano's well known results [9] with respect to integrable f-structures to F_1 and F_2, we establish the

Theorem 4. 2. *The f-structure F_1 which is defined by* (4. 1) *on the tangent bundle $T(M)$ over g.a.c.s. is integrable if and only if the quasi almost complex structure f is a complex structure and the equation $R_\varphi = 0$ holds good.*

Proof. The structures F_1 is integrable if and only if the Nijenhuis tensor of F_1 vanishes identically. Since $F_1 = l^-_-(f)$, it is enough to verify the Prop. 2. 1. in the case where $a = f$ and $b = c = d = 0$. The fifth equation of Prop. 2. 1 means $\dot{\nabla}^\# f = 0$, which implies that f is an almost complex structure of M. The first equation leads us to $f[fu, v]^\# + f[u, fv]^\# - [fu, fv]^\# + [u, v]^\# = 0$, which means that f is a complex structure of M by virtue of the fact that f is an almost complex structure of M. The second equation is, therefore, equivalent to $R_{\varphi}=0$. The rest of equations of Prop. 2.1 hold identically in the present case.

Theorem 4.3. *The f-structure F_2 which is defined by* (4.3) *on the tangent bundle $T(M)$ over g.a.c.s. is integrable if and only if the quasi almost complex structure f constructs a complex structure in each fibre and the quasi tensors R_{φ} and $\nabla^{\sharp} f$ vanish identically.*

Proof. Since $F_2=l^{+}_{+}(f)$, it is enough to verify the prop. 2.1 in the case where $d=f$ and $a=b=c=0$. The first, third and fifth equtions hold identically in this case. The second equation means $R_{\varphi}=0$ and sixth equation means $\nabla^{\sharp} f=0$. The fourth equation leads us to

$$f\dot{\nabla}^{\sharp} f\cdot(v, u)-f\dot{\nabla}^{\sharp} f\cdot(u, v)+\dot{\nabla}^{\sharp} f\cdot(u, fv)-\dot{\nabla}^{\sharp} f\cdot(v, fu)=0\,.$$

Q.E.D.

The preceding discussion is concerned with f-structures in the tangent bundle over *g.a.c.s.*. We are now in a position to consider the almost complex structures in the tangent bundle over *g.a.c.s.*.

Theorem 4.4. *In the tangent bundle $T(M)$ over g.a.c.s., there exist three kinds of almost complex structures F, F' and F'', i.e.,*

$$\begin{aligned}
F\ &= l^{-}_{-}(f)+l^{+}_{+}(f)\,,\\
F' &= l^{-}_{-}(f)-l^{+}_{+}(f)+\mu l^{+}_{-}(1)+\tau l^{+}_{+}(1)\,,\\
F'' &= \alpha l^{-}_{-}(f)+\beta l^{-}_{-}(1)+\rho l^{-}_{+}(1)+\sigma l^{-}_{+}(f)-\alpha l^{+}_{+}(1)-\beta l^{+}_{+}(f)\\
&\qquad -\frac{\rho(\alpha^2-\beta^2+1)+2\alpha\beta\sigma}{\rho^2+\sigma^2}\, l^{+}_{-}(1)+\frac{\sigma(\alpha^2-\beta^2+1)-2\alpha\beta\rho}{\rho^2+\sigma^2}\, l^{+}_{-}(f)\,,
\end{aligned}$$

where α, β, ρ, σ, μ and τ are scalar fields on $T(M)$ satisfying $\rho^2+\sigma^2\neq 0$.

Proof. By direct calculation we can easily verify

$$F^2=F'^2=F''^2=-E^{2n}\,.$$

Now the almost complex structures F, F' and F'', which are defined in this way, are the most general ones in the almost complex structures which are generated by 1 and f on $T(M)$ over φ-space. Because, if we put

$$H = \alpha l_{-}^{-}(1)+\beta l_{+}^{+}(1)+\rho l_{+}^{-}(1)+\sigma l_{-}^{+}(1)+\gamma l_{-}^{-}(f)+\delta l_{+}^{+}(f) \\ +\mu l_{+}^{-}(f)+\nu l_{-}^{+}(f)\,,$$

then the relation $H^2=-E^{2n}$ holds good if and only if H is equal to any one of F, F' and F''.

The structure F' depends upon the choice of φ, thereby, if necessary, we denote it by F'_{φ}. Then it is easily seen that $\varphi'=\varphi+\frac{\mu}{2}f-\frac{\tau}{2}f$ is also a φ-connection and the relation $\widetilde{F}_{\varphi'}=F'_{\varphi}$ holds good, where we put $\widetilde{F}=l_{-}^{-}(f)-l_{+}^{+}(f)$. That is to say, if $\mu=\tau=0$, the structure F' coincides with $\widetilde{F}$. If $\sigma=\beta=0$, the structure F'' coincides with J. In the case where $\sigma=1$ and $\alpha=\beta=\rho=0$, we denote F'' by F^*, i.e., $F^*=l_{+}^{-}(f)+l_{-}^{+}(f)$. Then we have

$$\text{(4.4)} \qquad \begin{cases} F\ (u^h)=(fu)^h\,, & F\ (u^v)=(fu)^v\,, \\ \widetilde{F}\ (u^h)=(fu)^h\,, & \widetilde{F}\ (u^v)=-(fu)^v\,, \\ F^*(u^h)=(fu)^v\,, & F^*(u^v)=(fu)^h\,. \end{cases}$$

For the integrability conditions of these almost complex structures, we have first

Theorem 4. 5. *A necessary and sufficient condition for the almost complex structure F to be integrable on the tangent bundle over g.a.c.s. is that, by any quasi vector fields u and v, the following equations are satisfied*:

$$\text{(4.5)} \qquad \begin{cases} f\nabla^{\sharp}f\cdot(u,v)-\nabla^{\sharp}f\cdot(u,fv)=0\,, \\ f\dot{\nabla}^{\sharp}f\cdot(u,v)-\dot{\nabla}^{\sharp}f\cdot(u,fv)=0\,, \\ R_{\varphi}(u,v)+fR_{\varphi}(fu,v)+fR_{\varphi}(u,fv)-R_{\varphi}(fu,fv)=0\,, \\ T_{\varphi}(u,v)+fT_{\varphi}(fu,v)+fT_{\varphi}(u,fv)-T_{\varphi}(fu,fv)=0\,. \end{cases}$$

Proof. It is enough to verify the Prop. 2.1 in the case where $a=f$, $d=f$ and $b=c=0$. In the case under consideration, the third equation of Prop. 2. 1 holds identically, and the second equation coincides with the third equation of (4. 5). The fifth equation of Prop. 2. 1 also leads us to the second equation of (4. 5). The fourth of Prop. 2. 1 implies that

$$f\dot{\nabla}^{\sharp}f\cdot(v,u)-f\dot{\nabla}^{\sharp}f\cdot(u,v)+\dot{\nabla}^{\sharp}f\cdot(u,fv)-\dot{\nabla}^{\sharp}f\cdot(v,fu)=0$$

which holds identically by virtue of the second of (4.5). Finally the first of Prop. 2.1 leads us to

$$f[fu, v]^{\#}+f[u, fv]^{\#}-[fu, fv]^{\#}+[u, v]^{\#}=0\,,$$

which implies the fourth equation (4.5) by virtue of the first equation of (4.5) and Prop. 1.2. The converse also follows at once.

As for the structure F^* and $\widetilde{F}$, similar discussion leads us to

Theorem 4.6. *A necessary and sufficient condition for the almost complex structure F^* on $T(M)$ over g.a.c.s. to be integrable is that the following relations hold good for any quasi vector fields u and v:*

$$(4.6)\qquad \begin{cases} fR_{\varphi}(fu, fv)+\dot{\nabla}^{\#}f\cdot(v, u)-\dot{\nabla}^{\#}f\cdot(u, v)=0\,,\\ fT_{\varphi}(u, v)+\nabla^{\#}f\cdot(v, u)-\nabla^{\#}f\cdot(u, v)=0\,. \end{cases}$$

The integrability condition for the almost complex structure $\widetilde{F}$ is also given by

$$(4.7)\qquad \begin{cases} f\nabla^{\#}f\cdot(u, v)-\nabla^{\#}f\cdot(u, fv)=0\,,\\ \dot{\nabla}^{\#}f\cdot(u, v)-\dot{\nabla}^{\#}f\cdot(v, u)=0\,,\\ R_{\varphi}(u, v)-fR_{\varphi}(fu, v)-fR_{\varphi}(u, fv)-R_{\varphi}(fu, fv)=0\,,\\ T_{\varphi}(u, v)+fT_{\varphi}(fu, v)+fT_{\varphi}(u, fv)-T_{\varphi}(fu, fv)\\ \qquad\qquad -2f\nabla^{\#}f(v, u)-2f\nabla^{\#}f(u, v)=0\,. \end{cases}$$

If the quasi almost complex structure f is an almost complex structure on M and the connection φ is given by $\varphi^i{}_j=\gamma^i{}_{jk}y^k$ where $\gamma^i{}_{jk}$ is a linear connection of M, then the almost complex structure F on $T(M)$ is a well known one (Tanno [24]). Moreover the integrability condition of the structure F in this case becomes simple, i.e.

$$(4.5')\qquad \begin{cases} f:\ \textit{integrable}\\ R(u, v)+fR(fu, v)+fR(u, fv)-R(fu, fu)=0\,,\\ f\nabla^{\#}f\cdot(u, v)-\nabla^{\#}f\cdot(u, fv)=0\,, \end{cases}$$

where R is a curvature tensor of γ, u and v are any vector fields on M.

§ 5. Generalized metric spaces

Let M be, in this section, a homogeneous φ-space and let there be given a quasi almost complex structure f and positive definite generalized metrics g and h (Moòr [16]). Of course g and h are assumed to be positively homogeneous of degree 0 with respect to y. Now we put

$$G = l_{--}(g), \quad H = l_{++}(h), \quad G^* = G+H. \tag{5.1}$$

Then we see, for any vector fields $U=u_1^h+u_2^v$ and $V=v_1^h+v_2^v$,

$$G^*(U, V) = g(u_1, v_1)+h(u_2, v_2). \tag{5.2}$$

Hence let us adopt (5.2) as the definition of an inner product of vectors U and V, then $T(M)$ is a Riemannian space with respect to G^*. We call this metric G^* *a canonical metric* with respect to g and h.

If the manifold M is a Finsler space whose Finsler metric is g, and it is assumed that $h=g$, then the canonical metric G^* is the so-called *lifted metric* (Matsumoto [14]). If the metric g is, moreover, a Riemannian metric, and it is assumed that $\varphi^i{}_j=\left\{{i \atop jk}\right\} y^k$ where $\left\{{i \atop jk}\right\}$ is the Christoffel's symbol, then the lifted metric is the one introduced by Sasaki [22]. In this paper the canonical metric G^* satisfying $h=g$ is called simply a *lifted metric.*

The canonical metric G^* constructs an almost hermitian structure together with the natural almost complex structure J^* if and only if the metric G^* is a lifted one. Because, the condition to be verified is $G^*(U, V)=G^*(J^*U, J^*V)$, where U and V are any vector fields. This is reduced to

$$g(u_1, v_1)+h(u_2, v_2) = g(u_2, v_2)+h(u_1, v_1),$$

where $U=u_1^h+u_2^v$ and $V=v_1^h+v_2^v$. Since this condition must be satisfied by any quasi vector fields u_1, u_2, v_1 and v_2, our demanding condition becomes $h=g$. Then we obtain

Theorem 5.1. *The lifted metric G^* constructs an almost Kähler structure together with the natural almost complex structure J^* if and only if the followings hold identically for any quasi vector fields u_1, u_2 and u_2:*

$$(5.3)\qquad \begin{cases} g \text{ is a Finsler metric,} \\ S_{1,2,3}[g(R_\varphi(u_1, u_2), u_3)] = 0\,, \\ \nabla^\# g\cdot(u_1, u_3, u_2) - \nabla^\# g\cdot(u_2, u_3, u_1) - g(T_\varphi(u_1, u_2), u_3) = 0\,. \end{cases}$$

Proof. It is well known that the lifted metric G^* is an almost Kähler structure together with the structure J^* if and only if the 2-form $\omega(U, V) = G^*(J^*U, V) = -g(u_2, v_1) + g(u_1, v_2)$ is closed. Thereby, taking account of $d\omega$, we obtain

$$\begin{aligned} d\omega(u_1^v, u_2^v, u_3^v) &= 0\,, \\ d\omega(u_1^h, u_2^h, u_3^h) &= S_{1,2,3}[g(R_\varphi(u_1, u_2), u_3)]\,, \\ d\omega(u_1^h, u_1^h, u_3^v) &= \nabla^\# g\cdot(u_2, u_3, u_1) - \nabla^\# g\cdot(u_1, u_3, u_2) + g(T_\varphi(u_1, u_2), u_3)\,, \\ d\omega(u_1^h, u_2^v, u_3^v) &= \dot{\nabla}^\# g\cdot(u_1, u_2, u_3) - \dot{\nabla}^\# g\cdot(u_1, u_3, u_2)\,. \end{aligned}$$

The right-hand side of the last equation vanishes identically when and only when $\dot{\partial}_k g_{ij} = \dot{\partial}_j g_{ik}$ holds in terms of canonical coordinates. Since g is positively homogeneous of degree 0 with respect to y, the last condition reduces to $y^j \dot{\partial}_k g_{ij} = 0$, that is to say, g is a Finsler metric (Moòr [16]). The converse follows from the properties of a Finsler metric. Q.E.D.

A manifold M^{2m} is said to admit *a generalized almost hermitian structure* or simply *g.a.h.str.* (f, g) when and only when the manifold M^{2m} admits a generalized metric g and a quasi almost complex structure f which are both positively homogeneous of degree 0 with respect to y and satisfy $g(u, v) = g(fu, fv)$.

Theorem 5.2. *In order that a manifold M^{2m} admits g.a.h.str. (f, g), it is necessary and sufficient that the tangent bundle $T(M^{2m})$ over M^{2m} admits a complex m-dimensional distribution Π^m which is horizontal and null with respect to a canonical metric G^*.*

Proof. Let us assume M^{2m} admits *g.a.h.str.* (f, g), then we shall consider quasi eigen vector space $\pi^m(x, y)$ corresponding to the eigen value $-\sqrt{-1}$ of f. Thus we have a complex distribution Π^m which is constructed, in any point $z = (x, y)$ of $T(M^{2m})$, by the horizontally lifted vectors $\Lambda_{(\alpha)} = \lambda_{(\alpha)}{}^h$, where $\lambda_{(\alpha)}$ are basic quasi vector fields belonging to $\pi^m(x, y)$ $(\alpha = 1, 2, \cdots, m)$. Then Π^m is horizontal and satisfies

$$G^*(\Lambda_{(\alpha)}, \Lambda_{(\beta)}) = g(\lambda_{(\alpha)}, \lambda_{(\beta)}) = g(\sqrt{-1}f\lambda_{(\alpha)}, \sqrt{-1}f\lambda_{(\beta)})$$
$$= -g(\lambda_{(\alpha)}, \lambda_{(\beta)}).$$

Consequently Π^m is null with respect to G^*.

Conversely, if Π^m is a horizontal null distribution with respect to G^* in $T(M^{2m})$, then $T(M^{2m})$ admits a horizontal Π_m-structure (Ichijyô [7]). Thus Prop. 4.1 shows us that M^{2m} admits a quasi almost complex structure f, which is positively homogeneous of degree 0 with respect to y by virtue of the fact that Π^m is horizontal and φ is positively homogeneous of degree 0. The f-structure $F_1 = l_{-}^{-}(f)$ satisfies $\sqrt{-1}F_1\Lambda_{(\alpha)} = \Lambda_{(\alpha)}$ from (4.2). If we put $\Lambda_{(\alpha)} = (a_{(\alpha)} + \sqrt{-1}b_{(\alpha)})^h$ where $a_{(\alpha)}$ and $b_{(\alpha)}$ are real quasi vector fields, then we have $a_{(\alpha)} = -fb_{(\alpha)}$ and $b_{(\alpha)} = fa_{(\alpha)}$. Hence, from these results and the relation $G^*(\Lambda_{(\alpha)}, \Lambda_{(\beta)}) = g(\lambda_{(\alpha)}, \lambda_{(\beta)}) = 0$, we have $g(a_{(\alpha)}, a_{(\beta)}) = g(b_{(\alpha)}, b_{(\beta)})$ and $g(b_{(\alpha)}, a_{(\beta)}) = -g(a_{(\alpha)}, b_{(\beta)})$. Thus we obtain $g(fa_{(\alpha)}, fa_{(\beta)}) = g(a_{(\alpha)}, a_{(\beta)})$, $g(fa_{(\alpha)}, fb_{(\beta)}) = g(a_{(\alpha)}, b_{(\beta)})$ and $g(fb_{(\alpha)}, fb_{(\beta)}) = g(b_{(\alpha)}, b_{(\beta)})$. Consequently we obtain $g(fu, fv) = g(u, v)$. Q.E.D.

If a manifold M^{2m} admits *g.a.h.str.* (f, g), then, in the tangent bundle $T(M^{2m})$ over M^{2m}, the metric $G^* = l_{--}(g) + l_{++}(h)$ together with $F_1 = l_{-}^{-}(f)$ constructs a so-callec $(F-G)$-structure. Indeed, F_1 and G^* become an $(F-G)$-structure if and only if F_1 is an f-structure and G^* is a Riemannian metric satisfying

$$(5.4) \qquad G^*(F_1X, F_1Y) + G^*(X, MY) = G^*(X, Y), \quad (M = 1 + F_1^2)$$

for any vector fields X and Y (Yano [27], Ichijyô [7]). Now, from the relations $M(u^h) = 0$ and $M(u^v) = u^v$, the *g.a.h.str.* (f, g) guarantees that (5.4) holds identically. Besides, the Prop. 4.1. and (5.4) show us directly that the converse of the above statement holds good.

Now we shall be concerned with the almost complex structures F, $\tilde{F}$ and F^* defined by (4.4).

The structure F and the canonical metric G^* become an almost hermitian structure if g, h and f satisfy

$$(5.5) \qquad g(fu, fv) = g(u, v), \quad h(fu, fv) = h(u, v),$$

for any quasi vector fields u and v. Because, the relations (5.2) and $(4.4)_1$ show us that the condition $G^*(FX, FY)=G^*(X, Y)$ is equivalent to (5.5). Hence the converse is also true. We establish moreover

Theorem 5.3. *In order that the almost complex structure F and the canonical metric G^* construct an almost Kähler structure, it is necessary and sufficient that the equation* (5.5) *and the following relations hold good for any quasi vector fields u_1, u_2 and u_3:*

(5.6)
$$\begin{cases} S_{1,2,3}[\nabla^{\#}g\cdot(fu_1, u_2, u_3)+g(\nabla^{\#}f\cdot(u_3, u_2), u_1)+g(fT_{\varphi}(u_1, u_2), u_3)] = 0\,, \\ \dot{\nabla}^{\#}g\cdot(fu_1, u_2, u_3)+g(\dot{\nabla}^{\#}f\cdot(u_1, u_3), u_2)-h(fR_{\varphi}(u_1, u_2), u_3) = 0\,, \\ \nabla^{\#}h\cdot(fu_1, u_2, u_3)+h(\nabla^{\#}f\cdot(u_1, u_3), u_2) = 0\,, \\ S_{1,2,3}[\dot{\nabla}^{\#}h\cdot(fu_1, u_2, u_3)+h(\dot{\nabla}^{\#}f\cdot(u_3, u_2), u_1)] = 0\,. \end{cases}$$

Proof. For the same reason as in Theorem 5.1, it is enough to show that the condition for the 2-form $\omega(X, Y)=G^*(FX, Y)$ to be closed is given by (5.6). To see this, taking account of $g(fu, v) =-g(u, fv)$, $\omega(u^h, v^h)=g(fu, v)$, $\omega(u^h, v^v)=0$, $\omega(u^v, v^h)=0$ and $\omega(u^v, v^v)=h(fu, v)$, we obtain by direct calculation

$$\begin{aligned} d\omega(u_1^h, u_2^h, u_3^h) &= S_{1,2,3}[\nabla^{\#}g\cdot(fu_1, u_2, u_3)+g(\nabla^{\#}f\cdot(u_3, u_2), u_1) \\ &\qquad +g(fT_{\varphi}(u_1, u_2), u_3)]\,, \\ d\omega(u_1^h, u_2^h, u_3^v) &= \dot{\nabla}^{\#}g\cdot(fu_1, u_2, u_3)+g(\dot{\nabla}^{\#}f\cdot(u_1, u_3), u_2) \\ &\qquad -h(fR_{\varphi}(u_1, u_2), u_3)\,, \\ d\omega(u_1^h, u_2^v, u_3^v) &= \nabla^{\#}h\cdot(fu_2, u_3, u_1)+h(\nabla^{\#}f\cdot(u_2, u_1), u_3)\,, \\ d\omega(u_1^v, u_2^v, u_3^v) &= S_{1,2,3}[\dot{\nabla}^{\#}h\cdot(fu_1, u_2, u_3)+h(\dot{\nabla}^{\#}f\cdot(u_3, u_2), u_1)]\,. \end{aligned}$$

Therefore, it follows from (5.5) that the proof is complete.

Let us consider the similar consideration on the almost complex structure F^* and $\widetilde{F}$ in the tangent bundle of *g.a.c.s.*.

If the structure F^* and the canonical metric G^* construct an almost hermitian structure in $T(M^{2m})$, then it follows that

$$G^*(u^v, v^v) = G^*((fu)^h, (fv)^h) = g(fu, fv)\,,$$

Hence we obtain, for any quasi vector fields u_1 and u_2,

$$h(u_1, u_2) = g(fu_1, fu_2)\,. \tag{5.7}$$

Conversely, if (5.7) holds, then $G^*(F^*u^v, F^*v^v)=G^*(u^v, v^v)$. Moreover it is easily seen that

$$G^*(F^*u^h, F^*v^h) = h(fu, fv) = g(f^2u, f^2v) = G^*(u^h, v^h),$$
$$G^*(F^*u^h, F^*v^v) = G^*(u^h, v^h).$$

Thus F^* and G^* construct an almost hermitian structure.

Theorem 5.4. *In order that the almost complex structure F^* and the canonical metric G^* construct an almost Kähler structure, it is necessary and sufficient that the equation* (5.7) *and the following relation hold good for any quasi vector fields u_1, u_2 and u_3:*

$$(5.8)\quad \begin{cases} S_{1,2,3}[g(fR_\varphi(u_1, u_2), u_3)] = 0, \\ \nabla^\sharp g\cdot(fu_3, u_1, u_2)-\nabla^\sharp g\cdot(fu_3, u_2, u_1)-g(\nabla^\sharp f\cdot(u_3, u_1), u_2) \\ \qquad +g(\nabla^\sharp f\cdot(u_3, u_2), u_1)-g(T_\varphi(u_1, u_2), fu_3) = 0, \\ \dot{\nabla}^\sharp g\cdot(fu_3, u_1, u_2)-\dot{\nabla}^\sharp g\cdot(u_1, fu_2, u_3)-g(u_1, \dot{\nabla}^\sharp f\cdot(u_2, u_3)) \\ \qquad +g(u_1, \dot{\nabla}^\sharp f\cdot(u_3, u_2)) = 0. \end{cases}$$

Proof. It is enough to show that the condition for the 2-form $\omega(X, Y)=G^*(F^*X, Y)$ to be closed is given by (5.8). To do this, taking account of $\omega(u^h, v^h)=0$, $\omega(u^v, v^v)=0$, $\omega(u^h, v^v)=h(fu, v)=-g(u, fv)$ and $\omega(u^v, v^h)=g(fu, v)$, we obtain that $d\omega(u_1^v, u_2^v, u_3^v)$ vanishes identically and $d\omega(u_1^h, u_2^h, u_3^h)=0$, $d\omega(u_1^h, u_2^h, u_3^v)=0$ and $d\omega(u_1^h, u_2^v, u_3^v)=0$ coincide with $(5.8)_1$, $(5.8)_2$ and $(5.8)_3$ respectively. Q.E.D.

As for the almost complex structure $\widetilde{F}$, the similar theorems will be obtained by slight modification.

§ 6. Finsler spaces and Minkowski spaces

Let M be a manifold with a Finsler metric g, G^* be its lifted metric and φ be the φ-connection derived from the Finsler metric g.

First we shall show, from our stand point, the following theorem obtained by Yano-Davies [28] and Matsumoto [14]:

"*In the tangent bundle $T(M)$ over a Finsler space M, the natural almost complex structure J^* forms an almost Kähler structure together with the lifted metric G^*.*"

To do this, it is sufficient to verify the relations $(5.3)_2$ and

$(5.3)_3$ in Theorem 5.1. In terms of the canonical coordinates, from (1.13), the left-hand side of $(5.3)_2$ is reduced to

$$g_{km}R^m{}_{ji}+g_{jm}R^m{}_{ik}+g_{im}R^m{}_{kj} = y^m(R_{ijmk}+R_{kimj}+R_{jkmi}),$$

which vanishes identically by virtue of the well known relation $R_{ijmk}y^m=-R_{ijkm}y^m$ and the Bianchi's identity in Finsler geometry. It follows further, in the present case, that $T_\varphi=0$ and

$$\nabla_j{}^\sharp g_{ik} = -2C^*{}_{jik|l}y^l \ (\equiv -2C^*{}_{jik|0}), \tag{6.1}$$

where the symbol $|$ means the covariant derivative of Cartan and we put $C^*{}_{ijk}=\frac{1}{2}\dot{\partial}_i g_{jk}=\frac{1}{2}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L$, from which (5.3) holds identically.

This theorem implies directly that the tangent bundle $T(M)$ over a Finsler space M is a Kähler space with respect to the lifted metric and the natural almost complex structure J^* if and only if the horizontal distribution Φ^h is integrable, i.e., $R_\varphi=0$. Hence the tangent bundle $T(M)$ over a Minkowski space M is a Kähler space with respect to the lifted metric G^* and the natural almost complex structure J^*.

Now let M^{2m} be a $2m$-dimensional Finsler space and admit a quasi almost complex structure f which constructs *g.a.h.str.* together with the Finsler metric g.

In the case where the quasi tensor field f is independent of y, i.e., f is an almost complex structure of M, E. Heil obtained the following theorem in his paper [5]: *If a Finsler space M^{2m} admits a complex structure f which constructs g.a.h.str. together with the given Finsler metric g, then the metric g is a Riemannian and the manifold M^{2m} is an Hermitian manifold.* We prove this algebraically.

Indeed, from the assumption, it follows that $\dot{\partial}_k f^i{}_j=0$ and $g_{ij}=g_{pq}f^p{}_i f^q{}_j$, which lead us to $C^*{}_{ijk}=C^*{}_{pqk}f^p{}_i f^q{}_j$. On the other hand, f is integrable, so there exists a canonical coordinate system with respect to which the components of f are written in the form

$$(f^i{}_j)=\begin{pmatrix} 0, & -\delta^\alpha{}_\beta \\ \delta^\alpha{}_\beta, & 0 \end{pmatrix}.$$

Hence we have $C^*_{\alpha\bar{\beta}k}=C^*_{pqk}f^p{}_\alpha f^q{}_{\bar{\beta}}=-C^*_{\bar{\alpha}\beta k}$ and $C^*_{\alpha\beta k}=C^*_{pqk}f^p{}_\alpha f^p{}_\beta=C^*_{\bar{\alpha}\bar{\beta}k}$. These relations lead us to

$$C^*_{\alpha\beta\bar{\gamma}}=C^*_{\alpha\bar{\gamma}\beta}=-C^*_{\bar{\alpha}\gamma\beta}=-C^*_{\gamma\beta\bar{\alpha}}=-C^*_{\bar{\gamma}\bar{\beta}\bar{\alpha}}=-C^*_{\bar{\alpha}\bar{\beta}\bar{\gamma}}=-C^*_{\alpha\beta\bar{\gamma}}\,.$$

Therefore we obtain $C^*_{\alpha\beta\bar{\gamma}}=C^*_{\bar{\alpha}\bar{\beta}\bar{\gamma}}=0$. Similarly we obtain also $C^*_{\alpha\beta\gamma}=C^*_{\bar{\alpha}\bar{\beta}\gamma}=0$. Consequently it follows that $C^*_{ijk}=0$, i.e., g is a Riemannian metric and M is an Hermitian manifold.

In the tangent bundle over M, we shall obtain the

Theorem 6.1. *Let M^{2m} be a Finsler space and admit an almost complex structure f which constructs g.a.h.str. together with the given Finsler metric g. If the tanget bundle $T(M^{2m})$ over the M^{2m} is almost Kählerian with respect to $F=l^-_-(f)+l^+_+(f)$ and the lifted metric G^*, then the metric g is a flat Riemannian metric.*

Proof. The condition for the given structures F and G^* to be almost Kählerian is written, from Theorem 5.3, as

$$(6.2)\quad \begin{cases} \dot{\nabla}^\# g\cdot(fu_1, u_2, u_3)+g(\dot{\nabla}^\# f\cdot(u_1, u_3), u_2)-g(fR_\varphi(u_1, u_2), u_3)=0\,, \\ \nabla^\# g\cdot(fu_1, u_2, u_3)+g(\nabla^\# f\cdot(u_1, u_3), u_2)=0\,, \\ S_{1,2,3}[g(fR_\varphi(u_1, u_2), u_3)]=0\,. \end{cases}$$

Since $\dot{\nabla}^\# f=0$ and $R_\varphi(u_1, u_2)=-R_\varphi(u_2, u_1)$, $(6.2)_1$ gives us $\dot{\nabla}^\# f\cdot(fu_1, u_2, u_3)=-\dot{\nabla}^\# g\cdot(fu_2, u_1, u_3)$. On the other hand $\dot{\nabla}^\# g(u_1, u_2, u_3)=2C^*(u_1, u_2, u_3)=\dot{\nabla}^\# g(u_1, u_3, u_2)$, then it follows that

$$\begin{aligned} g(fR_\varphi(u_2, u_3), u_1) &= \dot{\nabla}^\# g\cdot(fu_2, u_1, u_3) = -\dot{\nabla}^\# g\cdot(fu_1, u_2, u_3) \\ &= -g(fR_\varphi(u_1, u_2), u_3)\,. \end{aligned}$$

Thus $(6.2)_3$ gives us that $\dot{\nabla}^\# g=0$ and $R_\varphi=0$, which show us, by virtue of (1.13), g is a flat Riemannian metric.

Theorem 6.2. *Let a manifold M^{2m} admit g.a.h.str. (f, g) where g is a Minkowski metric. The tangent bundle $T(M^{2m})$ is almost Kählerian with respect to the lifted metric G^* and $F=l^-_-(f)+l^+_+(f)$ if and only if there exists a canonical coordinate system (x, y) in terms of which the components of f are independent of x, and the components of gf are constant.*

Proof. In a Minkowski space, the relations

(6. 3) $$R_{\varphi}=0\,,\quad \nabla^{\#}C^{*}=0\,,\quad \nabla^{\#}g=0\,,$$

hold good and there exists a coordinate system in terms of which the components of g are independent of x. Therefore, in terms of this coordinate system, we have $\gamma^{*i}{}_{jk}=0$ and $\varphi^{i}{}_{j}=0$. Now, $(6.2)_2$ gives us

(6. 4) $$\nabla^{\#}f=0\,,$$

which implies that $f^{i}{}_{j}$ are independent of x. Next, $(6.2)_1$ leads us to

(6. 5) $$\dot{\nabla}^{\#}(gf)=0\,,$$

which implies that $g_{il}f^{l}{}_{j}$ are independent of y, i.e., the components of gf are constant. The sufficiency follows at once from the above calculation.

Theorem 6. 3. *If the almost Kähler structure in Theorem* 6. 2 *is integrable, then an f-structure $F_2=l^{\ddagger}_{+}(f)$ is also integrable and each fibre is Kählerian.*

Proof. Under our assumption, the relation (6. 3), (6. 4) and (6. 5) hold good. The Theorem 4. 5 shows that the following holds good:

(6. 6) $$f\dot{\nabla}^{\#}f\cdot(u_1,u_2)-\dot{\nabla}^{\#}f\cdot(u_1,fu_2)=0\,.$$

From the Theorem 4. 3 and the relations (6. 3), (6. 4) and (6. 6), it follows that F_2 is integrable. Moreover (6. 5) and (6. 6) show us that in each fibre $l^{\ddagger}_{+}(f)$ and $l_{++}(g)$ form a Kähler structure.

Finally let us calculate, in the tangent bundle of a Finsler space, the Christoffel's symbol $\left\{{A \atop BC}\right\}$ corresponding to the lifted metric G^* of a given Finsler metric g.

As is shown in section 3, $\left\{{A \atop BC}\right\}$ is written as $\overset{0}{\Gamma}+K$. As to $\overset{0}{\Gamma}$, we adopt the one given by (3. 12) where we take $\gamma^{*i}{}_{jk}$ (Cartan) as $\gamma^{i}{}_{jk}$ and $\gamma^{*i}{}_{jk}y^k$ as $\varphi^{i}{}_{j}$. After some complicated calculations we have the components of $a, b, c\cdots$ and h which compose a tensor K under consideration as follows:

$$
(6.7)\quad \begin{cases} a^i{}_{kj} = 0, \quad g^i{}_{kj} = C^{*i}{}_{kj|0}, \quad h^i{}_{kj} = C^{*i}{}_{kj}, \quad d^i{}_{kj} = f^i{}_{kj} = 0, \\ c^i{}_{kj} = e^i{}_{kj} = C^{*i}{}_{kj} + \frac{1}{2} g^{il} g_{km} R^m{}_{jl}, \\ b^i{}_{kj} = -\varphi^i{}_m \gamma^{*m}{}_{kj} - y^l \partial_l \gamma^{*i}{}_{kj} + y^l \varphi^p{}_l \dot{\partial}_p \gamma^{*i}{}_{kj} - C^{*i}{}_{kj} \\ \qquad -\frac{1}{2}(\partial_k \varphi^i{}_j + \partial_j \varphi^i{}_k) + \frac{1}{2}(\gamma^{*i}{}_{kq} - C^{*i}{}_{kq|0})\varphi^q{}_j \\ \qquad +\frac{1}{2}(\gamma^{*i}{}_{jp} - C^{*i}{}_{jp|0})\varphi^p{}_k . \end{cases}
$$

Theorem 6.4. *In the tangent bundle over a manifold with a Finsler metric g, the Riemann connection* $\left\{ {A \atop BC} \right\}$ *corresponding to the lifted metric* G^* *of g is of h-f-type if and only if g is a flat Riemann metric.*

Proof. Comparing (3.13) with (6.7), we obtain that

$$
(6.8)\quad \begin{cases} \dot{\partial}_j \varphi^i{}_k - \gamma^{*i}{}_{kj} = 0, \\ -\frac{1}{2} R^i{}_{kj} + c^{*i}{}_{kj} + \frac{1}{2}(c^{*i}{}_{kq|0}\varphi^q{}_j + c^{*i}{}_{jq|0}\varphi^q{}_k) = 0 . \end{cases}
$$

In Finsler geometry it is well known that the first equation of (6.8) is equivalent to $C^{*i}{}_{kj|0}=0$. Thus we have $R^i{}_{kj}=2C^{*i}{}_{kj}$. Since $\left\{ {A \atop BC} \right\}$ gives a symmetric connection, Theorem 3.1 gives us $R^i{}_{kj}=0$. Therefore we obtain $C^{*i}{}_{kj}=0$, i.e., g is a Riemann metric. Moreover from (1.13), $R^i{}_{kj}=0$ shows us that g is flat. The converse is evident.

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REFERENCES

[1] Akbar-Zadeh et E. Bonan : Structure presque Kählerienne naturelle sur la fibrè tangent à une variété finslerienne, C. R. Acad., 258 (1964) 5581-5582.

[2] Cartan, E. : Sur les espace de Finsler, C. R. Acad., 196 (1933) 582-586.

[3] ———— : Les espace de Finsler, Actualités 79, Paris (1934).

[4] Dombrowski, P. : On the geometry of the tangent bundle, J. reine angew. Math., 210 (1962) 73-88.

[5] Heil, E. : A relation between Finsler and Hermitian metrics, Tensor (N. S.), 16 (1965) 1-3.

[6] Ichijyô, Y. : On almost contact metric manifolds admitting parallel fields of null planes, Tôhoku Math. J., 16 (1964) 123-129.

[7] Ichijyô, Y. : Analytic manifolds admitting parallel fields of complex planes, J. Math. Kyoto Univ., 4 (1965), 369–380.

[8] ———— : Differentiable manifolds admitting complex distributions, J. Math. Kyoto Univ., 5 (1966) 67–85.

[9] Ishihara, S. and K. Yano : On the integrabilty conditions of a structure f satisfying $f^3+f=0$, Quart. J. Math., 15 (1964) 217–222.

[10] Kashiwabara, S. : On Euclidean connections in a Finsler manifold, Tôhoku Math. J., 10 (1958) 69–80.

[11] Kawaguchi, A. : On the theory of non-linear connections I, II, Tensor (N. S.), 2 (1952) 123–142, Ibid. 6 (1956) 165–199.

[12] Matsumoto, M. : Affine transformations of Finsler spaces, J. Math. Kyoto Univ., 3 (1963) 1–35.

[13] ———— : Linear transformations of Finsler connections, Ibid., 3 (1964) 145–167.

[14] ———— : Connections, metrics and almost complex structures of tangent bundles, Ibid., 5 (1966) 251–278.

[15] Matsumoto, M. and T. Okada : Connection in Finsler spaces, Seminor in differential geometry, 4 (1965) Kyoto Univ. (Japanese).

[16] Moòr, A. : Entwicklung einer Geometrie der allgemeinen metrischen Linienelementraüme, Acta Scie. Math., 17 (1956) 85–120.

[17] Nagano, T. : Isometries on complex product spaces, Tensor (N. S.) 9 (1959) 47–61.

[18] Nomizu, K. : Lie groups and differential geometry, Publ. Math. Soc. Japan, 2 (1956).

[19] Okada, T. : Theory of pair connections, Sci. eng. review of Doshisha Univ., 5 (1964) 35–54.

[20] Rizza, G. B. : Structure di Finsler sulle varietà quasi complesse, Rendioconti Accad. Naz. Lincei, 33 (1962) 271–275.

[21] Rund, H. : The differential geometry of Finsler spaces, Springer (1959).

[22] Sasaki, S. : On the differential geometry of tangent bundles of Riemannian manifolds I, II, Tôhoku Math. J., 10 (1958) 338–354. Ibid. 14 (1962) 146–155.

[23] Tachibana, S. and M. Okumura : On the almost complex structure of tangent bundles of Riemannian spaces, Tôhoku Math. J., 14 (1962) 156–161.

[24] Tanno, S. : Almost complex structures in bundle spaces over almost contact manifolds, J. Math. Soc. Japan, 17 (1965) 167–186.

[25] Walker, A. G. : Connections for parallel distributions in the large, Quart. J. Math. Oxford (2), 6 (1955) 310–308.

[26] Willmore, T. J. : Parallel distributions on manifolds, Proc. London Math. Soc. (3), 6 (1956) 191–204.

[27] Yano, K. : On the structure defined by a tensor field f of type (1, 1) satisfying $f^3+f=0$, Tensor (N. S.), 14 (1963) 99–109.

[28] Yano, K. and E. T. Davies : On the tangent bundle of Finsler and Riemannian manifolds, Rend. Circ. Mate. Palermo, (2), 12 (1963) 211–228.

[29] Yano, K. and T. Okubo : Fibred spaces and non-linear connections, Annali di Mate., 55 (1961) 203–244.

[30] Yano, K. and S. Kobayashi : Prolongations of tensor fields and connections to tangent bundles I, II, J. Math. Soc. Japan, 18 (1966) 194–210, Ibid. 236–246.