

ALMOST COMPLEX STRUCTURES ON TENSOR BUNDLES

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1. Introduction

It is well known that the tangent bundle of a C^∞ manifold M admits an almost complex structure if M admits an affine connection [1], [5] or an almost complex structure [7], [8]. The main purpose of this paper is to investigate a similar problem for tensor bundles $T_r^s M$. We prove that if a Riemannian manifold M admits an almost complex structure then so does $T_r^s M$ provided $r + s$ is odd. If $r + s$ is even a further condition is required on M . The proofs depend on some generalizations of the notions of lifting vector fields and derivations on M , which were defined previously only for tangent bundles and cotangent bundles [4], [7], [8], [9], [10].

2. Notations and definitions

- (i) M is a C^∞ paracompact manifold of finite dimension n .
- (ii) $F(M)$ is the ring of real-valued C^∞ functions on M .
- (iii) For $r + s > 0$, $T_r^s M$ is the bundle over M of tensors of type (r, s) , contravariant of order r and covariant of order s . π is the projection of $T_r^s M$ onto M . We write $T_0^r M = T^r M$, $T_r^0 M = T_s M$.
- (iv) $\mathcal{T}_r^s(M)$ is the module over $F(M)$ of C^∞ tensor fields of type (r, s) . We write $\mathcal{T}_0^r(M) = \mathcal{T}^r(M)$, $\mathcal{T}_r^0(M) = \mathcal{T}_s(M)$, and $\mathcal{T}^0(M) = F(M)$. $\mathcal{T}(M)$ is the direct sum $\sum_{r,s} \mathcal{T}_r^s(M)$. T_p is the value at $p \in M$ of a tensor field T on M , and $\mathcal{T}_r^s(p)$ is the vector space of tensors of type (r, s) at p .
- (v) Let $S \in \mathcal{T}_r^s(p)$ and $T \in \mathcal{T}_s^r(p)$. Then the real number $S(T) = T(S)$ is defined, in the usual way, by contraction. It follows that if $S \in \mathcal{T}_r^s(M)$ then S is a differentiable function on $T_s^r M$.
- (vi) A map $D: \mathcal{T}(M) \rightarrow \mathcal{T}(M)$ is a derivation on M if
 - (a) D is linear with respect to constant coefficients,
 - (b) for all r, s , $D\mathcal{T}_r^s(M) \subset \mathcal{T}_r^s(M)$,
 - (c) for all C^∞ tensor fields T_1 and T_2 on M ,

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes DT_2,$$

(d) D commutes with contraction.

A derivation is determined by its action on $F(M)$ and $\mathcal{F}^1(M)$. In particular, $\mathcal{F}_1^1(M)$ may be identified with the set of derivations which map $F(M)$ to zero. The set of derivations on M forms a module $\mathcal{D}M$ over $F(M)$.

(vii) The notation for covariant derivatives and curvature tensors is that of [2]. The linear connections considered on M are assumed to have zero torsion.

3. Vector fields on $T_r^s M$

In this section we show how vector fields on $T_r^s M$ can be induced from vector fields, tensor fields of type (r, s) , and derivations on M .

We first prove a lemma which, together with its corollary, will be of use later.

Lemma 1. *Let $p \in M$ and $S \in \pi^{-1}(p)$. If W is a vertical vector at S (i.e. tangential to $\pi^{-1}(p)$ at S) and $W(\alpha) = 0$ for all $\alpha \in \mathcal{F}_r^s(p)$ then $W = 0$.*

Proof. The vector space $\mathcal{F}_r^s(p)$ is dual to $\mathcal{F}_r^s(p)$ and hence α contains a system of coordinates on $\pi^{-1}(p)$. The result follows immediately.

Corollary 1. *Let $W \in \mathcal{F}^1(T_r^s M)$. If $W(\alpha) = 0$ for all $\alpha \in \mathcal{F}_r^s(M)$ then $W = 0$.*

Proof. The assumption on W implies that for $\beta \in \mathcal{F}_{r-1}^s(M)$ and $f \in F(M)$,

$$0 = \frac{1}{2}W(df^2 \otimes \beta) = W((f \circ \pi)df \otimes \beta) = W(f \circ \pi)df \otimes \beta .$$

Hence $d\pi W = 0$, and so W is a vertical vector field. Thus $W = 0$ by Lemma 1, the values of W on the zero section of $\mathcal{F}_r^s M$ being zero by continuity.

Proposition 1. *Let $T \in \mathcal{F}_r^s(M)$. Then there is a unique C^∞ vector field T^v on $T_r^s M$ such that for $\alpha \in \mathcal{F}_r^s(M)$,*

$$(1) \quad T^v(\alpha) = \alpha(T) \circ \pi .$$

Proof. For $p \in M$, $\pi^{-1}(p)$ is a vector space and so T_p determines a unique vertical vector field T_p^v on $\pi^{-1}(p)$ such that for $\alpha \in \mathcal{F}_r^s(p)$, $T_p^v(\alpha) = \alpha(T_p)$. The cross section T on $T_r^s M$ then determines a C^∞ vertical vector field which satisfies (1). T^v will be called the *vertical lift* of T .

Corollary 2. *Let $S \in \pi^{-1}(p)$, and let T_S^v be the value of T^v at S . Then the map $T_p \rightarrow T_S^v$ is a linear isomorphism of $\pi^{-1}(p) \rightarrow (\pi^{-1}(p))_S$, where $(\pi^{-1}(p))_S$ is the tangent space to the fibre $\pi^{-1}(p)$ at S .*

Proposition 2. *Let D be a derivation on M . Then there is a unique vector field \bar{D} on $T_r^s M$ such that for $\alpha \in \mathcal{F}_r^s(M)$*

$$(2) \quad \bar{D}\alpha = D\alpha .$$

Proof. Let $\{x^i\}$ ($i = 1, 2, \dots, n$) be a coordinate system on a neighbourhood U of $p \in M$, and $\{\omega^\theta\}$ ($\theta = 1, 2, \dots, n^{r+s}$) a basis for $\mathcal{F}_r^s(U)$. Then $\{x^i \circ \pi, \omega^\theta\}$ is a coordinate system on $\pi^{-1}(U)$. Define \bar{D} on $\pi^{-1}(U)$ by

$$(3) \quad \bar{D}(x^i \circ \pi) = (Dx^i) \circ \pi,$$

$$(4) \quad \bar{D}(\omega^\theta) = D(\omega^\theta).$$

Thus a C^∞ vector field \bar{D} is defined on $\pi^{-1}(U)$. Moreover, for $\alpha \in \mathcal{F}_r^s(U)$ we have $\bar{D}\alpha = D\alpha$. Hence, using Corollary 1, it follows that \bar{D} is defined over T_r^sM as the unique solution of (2).

Corollary 3. *If $f \in F(M)$ then $\bar{D}(f \circ \pi) = (Df) \circ \pi$.*

Corollary 4. *\bar{D} is a vertical vector field if and only if $D \in \mathcal{F}_1^1(M)$.*

Corollary 5. *If D_1, D_2 are derivations on M , and $f_1, f_2 \in F(M)$, then $f_1D_1 + f_2D_2$ is a derivation on M , and*

$$\overline{f_1D_1 + f_2D_2} = (f_1 \circ \pi)\bar{D}_1 + (f_2 \circ \pi)\bar{D}_2.$$

Thus if $F(M)$ is identified with $F(M) \circ \pi = \{f \circ \pi : f \in F(M)\}$ then $D \rightarrow \bar{D}$ is a linear map of $\mathcal{D}M \rightarrow \mathcal{F}^1(T_r^sM)$.

Corollary 6. *If $p \in M$ and $A \in \mathcal{F}_1^1(p)$ then for $S \in T_r^s(p)$,*

$$(5) \quad \bar{A}_S = -(AS)_S^s,$$

where the suffix S denotes evaluation at S .

Proof. Let $\alpha \in \mathcal{F}_r^s(p)$. Then

$$\bar{A}_S(\alpha) = (A\alpha)(S) = -(AS)_S^s(\alpha).$$

The result follows from Lemma 1.

Corollary 7. *Let $X \in \mathcal{F}^1(M)$ and \mathcal{L}_X denote Lie derivation with respect to X . Then $\bar{\mathcal{L}}_X$ is a vector field on T_r^sM . In conformity with the notation of [4], [8], [9], [10], we call $\bar{\mathcal{L}}_X$ the complete lift of X and write $\bar{\mathcal{L}}_X = X^c$.*

Remark 1. If $f \in F(M)$ then

$$\mathcal{L}_{fX} = f\mathcal{L}_X - X \otimes df,$$

where $X \otimes df$ is regarded as a derivation on M . Thus

$$(6) \quad (fX)^c = (f \circ \pi)X^c - \overline{X \otimes df}.$$

Now if T_r^sM is the tangent bundle T^1M then for $\alpha \in \mathcal{F}_1(M)$,

$$\overline{X \otimes df}(\alpha) = -\alpha(X)df.$$

Hence by Proposition 1,

$$\overline{X \otimes df} = -dfX^\circ,$$

where X° is the vertical lift of X to T^1M . We then have

$$(7) \quad (fX)^\circ = fX^\circ + dfX^\circ.$$

Equation (7) was used extensively in [8] but does not appear to extend to tensor bundles of high order. Equation (6) is perhaps a useful generalization.

Lemma 2. *Let $p \in M$ and $A \in \mathcal{T}_1^1(p)$. Suppose there exist non-negative integers a and b , not both zero, such that $A\mathcal{T}_a^a(p) = 0$. Then $A = kI$ where k is some real number. If $a \neq b$ then $A = 0$.*

Proof. We prove the lemma for the case $a > 0$. The proof for $a = 0$ and $b > 0$ is essentially the same with covariance and contravariance exchanged.

Let $S \in \mathcal{T}_a^{a-1}(p)$ be non-zero, and let $X \in \mathcal{T}^1(p)$. Then

$$AS \otimes X + S \otimes AX = 0.$$

Choose $\omega \in \mathcal{T}_{a-1}^b(p)$ such that $\omega(S) \neq 0$. Then $(A - kI)X = 0$, where $k = -\omega(AS)/\omega(S)$. It follows immediately that $A = kI$. Then for $T \in \mathcal{T}_a^a(p)$

$$0 = AT = k(a - b)T.$$

Hence, if $a \neq b$ then $k = 0$ and $A = 0$.

Remark 2. $A = kI$ for some k is a necessary and sufficient condition for $A\mathcal{T}_a^a(p) = 0$, $a \neq 0$.

Corollary 8. *Let $D \in \mathcal{D}M$ and suppose there exist non-negative integers a and b , not both zero, such that $D\mathcal{T}_a^a(M) = 0$. Then $D = fI$, where $f \in F(M)$. If $a \neq b$ then $D = 0$.*

Proof. Let $h \in F(M)$ and $T \in \mathcal{T}_a^a(M)$. Then

$$(Dh)T = 0.$$

It follows immediately that $DF(M) = 0$ and hence $D \in \mathcal{T}_1^1(M)$. Then by Lemma 2, $D = fI$ for some $f \in F(M)$, and if $a \neq b$, then f is zero by Lemma 2. This completes the proof.

Remark 3. $D = fI$ for some $f \in F(M)$ is a necessary and sufficient condition for $D\mathcal{T}_a^a(M) = 0$, $a \neq 0$.

Corollary 9. *The map $D \rightarrow \bar{D}$ of $\mathcal{D}M \rightarrow \mathcal{T}^1(T^1M)$ is a monomorphism when $r \neq s$ and has kernel $\{fI : f \in F(M)\}$ when $r = s$.*

Proof. This follows from Corollaries 1, 5 and 8.

Corollary 10. *If $r \neq s$ then T_s^rM admits a vertical vector field which vanishes only on the zero section of T_s^rM .*

Proof. The vector field \bar{I} has the required properties.

Corollary 11. *Let $p \in M$, $A \in \mathcal{T}_1^1(p)$ and $T \in \mathcal{T}_s^r(P)$, $r \neq s$. Then $\bar{A} = T^\circ$ implies $A = 0$ and $T = 0$.*

Proof. Suppose $\bar{A} = T^v$. Then by Corollaries 2 and 6, $AS = -T$ for all $S \in \mathcal{F}_s^r(p)$. Since A is linear it follows that $T = 0$ and $A\mathcal{F}_s^r(p) = 0$. Hence $A = 0$ by Lemma 2.

Suppose now that ∇ is a linear connection (with zero torsion) on M , and let $X \in \mathcal{F}^1(M)$. Then $\nabla X \in \mathcal{F}_1^1(M)$, and hence, by Corollary 4, $\overline{\nabla X}$ is a C^∞ vertical vector field on $T_s^r M$.

Another C^∞ vector field \bar{V}_X on $T_s^r M$ is determined by the derivation ∇_X . In conformity with [4] we write $\bar{V}_X = X^h$, and call X^h the horizontal lift of X . If $f \in F(M)$ then using Corollary 3,

$$X^h(f \circ \pi) = \bar{V}_X(f \circ \pi) = (\nabla_X f) \circ \pi = (Xf) \circ \pi .$$

Hence

$$(8) \quad d\pi X^h = X .$$

The horizontal lift clearly satisfies

$$(fX + gY)^h = (f \circ \pi)X^h + (g \circ \pi)Y^h ,$$

for $f, g \in F(M)$ and $X, Y \in \mathcal{F}^1(M)$. Thus the horizontal lift is a linear map of $\mathcal{F}^1(M) \rightarrow \mathcal{F}^1(T_s^r M)$ if, as before, $F(M)$ and $F(M) \circ \pi$ are identified. Since $\bar{V}_X = 0$ if and only if $X = 0$, the horizontal lift is a monomorphism, and so determines a horizontal subspace H_S of dimension $n (= \dim M)$ at each point $S \in T_s^r M$. Then C^∞ distribution H on $T_s^r M$ so obtained is usually called the horizontal distribution determined by the connection ∇ .

If $S \in T_s^r M$ then the tangent space $(T_s^r M)_S$ is the direct sum $V_S + H_S$, where V_S is the subspace of vertical vectors at S . Thus, if $W \in (T_s^r M)_S$ then

$$W = h(W) + v(W) ,$$

where h and v are the projections onto the horizontal and vertical subspaces at S . Clearly $X^h = h(X^h)$ and $T^v = v(T^v)$ for any vector X and tensor T of type (r, s) at $\pi(S)$.

4. Lie brackets

We now determine, for later use, the Lie brackets of some particular types of vector fields on $T_s^r M$. These results generalize some of those already obtained for tangent bundles and cotangent bundles [1], [4], [7], [8], [9], [10].

Lemma 3. Let $T_1, T_2 \in \mathcal{F}_s^r(M)$ and $X, X_1, X_2 \in \mathcal{F}^1(M)$, and let D, D_1, D_2, A be derivations on M , where $A \in \mathcal{F}_1^1(M)$. Let R denote the curvature tensor field of the connection ∇ . Then

$$(9) \quad [T_1^v, T_2^v] = 0 ,$$

$$(10) \quad [\bar{D}_1, \bar{D}_2] = \overline{[D_1, D_2]} ,$$

$$(11) \quad [\bar{D}, T^v] = (DT)^v ,$$

$$(12) \quad [X^h, T^v] = (\nabla_X T)^v ,$$

$$(13) \quad [X_1^h, X_2^h] = \overline{R(X_1, X_2)} + [X_1, X_2]^h ,$$

$$(14) \quad [X^h, \bar{A}] = \overline{\nabla_X A} ,$$

$$(15) \quad [X_1^c, X_2^c] = [X_1, X_2]^c .$$

Proof. Several equations can be proved by application of Corollary 1.

If $p \in M$ then $\pi^{-1}(p)$ is a vector space, and has the structure of an abelian Lie group. If $S \in \mathcal{F}_s^r(M)$ then S^v is an invariant vector field on $\pi^{-1}(p)$ and equation (9) follows immediately.

We have, from Proposition 2,

$$[\bar{D}_1, \bar{D}_2]\alpha = (\bar{D}_1\bar{D}_2 - \bar{D}_2\bar{D}_1)\alpha = [D_1, D_2]\alpha .$$

Since $[D_1, D_2]$ is a derivation on M , from Proposition 2 we have

$$[D_1, D_2]\alpha = \overline{[D_1, D_2]}\alpha ,$$

and hence equation (10).

$$[\bar{D}, T^v]\alpha = (D(\alpha(T)) - (D\alpha)(T)) \circ \pi = (\alpha(DT)) \circ \pi = (DT)^v(\alpha) ,$$

which gives equation (11). Since $X^h = \bar{\nabla}_X$, equation (12) is a special case of (11).

Since $R(X_1, X_2) \in \mathcal{F}_1^1(M)$, we have

$$[X_1^h, X_2^h] = [\bar{\nabla}_{X_1}, \bar{\nabla}_{X_2}] = \overline{[\nabla_{X_1}, \nabla_{X_2}]} = \overline{R(X_1, X_2)} + \bar{\nabla}_{[X_1, X_2]} ,$$

from which follows immediately equation (13).

$$[X^h, \bar{A}]\alpha = \nabla_X(A\alpha) - A(\nabla_X\alpha) = (\nabla_X A)\alpha = \overline{(\nabla_X A)}\alpha ,$$

which gives equation (14). Since $X^c = \bar{\mathcal{L}}_X$, equation (15) is a special case of (10).

5. Almost complex structures

We now consider the main problem, that is, to determine a class of tensor bundles which admit almost complex structures. For this purpose it is sufficient to consider contravariant tensor bundles since a Riemannian metric tensor field induces a fibre preserving diffeomorphism of $T_r^*M \rightarrow T^{r+s}M$. Also

the tangent bundle T^rM of a Riemannian space always admits an almost complex structure [1], [5]. Hence we shall restrict attention to T^rM , $r > 1$.

Lemma 4. *Let ∇ and g be, respectively, a symmetric connection and a Riemannian metric tensor field on M , and $E \in \mathcal{F}^{r-1}(M)$ be nowhere zero on M . Then T^rM admits three distributions which are mutually orthogonal with respect to a Riemannian metric tensor field \bar{g} induced on T^rM by ∇ and g .*

Proof. For each $p \in M$ a scalar product \langle, \rangle is defined on the vector space $\pi^{-1}(p)$ by $\langle T_1, T_2 \rangle = t_1(T_2)$, where, for any tensor T with components $T^{i_1 i_2 \dots i_r}$, t is the covariant tensor associated to T by g . Thus t has components

$$t_{i_1 i_2 \dots i_r} = T^{j_1 j_2 \dots j_r} g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_r j_r} ,$$

where each repeated suffix indicates summation over its range. If $S \in T^rM$, then a scalar product, denoted by the same symbol \langle, \rangle , is defined on the vector space $(T^rM)_S$ by the three equations

$$(16) \quad \langle T_1^v, T_2^v \rangle = \langle T_1, T_2 \rangle \circ \pi ,$$

$$(17) \quad \langle T^v, X^h \rangle = 0 ,$$

$$(18) \quad \langle X_1^h, X_2^h \rangle = \langle X_1, X_2 \rangle \circ \pi ,$$

where X^h is the horizontal lift of X with respect to ∇ . These equations are easily seen to determine \bar{g} on T^rM with respect to which the horizontal distribution H , induced by ∇ , is orthogonal to the fibres of T^rM [3].

We now make use of E . For $X \in \mathcal{F}^1(M)$, define the vertical lift X_E^v of X with respect to E by

$$X_E^v = (E \otimes X)^v .$$

The map $X \rightarrow X_E^v$ is then a monomorphism of $\mathcal{F}^1(M) \rightarrow \mathcal{F}^1(T^rM)$. Hence an n -dimensional C^∞ vertical distribution V^E is defined on T^rM . Let V^\perp be the distribution on T^rM which is orthogonal to H and V^E . Then H, V^E and V^\perp are the required distributions and the proof is complete.

We now give an alternative characterization of V^\perp .

Lemma 5. *Let $p \in M, S \in \pi^{-1}(p)$, and $\mathcal{F}_E^r(p)$ be the subspace of $\mathcal{F}^r(p)$ defined by*

$$\mathcal{F}_E^r(p) = \{T: \langle T, E \otimes X \rangle = 0 \text{ for all } X \in \mathcal{F}^1(p)\} .$$

Then $V_S^\perp = (\mathcal{F}_E^r(p))_S^v$.

Let $E^\perp(p)$ be the subspace of $\mathcal{F}^{r-1}(p)$ defined by

$$E^\perp(p) = \{T: \langle T, E \rangle = 0\} .$$

Then $\mathcal{F}_E^r(p) = E^\perp(p) \otimes \mathcal{F}^1(p)$.

Proof. The first part of the lemma follows from the fact that the vertical lift preserves scalar products. To prove the second part it is sufficient to note that $E^\perp(p) \otimes \mathcal{T}^1(p) \subset \mathcal{T}_E^r(p)$, and

$$\dim(E^\perp(p) \otimes \mathcal{T}^1(p)) = n(n^{r-1} - 1) = n^r - n = \dim \mathcal{T}_E^r(p).$$

Theorem. *If M admits an almost complex structure and a nowhere zero tensor field $E \in \mathcal{T}^{r-1}(M)$, then T^rM admits an almost complex structure.*

Proof. Let F be an almost complex structure on M . We define a C^∞ tensor field J of type $(1,1)$ on T^rM by its action on the distributions H, V^E and V^\perp . Thus for $X \in \mathcal{T}^1(M)$ and $T \in \mathcal{T}^r(M)$ define J by

$$(19) \quad J(X^h) = X_E^v, J(X_E^v) = -X^h, J(T^v) = \bar{T}^v,$$

where \bar{T} is obtained by contracting $T \otimes F$, and has components $T^{i_1 i_2 \dots i_{r-1} l} F_l^j$, where $T^{i_1 i_2 \dots i_r}$ and F_j^i are local components of T and F respectively. The restrictions of J to $H + V^E$ and V^\perp are endomorphisms, and hence J is a tensor field on T^rM . It is easily seen that J is C^∞ and $J^2 = -I$, I being the unit tensor. Hence J is an almost complex structure on T^rM .

Corollary 12. *Suppose a Riemannian manifold M admits an almost complex structure. Then T^rM admits an almost complex structure if (i) r is odd or (ii) r is even and M admits a nowhere zero vector field.*

Proof. (i) For $r = 2s + 1$ choose $E = (\otimes g^{-1})^s$, where g^{-1} is the inverse of a metric tensor field g on M , and $(\otimes g^{-1})^s$ is the tensor product of g^{-1} with itself s times.

(ii) For $r = 2s, s > 1$, choose $E = (\otimes g^{-1})^{s-1} \otimes X$, where M is assumed to admit a nowhere zero vector field X . For $r = 2$ choose $E = X$.

6. Integrability of the almost complex structure J

We now establish necessary and sufficient conditions for the integrability of J .

Let e be the covariant tensor field of order $r - 1$ associated to E by g ; thus, with respect to local coordinates, e has components $e_{i_1 i_2 \dots i_{r-1}}$ given by

$$e_{i_1 i_2 \dots i_{r-1}} = g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_{r-1} j_{r-1}} E^{j_1 j_2 \dots j_{r-1}}.$$

Proposition 3. *Suppose M admits an almost complex structure F and a nowhere zero tensor field $E \in \mathcal{T}^{r-1}(M)$. Then the induced almost complex structure J is integrable if and only if, for $X, Y \in \mathcal{T}^1(M)$,*

$$R(X, Y) = 0, \quad \nabla_X E = 0, \quad \nabla_X F = 0, \quad \nabla_X \frac{e}{\langle E, E \rangle} = 0.$$

Proof. Let N be the Nijenhuis 2-form on T^rM with values in $\mathcal{F}^1(T^rM)$, defined by

$$N(W_1, W_2) = [W_1, W_2] + J[JW_1, W_2] + J[W_1, JW_2] - [JW_1, JW_2]$$

for $W_1, W_2 \in \mathcal{F}^1(T^rM)$. Then J is integrable if and only if $N = 0$.

Suppose $N = 0$. Then for $X, Y \in \mathcal{F}^1(M)$, $N(X_E^v, Y_E^v) = 0$. Hence, putting $W_1 = X_E^v, W_2 = Y_E^v$ we have, from (9), (12), (13), and the definition of J ,

$$\begin{aligned} \overline{R(X, Y)} &= J(\nabla_Y(E \otimes X))^v - J(\nabla_X(E \otimes Y))^v - [X, Y]^h \\ (20) \quad &= J((\nabla_Y E) \otimes X)^v - J((\nabla_X E) \otimes Y)^v - (\nabla_Y X)^h \\ &\quad + (\nabla_X Y)^h - [X, Y]^h \\ &= J((\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y)^v \end{aligned}$$

since ∇ has zero torsion. Now since $E \otimes \mathcal{F}^1(M)$ is a subspace of $\mathcal{F}^r(M)$ there is a unique $T \in \mathcal{F}^r(M)$ orthogonal to this subspace and a unique $Z \in \mathcal{F}^1(M)$ such that

$$(\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y = T + E \otimes Z .$$

Then from (19) and (20)

$$\overline{R(X, Y)} = \tilde{T}^v - Z^h .$$

Since $\overline{R(X, Y)}$ is vertical, $Z^h = 0$ and hence $Z = 0$. It follows from Corollary 11 that

$$(21) \quad R(X, Y) = 0 ,$$

$$(22) \quad T = 0 .$$

We thus have for all $X, Y \in \mathcal{F}^1(M)$,

$$(\nabla_X E) \otimes Y = (\nabla_Y E) \otimes X .$$

Since M is assumed to admit an almost complex structure, $\dim M \geq 2$. Hence by choosing X, Y to be linearly independent it follows that

$$(23) \quad \nabla_X E = 0 .$$

We next consider the case $N(X_E^v, T^v) = 0$, where $X_E^v \in V^E$ and $T^v \in V^\perp$. Then from (9), (12) and the definition of J we have

$$(24) \quad J(\nabla_X T)^v = (\nabla_X \tilde{T})^v .$$

It follows that $(\nabla_X T)^v \in V^\perp$. Choose $T = S \otimes Y$ where $S \in \mathcal{F}^{r-1}(M), Y \in \mathcal{F}^1(M)$

and $\langle S, E \rangle = 0$ (since M is paracompact such an S exists and can be chosen to be non-zero in a neighbourhood of a point). Then by Lemma 5, $T^v \in V^\perp$ and (24) imply that

$$(\nabla_x S) \otimes FY + S \otimes F\nabla_x Y = (\nabla_x S) \otimes FY + S \otimes \nabla_x(FY) .$$

Hence

$$S \otimes (\nabla_x F)Y = 0 ,$$

and it follows immediately that

$$(25) \quad \nabla_x F = 0 .$$

Finally, from Lemma 5 the condition $(\nabla_x T)^v \in V^\perp$ implies that

$$(26) \quad 0 = e(\nabla_x S) = -(\nabla_x e)S .$$

But S is any tensor field which satisfies $\langle S, E \rangle = 0$. Hence we deduce that

$$(27) \quad \nabla_x e = \alpha(X)e ,$$

where $\alpha \in \mathcal{F}_1(M)$. Then α is determined by

$$\alpha(X) = \frac{(\nabla_x e)(E)}{e(E)} = \frac{X(e(E))}{e(E)} = \frac{X \langle E, E \rangle}{\langle E, E \rangle} .$$

Thus

$$(28) \quad \alpha = d \log e(E) = d \log \langle E, E \rangle .$$

(If ∇ is the Riemannian connection associated with g then (23) implies (27) and $\alpha = 0$.) Hence, from (27) and (28), the tensor field $\frac{e}{\langle E, E \rangle}$ has zero covariant derivative. This proves the necessity of the conditions in Proposition 3.

To prove the sufficiency we note that

$$\begin{aligned} N(X_E^v, Y_E^v) &= N(Y^h, X^h) = JN(Y_E^v, X^h) , \\ N(X_E^v, T^v) &= JN(T^v, X^h), \quad N(T_1^v, T_2^v) = 0 . \end{aligned}$$

Thus $N = 0$ if $N(X_E^v, Y_E^v) = N(X_E^v, T^v) = 0$. Suppose $\nabla_x E = 0$ and $R(X, Y) = 0$ for all $X, Y \in \mathcal{F}^1(M)$. Then

$$\begin{aligned} N(X_E^v, Y_E^v) &= -J[X^h, Y_E^v] - J[X_E^v, Y^h] - [X^h, Y^h] \\ &= (\nabla_x Y)^h - (\nabla_y X)^h - [X, Y]^h = 0 . \end{aligned}$$

Suppose $\nabla_x \frac{e}{\langle E, E \rangle} = 0$. Then (27) follows and hence if $T^v \in V^\perp$ then $(\nabla_x T)^v \in V^\perp$. If we next assume $\nabla_x F = 0$ then we have

$$N(X_E^v, T^v) = (\nabla_x \tilde{T})^v - J(\nabla_x T)^v = 0,$$

which proves the sufficiency.

7. Kählerian structure on T^rM

We now determine necessary and sufficient conditions for the metric \bar{g} on T^rM , defined in §5, to be Kählerian with respect to J .

Proposition 4. \bar{g} is Hermitian with respect to J if and only if $\langle E, E \rangle = 1$ and g is Hermitian with respect to F .

Proof. Suppose \bar{g} is Hermitian with respect to J . Then for $X, Y \in \mathcal{F}^1(M)$,

$$\begin{aligned} \langle X, Y \rangle \circ \pi &= \langle X^h, Y^h \rangle = \langle JX_E^v, JY_E^v \rangle = \langle X_E^v, Y_E^v \rangle \\ &= \langle E \otimes X, E \otimes Y \rangle \circ \pi = \langle E, E \rangle \langle X, Y \rangle \circ \pi. \end{aligned}$$

Hence $\langle E, E \rangle = 1$. Now let $p \in M$ and let $S \in \mathcal{F}^{r-1}(p)$ be non-zero such that $\langle S, E \rangle = 0$. Then by Lemma 5 and the definition of J we have, for $X, Y \in \mathcal{F}^1(p)$,

$$\begin{aligned} \langle S, S \rangle \langle X, Y \rangle \circ \pi &= \langle S \otimes X, S \otimes Y \rangle \circ \pi \\ &= \langle (S \otimes X)^v, (S \otimes Y)^v \rangle = \langle J(S \otimes X)^v, J(S \otimes Y)^v \rangle \\ &= \langle S \otimes FX, S \otimes FY \rangle \circ \pi = \langle S, S \rangle \langle FX, FY \rangle \circ \pi. \end{aligned}$$

Thus at p , $\langle X, Y \rangle = \langle FX, FY \rangle$. Since p is arbitrary, g is Hermitian with respect to F . The sufficiency of the above conditions is easily proved by the same method.

Proposition 5. Suppose \bar{g} is Hermitian with respect to J . Then \bar{g} is Kählerian with respect to J if and only if ∇ is the Riemannian connection associated with g , $R = 0$, $\nabla E = 0$ and $\nabla F = 0$.

Proof. Let α be the field of 2-forms on T^rM defined for all $W_1, W_2 \in \mathcal{F}^1(T^rM)$ by $\alpha(W_1, W_2) = \langle W_1, JW_2 \rangle$. Then \bar{g} is Kählerian with respect to J if and only if α is closed and J is integrable [6, Chapter VII]. As usual it is sufficient to consider the action of α and $d\alpha$ on the three distributions H, V^E and V^\perp on T^rM . Then for $X, Y \in \mathcal{F}^1(M)$ and $T_1^v, T_2^v \in V^\perp$ we have

$$\begin{aligned} \alpha(X_E^v, Y_E^v) &= \alpha(X^h, Y^h) = \alpha(T_1^v, X_E^v) = \alpha(T_1^v, X^h) = 0, \\ (29) \quad \alpha(X_E^v, Y^h) &= \langle E \otimes X, E \otimes Y \rangle \circ \pi = \langle X, Y \rangle \circ \pi, \\ \alpha(T_1^v, T_2^v) &= \langle T_1, \tilde{T}_2 \rangle \circ \pi. \end{aligned}$$

Suppose \bar{g} is Kählerian with respect to J . Then by Propositions 3 and 4, $R = 0$, $\nabla_X E = 0$, and $\nabla_X e = 0$, for all $X \in \mathcal{S}^1(M)$. Let $p \in M$, $X \in \mathcal{S}^1(p)$, and choose $T \in \mathcal{S}^{r-1}(M)$ such that $\langle T, E \rangle = 0$ and $\langle T, T \rangle = 1$ on some neighbourhood U of p . Since $R = 0$ parallel vector fields Y and Z exist on U with arbitrary initial values at p . Then using (9), (12) and Lemma 5 we have, on $\pi^{-1}(p)$,

$$\begin{aligned}
 0 &= d\alpha((T \otimes Y)^v, (T \otimes X)^v, X^h) \\
 &= X \langle T \otimes Y, T \otimes FX \rangle + \langle T \otimes FY, \nabla_X(T \otimes Z) \rangle \\
 &\quad - \langle \nabla_X(T \otimes Y), T \otimes FZ \rangle \\
 (30) \quad &= X \langle Y, FZ \rangle + 2 \langle T, \nabla_X T \rangle \langle FY, Z \rangle \\
 &\quad + \langle FY, \nabla_X Z \rangle - \langle \nabla_X Y, FZ \rangle \\
 &= (\nabla_X g)(Y, FZ) - 2 \langle T, \nabla_X T \rangle \langle Y, FZ \rangle .
 \end{aligned}$$

Since F is non-singular it follows that

$$\nabla_X g = \alpha(X)g ,$$

for some $\alpha \in \mathcal{S}^1(p)$. Then since $\nabla_X E = 0$ and $\nabla_X e = 0$ it follows easily that for all $X \in \mathcal{S}^1(p)$,

$$0 = \nabla_X e = (r - 1)\alpha(X)e .$$

The tensor e is non-zero and so $\alpha = 0$. Thus $\nabla g = 0$ at p and hence on M since p is arbitrary. It follows that ∇ , having no torsion, is the Riemannian connection associated with g .

We now prove the sufficiency of the above conditions by showing that the 2-form α is exact. Let $X \in \mathcal{S}^1(M)$, and $T^v \in V^\perp$. Define a 1-form β on $T^r M$ as follows: at each point $S \in T^r M$,

$$\beta(X^h) = \langle S, E \otimes X \rangle, \quad \beta(X_E^v) = 0, \quad \beta(T^v) = \frac{1}{2} \langle S, \bar{T} \rangle .$$

Then using (29) it follows after some calculation that $\alpha = d\beta$. Hence $d\alpha = 0$, and this together with Proposition 3 proves the sufficiency.

8. Integrability of $H + V^E$ and $H + V^\perp$

Proposition 6. $H + V^E$ is integrable if and only if $R = 0$ and for $X \in \mathcal{S}^1(M)$, $\nabla_X E = \alpha(X)E$, where $\alpha(X) = \frac{\langle E, \nabla_X E \rangle}{\langle E, E \rangle}$.

Proof. It follows from (12) and (13) that $H + V^E$ is an integrable distribution if and only if for $X_1, X_2 \in \mathcal{S}^1(M)$,

$$(31) \quad (\nabla_{X_1}(E \otimes X_2))^v \in V^E ,$$

$$(32) \quad \overline{R(X_1, X_2)} \in V^E .$$

Let Y_1 and Y_2 be orthogonal vectors at $p \in M$, and let $\langle T, E \rangle = 0$ at p . Then from (16), (32) and Corollary 6,

$$\begin{aligned} 0 &= \langle R(X_1, X_2)(T \otimes Y_1), T \otimes Y_2 \rangle \\ &= \langle T, T \rangle \langle R(X_1, X_2)Y_1, Y_2 \rangle . \end{aligned}$$

Hence $R(X_1, X_2)Y_1 = cY_1$ where c is some real number which depends on X_1 and X_2 . Since Y_1 is arbitrary it follows that $R(X_1, X_2) = cI$ at p . Then at any point $S \in \pi^{-1}(p)$ we have $\overline{R(X_1, X_2)} = -crS^v$, and by choosing $S^v \in V^\perp$ it follows that $\overline{R(X_1, X_2)} = 0$ at S ; hence $c = 0$. Since p, X_1 and X_2 are arbitrary we have $R = 0$ on M .

Using (30) and Lemma 5 we obtain $\nabla_X E = \alpha(X)E$ and α is then uniquely determined by this equation.

The proof of the sufficiency is immediate.

Proposition 7. $H + V^\perp$ is integrable if and only if $R = 0$ and for $X \in \mathcal{F}^1(M)$, $\nabla_X e = \alpha(X)e$, where $\alpha = \frac{\langle e, \nabla_X e \rangle}{\langle e, e \rangle}$.

Proof. The proof is similar to that of Proposition 6 and we shall use the same notation. It follows from (12), (13) and Lemma 5 that $H + V^\perp$ is an integrable distribution if and only if for $S^v \in V^\perp$,

$$(33) \quad (\nabla_{X_1}(S \otimes X_2))^v \in V^\perp ,$$

$$(34) \quad \overline{R(X_1, X_2)} \in V^\perp .$$

then from (16), (34) and Corollary 6,

$$\begin{aligned} 0 &= \langle R(X_1, X_2)(E \otimes Y_1), E \otimes Y_2 \rangle \\ &= \langle E, E \rangle \langle R(X_1, X_2)Y_1, Y_2 \rangle . \end{aligned}$$

Hence, as before, $R = 0$.

From (33) we obtain

$$0 = \langle \nabla_{X_1} S, E \rangle \langle X_2, Y \rangle$$

for $Y \in \mathcal{F}^1(p)$. Hence

$$0 = \langle \nabla_{X_1} S, E \rangle = e(\nabla_{X_1} S) = -(\nabla_{X_1} e)S .$$

It follows that $\nabla_{X_1} e = \alpha(X_1)e$ at p . Since p and X_1 are arbitrary we obtain $\nabla_X e = \alpha(X)e$ on M , and α is then uniquely determined.

The proof of the sufficiency is immediate.

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