J. DIFFERENTIAL GEOMETRY
1 (1967) 355-368

ALMOST COMPLEX STRUCTURES ON TENSOR BUNDLES

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1. Introduction

It is well known that the tangent bundle of a C^∞ manifold M admits an almost complex structure if M admits an affine connection [1], [5] or an almost complex structure [7], [8]. The main purpose of this paper is to investigate a similar problem for tensor bundles T^r_sM. We prove that if a Riemannian manifold M admits an almost complex structure then so does T^r_sM provided $r + s$ is odd. If $r + s$ is even a further condition is required on M. The proofs depend on some generalizations of the notions of lifting vector fields and derivations on M, which were defined previously only for tangent bundles and cotangent bundles [4], [7], [8], [9], [10].

2. Notations and definitions

(i) M is a C^∞ paracompact manifold of finite dimension n.

(ii) $F(M)$ is the ring of real-valued C^∞ functions on M.

(iii) For $r + s > 0$, T^r_sM is the bundle over M of tensors of type (r, s), contravariant of order r and covariant of order s. π is the projection of T^r_sM onto M. We write $T^r_0M = T^rM$, $T^0_sM = T_sM$.

(iv) $\mathscr{T}^r_s(M)$ is the module over $F(M)$ of C^∞ tensor fields of type (r, s). We write $\mathscr{T}^r_0(M) = \mathscr{T}^r(M)$, $\mathscr{T}^0_s(M) = \mathscr{T}_s(M)$, and $\mathscr{T}^0_0(M) = F(M)$. $\mathscr{T}(M)$ is the direct sum $\sum_{r,s} \mathscr{T}^r_s(M)$. T_p is the value at $p \in M$ of a tensor field T on M, and $\mathscr{T}^r_s(p)$ is the vector space of tensors of type (r, s) at p.

(v) Let $S \in \mathscr{T}^s_r(p)$ and $T \in \mathscr{T}^r_s(p)$. Then the real number $S(T) = T(S)$ is defined, in the usual way, by contraction. It follows that if $S \in \mathscr{T}^s_r(M)$ then S is a differentiable function on T^r_sM.

(vi) A map $D: \mathscr{T}(M) \to \mathscr{T}(M)$ is a derivation on M if
  (a) D is linear with respect to constant coefficients,
  (b) for all r, s, $D\mathscr{T}^r_s(M) \subset \mathscr{T}^r_s(M)$,
  (c) for all C^∞ tensor fields T_1 and T_2 on M,

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes DT_2\,,$$

Communicated August 15, 1967.

(d) D commutes with contraction.
A derivation is determined by its action on $F(M)$ and $\mathscr{T}^1(M)$. In particular, $\mathscr{T}^1_1(M)$ may be identified with the set of derivations which map $F(M)$ to zero. The set of derivations on M forms a module $\mathscr{D}M$ over $F(M)$.

(vii) The notation for covariant derivatives and curvature tensors is that of [2]. The linear connections considered on M are assumed to have zero torsion.

3. Vector fields on $T^r_s M$

In this section we show how vector fields on T^r_sM can be induced from vector fields, tensor fields of type (r, s), and derivations on M.

We first prove a lemma which, together with its corollary, will be of use later.

Lemma 1. *Let $p \in M$ and $S \in \pi^{-1}(p)$. If W is a vertical vector at S* (i.e. *tangential to $\pi^{-1}(p)$ at S*) *and $W(\alpha) = 0$ for all $\alpha \in \mathscr{T}^s_r(p)$ then $W = 0$.*

Proof. The vector space $\mathscr{T}^s_r(p)$ is dual to $\mathscr{T}^r_s(p)$ and hence α contains a system of coordinates on $\pi^{-1}(p)$. The result follows immediately.

Corollary 1. *Let $W \in \mathscr{T}^1(T^r_sM)$. If $W(\alpha)=0$ for all $\alpha \in \mathscr{T}^s_r(M)$ then $W=0$.*

Proof. The assumption on W implies that for $\beta \in \mathscr{T}^s_{r-1}(M)$ and $f \in F(M)$,

$$0 = \frac{1}{2}W(df^2 \otimes \beta) = W((f \circ \pi)df \otimes \beta) = W(f \circ \pi)df \otimes \beta .$$

Hence $d\pi W = 0$, and so W is a vertical vector field. Thus $W = 0$ by Lemma 1, the values of W on the zero section of $\mathscr{T}^r_sM$ being zero by continuity.

Proposition 1. *Let $T \in \mathscr{T}^r_s(M)$. Then there is a unique C^∞ vector field T^v on T^r_sM such that for $\alpha \in \mathscr{T}^s_r(M)$,*

$$T^v(\alpha) = \alpha(T) \circ \pi . \tag{1}$$

Proof. For $p \in M$, $\pi^{-1}(p)$ is a vector space and so T_p determines a unique vertical vector field T^v_p on $\pi^{-1}(p)$ such that for $\alpha \in \mathscr{T}^s_r(p)$, $T^v_p(\alpha) = \alpha(T_p)$. The cross section T on T^r_sM then determines a C^∞ vertical vector field which satisfies (1). T^v will be called the *vertical lift* of T.

Corollary 2. *Let $S \in \pi^{-1}(p)$, and let T^v_S be the value of T^v at S. Then the map $T_p \to T^v_S$ is a linear isomorphism of $\pi^{-1}(p) \to (\pi^{-1}(p))_S$, where $(\pi^{-1}(p))_S$ is the tangent space to the fibre $\pi^{-1}(p)$ at S.*

Proposition 2. *Let D be a derivation on M. Then there is a unique vector field $\bar{D}$ on T^r_sM such that for $\alpha \in \mathscr{T}^s_r(M)$*

$$\bar{D}\alpha = D\alpha . \tag{2}$$

Proof. Let $\{x^i\}$ $(i = 1, 2, \cdots, n)$ be a coordinate system on a neighbourhood U of $p \in M$, and $\{\omega^\theta\}$ $(\theta = 1, 2, \cdots, n^{r+s})$ a basis for $\mathscr{T}^s_r(U)$. Then $\{x^i \circ \pi, \omega^\theta\}$ is a coordinate system on $\pi^{-1}(U)$. Define $\bar{D}$ on $\pi^{-1}(U)$ by

$$\bar{D}(x^i \circ \pi) = (Dx^i) \circ \pi \,, \tag{3}$$

$$\bar{D}(\omega^\theta) = D(\omega^\theta) \,. \tag{4}$$

Thus a C^∞ vector field $\bar{D}$ is defined on $\pi^{-1}(U)$. Moreover, for $\alpha \in \mathscr{T}^s_r(U)$ we have $\bar{D}\alpha = D\alpha$. Hence, using Corollary 1, it follows that $\bar{D}$ is defined over T^r_sM as the unique solution of (2).

Corollary 3. *If $f \in F(M)$ then $\bar{D}(f \circ \pi) = (Df) \circ \pi$.*

Corollary 4. *$\bar{D}$ is a vertical vector field if and only if $D \in \mathscr{T}^1_1(M)$.*

Corollary 5. *If D_1, D_2 are derivations on M, and $f_1, f_2 \in F(M)$, then $f_1D_1 + f_2D_2$ is a derivation on M, and*

$$\overline{f_1D_1 + f_2D_2} = (f_1 \circ \pi)\bar{D}_1 + (f_2 \circ \pi)\bar{D}_2 \,.$$

Thus if $F(M)$ is identified with $F(M) \circ \pi = \{f \circ \pi : f \in F(M)\}$ then $D \to \bar{D}$ is a linear map of $\mathscr{D}M \to \mathscr{T}^1(T^r_sM)$.

Corollary 6. *If $p \in M$ and $A \in \mathscr{T}^1_1(p)$ then for $S \in T^r_s(p)$,*

$$\bar{A}_S = -(AS)^v_S \,, \tag{5}$$

where the suffix S denotes evaluation at S.

Proof. Let $\alpha \in \mathscr{T}^s_r(p)$. Then

$$\bar{A}_S(\alpha) = (A\alpha)(S) = -(AS)^v_S(\alpha) \,.$$

The result follows from Lemma 1.

Corollary 7. *Let $X \in \mathscr{T}^1(M)$ and $\mathscr{L}_X$ denote Lie derivation with respect to X. Then $\bar{\mathscr{L}}_X$ is a vector field on T^r_sM. In conformity with the notation of* [4], [8], [9], [10], *we call $\bar{\mathscr{L}}_X$ the complete lift of X and write $\bar{\mathscr{L}}_X = X^c$.*

Remark 1. If $f \in F(M)$ then

$$\mathscr{L}_{fX} = f\mathscr{L}_X - X \otimes df \,,$$

where $X \otimes df$ is regarded as a derivation on M. Thus

$$(fX)^c = (f \circ \pi)X^c - \overline{X \otimes df} \,. \tag{6}$$

Now if T^r_sM is the tangent bundle T^1M then for $\alpha \in \mathscr{T}_1(M)$,

$$\overline{X \otimes df}(\alpha) = -\alpha(X)df \,.$$

Hence by Proposition 1,

$$\overline{X \otimes df} = -dfX^v ,$$

where X^v is the vertical lift of X to T^1M. We then have

$$(fX)^c = fX^c + dfX^v . \tag{7}$$

Equation (7) was used extensively in [8] but does not appear to extend to tensor bundles of high order. Equation (6) is perhaps a useful generalization.

Lemma 2. *Let $p \in M$ and $A \in \mathscr{T}^1_1(p)$. Suppose there exist non-negative integers a and b, not both zero, such that $A\mathscr{T}^a_b(p) = 0$. Then $A = kI$ where k is some real number. If $a \neq b$ then $A = 0$.*

Proof. We prove the lemma for the case $a > 0$. The proof for $a = 0$ and $b > 0$ is essentially the same with covariance and contravariance exchanged.

Let $S \in \mathscr{T}^{a-1}_b(p)$ be non-zero, and let $X \in \mathscr{T}^1(p)$. Then

$$AS \otimes X + S \otimes AX = 0 .$$

Choose $\omega \in \mathscr{T}^b_{a-1}(p)$ such that $\omega(S) \neq 0$. Then $(A - kI)X = 0$, where $k = -\omega(AS)/\omega(S)$. It follows immediately that $A = kI$. Then for $T \in \mathscr{T}^a_b(p)$

$$0 = AT = k(a - b)T .$$

Hence, if $a \neq b$ then $k = 0$ and $A = 0$.

Remark 2. $A = kI$ for some k is a necessary and sufficient condition for $A\mathscr{T}^a_a(p) = 0$, $a \neq 0$.

Corollary 8. *Let $D \in \mathscr{D}M$ and suppose there exist non-negative integers a and b, not both zero, such that $D\mathscr{T}^a_b(M) = 0$. Then $D = fI$, where $f \in F(M)$. If $a \neq b$ then $D = 0$.*

Proof. Let $h \in F(M)$ and $T \in \mathscr{T}^a_b(M)$. Then

$$(Dh)T = 0 .$$

It follows immediately that $DF(M) = 0$ and hence $D \in \mathscr{T}^1_1(M)$. Then by Lemma 2, $D = fI$ for some $f \in F(M)$, and if $a \neq b$, then f is zero by Lemma 2. This completes the proof.

Remark 3. $D = fI$ for some $f \in F(M)$ is a necessary and sufficient condition for $D\mathscr{T}^a_a(M) = 0$, $a \neq 0$.

Corollary 9. *The map $D \to \bar{D}$* of *$\mathscr{D}M \to \mathscr{T}^1(T^r_sM)$ is a monomorphism when $r \neq s$ and has kernel $\{fI : f \in F(M)\}$ when $r = s$.*

Proof. This follows from Corollaries 1, 5 and 8.

Corollary 10. *If $r \neq s$ then T^r_sM admits a vertical vector field which vanishes only on the zero section of T^r_sM.*

Proof. The vector field $\bar{I}$ has the required properties.

Corollary 11. *Let $p \in M$, $A \in \mathscr{T}^1_1(p)$ and $T \in \mathscr{T}^r_s(P)$, $r \neq s$. Then $\bar{A} = T^v$ implies $A = 0$ and $T = 0$.*

Proof. Suppose $\bar{A} = T^v$. Then by Corollaries 2 and 6, $AS = -T$ for all $S \in \mathscr{T}^r_s(p)$. Since A is linear it follows that $T = 0$ and $A\mathscr{T}^r_s(p) = 0$. Hence $A = 0$ by Lemma 2.

Suppose now that ∇ is a linear connection (with zero torsion) on M, and let $X \in \mathscr{T}^1(M)$. Then $\nabla X \in \mathscr{T}^1_1(M)$, and hence, by Corollary 4, $\overline{\nabla X}$ is a C^∞ vertical vector field on $T^r_s M$.

Another C^∞ vector field $\bar{\nabla}_X$ on $T^r_s M$ is determined by the derivation ∇_X. In conformity with [4] we write $\bar{\nabla}_X = X^h$, and call X^h the horizontal lift of X. If $f \in F(M)$ then using Corollary 3,

$$X^h(f \circ \pi) = \bar{\nabla}_X(f \circ \pi) = (\nabla_X f) \circ \pi = (Xf) \circ \pi \ .$$

Hence

$$d\pi X^h = X \ . \tag{8}$$

The horizontal lift clearly satisfies

$$(fX + gY)^h = (f \circ \pi)X^h + (g \circ \pi)Y^h \ ,$$

for $f, g \in F(M)$ and $X, Y \in \mathscr{T}^1(M)$. Thus the horizontal lift is a linear map of $\mathscr{T}^1(M) \to \mathscr{T}^1(T^r_s M)$ if, as before, $F(M)$ and $F(M) \circ \pi$ are identified. Since $\bar{\nabla}_X = 0$ if and only if $X = 0$, the horizontal lift is a monomorphism, and so determines a horizontal subspace H_S of dimension $n(= \dim M)$ at each point $S \in T^r_s M$. Then C^∞ distribution H on $T^r_s M$ so obtained is usually called the horizontal distribution determined by the connection ∇.

If $S \in T^r_s M$ then the tangent space $(T^r_s M)_S$ is the direct sum $V_S + H_S$, where V_S is the subspace of vertical vectors at S. Thus, if $W \in (T^r_s M)_S$ then

$$W = h(W) + v(W) \ ,$$

where h and v are the projections onto the horizontal and vertical subspaces at S. Clearly $X^h = h(X^h)$ and $T^v = v(T^v)$ for any vector X and tensor T of type (r, s) at $\pi(S)$.

4. Lie brackets

We now determine, for later use, the Lie brackets of some particular types of vector fields on $T^r_s M$. These results generalize some of those already obtained for tangent bundles and cotangent bundles [1], [4], [7], [8], [9], [10].

Lemma 3. *Let $T_1, T_2 \in \mathscr{T}^r_s(M)$ and $X, X_1, X_2 \in \mathscr{T}^1(M)$, and let D, D_1, D_2, A be derivations on M, where $A \in \mathscr{T}^1_1(M)$. Let R denote the curvature tensor field of the connection ∇. Then*

$$[T_1^v, T_2^v] = 0 \ , \tag{9}$$

(10) $$[\bar{D}_1, \bar{D}_2] = \overline{[D_1, D_2]} ,$$

(11) $$[\bar{D}, T^v] = (DT)^v ,$$

(12) $$[X^h, T^v] = (\nabla_X T)^v ,$$

(13) $$[X_1^h, X_2^h] = \overline{R(X_1, X_2)} + [X_1, X_2]^h ,$$

(14) $$[X^h, \bar{A}] = \overline{\nabla_X A} ,$$

(15) $$[X_1^c, X_2^c] = [X_1, X_2]^c .$$

Proof. Several equations can be proved by application of Corollary 1.

If $p \in M$ then $\pi^{-1}(p)$ is a vector space, and has the structure of an abelian Lie group. If $S \in \mathscr{T}_s^r(M)$ then S^v is an invariant vector field on $\pi^{-1}(p)$ and equation (9) follows immediately.

We have, from Proposition 2,

$$[\bar{D}_1, \bar{D}_2]\alpha = (\bar{D}_1\bar{D}_2 - \bar{D}_2\bar{D}_1)\alpha = [D_1, D_2]\alpha .$$

Since $[D_1, D_2]$ is a derivation on M, from Proposition 2 we have

$$[D_1, D_2]\alpha = \overline{[D_1, D_2]}\alpha ,$$

and hence equation (10).

$$[\bar{D}, T^v]\alpha = (D(\alpha(T)) - (D\alpha)(T)) \circ \pi = (\alpha(DT)) \circ \pi = (DT)^v(\alpha) ,$$

which gives equation (11). Since $X^h = \bar{\nabla}_X$, equation (12) is a special case of (11).

Since $R(X_1, X_2) \in \mathscr{T}_1^1(M)$, we have

$$[X_1^h, X_2^h] = [\bar{\nabla}_{X_1}, \bar{\nabla}_{X_2}] = \overline{[\nabla_{X_1}, \nabla_{X_2}]} = \overline{R(X_1, X_2)} + \bar{\nabla}_{[X_1, X_2]} ,$$

from which follows immediately equation (13).

$$[X^h, \bar{A}]\alpha = \nabla_X(A\alpha) - A(\nabla_X\alpha) = (\nabla_X A)\alpha = (\overline{\nabla_X A})\alpha ,$$

which gives equation (14). Since $X^c = \bar{\mathscr{L}}_X$, equation (15) is a special case of (10).

5. Almost complex structures

We now consider the main problem, that is, to determine a class of tensor bundles which admit almost complex structures. For this purpose it is sufficient to consider contravariant tensor bundles since a Riemannian metric tensor field induces a fibre preserving diffeomorphism of $T_s^r M \to T^{r+s}M$. Also the tangent bundle T^1M of a Riemannian space always admits an almost complex structure [1], [5]. Hence we shall restrict attention to T^rM, $r > 1$.

Lemma 4. *Let ∇ and g be, respectively, a symmetric connection and a Riemannian metric tensor field on M, and $E \in \mathscr{T}^{r-1}(M)$ be nowhere zero on M. Then T^rM admits three distributions which are mutually orthogonal with respect to a Riemannian metric tensor field $\bar{g}$ induced on T^rM by ∇ and g.*

Proof. For each $p \in M$ a scalar product $<, >$ is defined on the vector space $\pi^{-1}(p)$ by $< T_1, T_2 > = t_1(T_2)$, where, for any tensor T with components $T^{i_1 i_2 \cdots i_r}$, t is the covariant tensor associated to T by g. Thus t has components

$$t_{i_1 i_2 \ldots i_r} = T^{j_1 j_2 \cdots j_r} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_r j_r} ,$$

where each repeated suffix indicates summation over its range. If $S \in T^rM$, then a scalar product, denoted by the same symbol $<, >$, is defined on the vector space $(T^rM)_S$ by the three equations

$$< T_1^v, T_2^v > = < T_1, T_2 > \circ \pi , \tag{16}$$

$$< T^v, X^h > = 0 , \tag{17}$$

$$< X_1^h, X_2^h > = < X_1, X_2 > \circ \pi , \tag{18}$$

where X^h is the horizontal lift of X with respect to ∇. These equations are easily seen to determine $\bar{g}$ on T^rM with respect to which the horizontal distribution H, induced by ∇, is orthogonal to the fibres of T^rM [3].

We now make use of E. For $X \in \mathscr{T}^1(M)$, define the vertical lift X_E^v of X with respect to E by

$$X_E^v = (E \otimes X)^v .$$

The map $X \to X_E^v$ is then a monomorphism of $\mathscr{T}^1(M) \to \mathscr{T}^1(T^rM)$. Hence an n-dimensional C^∞ vertical distribution V^E is defined on T^rM. Let $V^\perp$ be the distribution on T^rM which is orthogonal to H and V^E. Then H, V^E and $V^\perp$ are the required distributions and the proof is complete.

We now give an alternative characterization of $V^\perp$.

Lemma 5. *Let $p \in M$, $S \in \pi^{-1}(p)$, and $\mathscr{T}_E^r(p)$ be the subspace of $\mathscr{T}^r(p)$ defined by*

$$\mathscr{T}_E^r(p) = \{T : < T, E \otimes X > = 0 \text{ for all } X \in \mathscr{T}^1(p)\} .$$

Then $V_S^\perp = (\mathscr{T}_E^r(p))_S^v$.

Let $E^\perp(p)$ be the subspace of $\mathscr{T}^{r-1}(p)$ defined by

$$E^\perp(p) = \{T : < T, E > = 0\} .$$

Then $\mathscr{T}_E^r(p) = E^\perp(p) \otimes \mathscr{T}^1(p)$.

Proof. The first part of the lemma follows from the fact that the vertical lift preserves scalar products. To prove the second part it is sufficient to note that $E^\perp(p) \otimes \mathcal{T}^1(p) \subset \mathcal{T}^r_E(p)$, and

$$\dim (E^\perp(p) \otimes \mathcal{T}^1(p)) = n(n^{r-1} - 1) = n^r - n = \dim \mathcal{T}^r_E(p) .$$

Theorem. *If M admits an almost complex structure and a nowhere zero tensor field $E \in \mathcal{T}^{r-1}(M)$, then T^rM admits an almost complex structure.*

Proof. Let F be an almost complex structure on M. We define a C^∞ tensor field J of type (1,1) on T^rM by its action on the distributions H, V^E and $V^\perp$. Thus for $X \in \mathcal{T}^1(M)$ and $T \in \mathcal{T}^r(M)$ define J by

$$J(X^h) = X^v_E,\ J(X^v_E) = -X^h,\ J(T^v) = \bar{T}^v , \tag{19}$$

where $\bar{T}$ is obtained by contracting $T \otimes F$, and has components $T^{i_1 i_2 \cdots i_{r-1} l} F_l^{i_r}$, where $T^{i_1 i_2 \cdots i_r}$ and F_j^i are local components of T and F respectively. The restrictions of J to $H + V^E$ and $V^\perp$ are endomorphisms, and hence J is a tensor field on T^rM. It is easily seen that J is C^∞ and $J^2 = -I$, I being the unit tensor. Hence J is an almost complex structure on T^rM.

Corollary 12. *Suppose a Riemannian manifold M admits an almost complex structure. Then T^rM admits an almost complex structure if* (i) *r is odd or* (ii) *r is even and M admits a nowhere zero vector field.*

Proof. (i) For $r = 2s + 1$ choose $E = (\otimes g^{-1})^s$, where g^{-1} is the inverse of a metric tensor field g on M, and $(\otimes g^{-1})^s$ is the tensor product of g^{-1} with itself s times.

(ii) For $r = 2s$, $s > 1$, choose $E = (\otimes g^{-1})^{s-1} \otimes X$, where M is assumed to admit a nowhere zero vector field X. For $r = 2$ choose $E = X$.

6. Integrability of the almost complex structure J

We now establish necessary and sufficient conditions for the integrability of J.

Let e be the covariant tensor field of order $r - 1$ associated to E by g; thus, with respect to local coordinates, e has components $e_{i_1 i_2 \cdots i_{r-1}}$ given by

$$e_{i_1 i_2 \cdots i_{r-1}} = g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_{r-1} j_{r-1}} E^{j_1 j_2 \cdots j_{r-1}} .$$

Proposition 3. *Suppose M admits an almost complex structure F and a nowhere zero tensor field $E \in \mathcal{T}^{r-1}(M)$. Then the induced almost complex structure J is integrable if and only if, for $X, Y \in \mathcal{T}^1(M)$,*

$$R(X, Y) = 0, \quad \nabla_X E = 0, \quad \nabla_X F = 0, \quad \nabla_X \frac{e}{\langle E, E \rangle} = 0 .$$

Proof. Let N be the Nijenhuis 2-form on T^rM with values in $\mathscr{T}^1(T^rM)$, defined by

$$N(W_1, W_2) = [W_1, W_2] + J[JW_1, W_2] + J[W_1, JW_2] - [JW_1, JW_2]$$

for $W_1, W_2 \in \mathscr{T}^1(T^rM)$. Then J is integrable if and only if $N = 0$.

Suppose $N = 0$. Then for $X, Y \in \mathscr{T}^1(M)$, $N(X_E^v, Y_E^v) = 0$. Hence, putting $W_1 = X_E^v$, $W_2 = Y_E^v$ we have, from (9), (12), (13), and the definition of J,

$$\begin{aligned}\overline{R(X, Y)} &= J(\nabla_Y(E \otimes X))^v - J(\nabla_X(E \otimes Y))^v - [X, Y]^h \\ &= J((\nabla_Y E) \otimes X)^v - J((\nabla_X E) \otimes Y)^v - (\nabla_Y X)^h \\ &\qquad + (\nabla_X Y)^h - [X, Y]^h \\ &= J((\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y)^v\end{aligned} \tag{20}$$

since ∇ has zero torsion. Now since $E \otimes \mathscr{T}^1(M)$ is a subspace of $\mathscr{T}^r(M)$ there is a unique $T \in \mathscr{T}^r(M)$ orthogonal to this subspace and a unique $Z \in \mathscr{T}^1(M)$ such that

$$(\nabla_Y E) \otimes X - (\nabla_X E) \otimes Y = T + E \otimes Z .$$

Then from (19) and (20)

$$\overline{R(X, Y)} = \tilde{T}^v - Z^h .$$

Since $\overline{R(X, Y)}$ is vertical, $Z^h = 0$ and hence $Z = 0$. It follows from Corollary 11 that

$$R(X, Y) = 0 , \tag{21}$$

$$T = 0 . \tag{22}$$

We thus have for all $X, Y \in \mathscr{T}^1(M)$,

$$(\nabla_X E) \otimes Y = (\nabla_Y E) \otimes X .$$

Since M is assumed to admit an almost complex structure, $\dim M \geq 2$. Hence by choosing X, Y to be linearly independent it follows that

$$\nabla_X E = 0 . \tag{23}$$

We next consider the case $N(X_E^v, T^v) = 0$, where $X_E^v \in V^E$ and $T^v \in V^\perp$. Then from (9), (12) and the definition of J we have

$$J(\nabla_X T)^v = (\nabla_X \tilde{T})^v . \tag{24}$$

It follows that $(\nabla_X T)^v \in V^\perp$. Choose $T = S \otimes Y$ where $S \in \mathscr{T}^{r-1}(M)$, $Y \in \mathscr{T}^1(M)$

and $<S, E> = 0$ (since M is paracompact such an S exists and can be chosen to be non-zero in a neighbourhood of a point). Then by Lemma 5, $T^v \in V^\perp$ and (24) imply that

$$(\nabla_X S) \otimes FY + S \otimes F\nabla_X Y = (\nabla_X S) \otimes FY + S \otimes \nabla_X(FY) .$$

Hence

$$S \otimes (\nabla_X F)Y = 0 ,$$

and it follows immediately that

$$\nabla_X F = 0 . \tag{25}$$

Finally, from Lemma 5 the condition $(\nabla_X T)^v \in V^\perp$ implies that

$$0 = e(\nabla_X S) = -(\nabla_X e)S . \tag{26}$$

But S is any tensor field which satisfies $<S, E> = 0$. Hence we deduce that

$$\nabla_X e = \alpha(X)e , \tag{27}$$

where $\alpha \in \mathscr{T}_1(M)$. Then α is determined by

$$\alpha(X) = \frac{(\nabla_X e)(E)}{e(E)} = \frac{X(e(E))}{e(E)} = \frac{X<E, E>}{<E, E>} .$$

Thus

$$\alpha = d \log e(E) = d \log <E, E> . \tag{28}$$

(If ∇ is the Riemannian connection associated with g then (23) implies (27) and $\alpha = 0$.) Hence, from (27) and (28), the tensor field $\dfrac{e}{<E, E>}$ has zero covariant derivative. This proves the necessity of the conditions in Proposition 3.

To prove the sufficiency we note that

$$N(X_E^v, Y_E^v) = N(Y^h, X^h) = JN(Y_E^v, X^h) ,$$
$$N(X_E^v, T^v) = JN(T^v, X^h), \quad N(T_1^v, T_2^v) = 0 .$$

Thus $N = 0$ if $N(X_E^v, Y_E^v) = N(X_E^v, T^v) = 0$. Suppose $\nabla_X E = 0$ and $R(X, Y) = 0$ for all $X, Y \in \mathscr{T}^1(M)$. Then

$$N(X_E^v, Y_E^v) = -J[X^h, Y_E^v] - J[X_E^v, Y^h] - [X^h, Y^h]$$
$$= (\nabla_X Y)^h - (\nabla_Y X)^h - [X, Y]^h = 0 .$$

Suppose $\nabla_X \dfrac{e}{<E, E>} = 0$. Then (27) follows and hence if $T^v \in V^\perp$ then $(\nabla_X T)^v \in V^\perp$. If we next assume $\nabla_X F = 0$ then we have

$$N(X_E^v, T^v) = (\nabla_X \bar{T})^v - J(\nabla_X T)^v = 0 ,$$

which proves the sufficiency.

7. Kählerian structure on T^rM

We now determine necessary and sufficient conditions for the metric $\bar{g}$ on T^rM, defined in §5, to be Kählerian with respect to J.

Proposition 4. *$\bar{g}$ is Hermitian with respect to J if and only if $<E, E>=1$ and g is Hermitian with respect to F.*

Proof. Suppose $\bar{g}$ is Hermitian with respect to J. Then for $X, Y \in \mathscr{T}^1(M)$,

$$\begin{aligned} <X, Y> \circ \pi &= <X^h, Y^h> = <JX_E^v, JY_E^v> = <X_E^v, Y_E^v> \\ &= <E \otimes X, E \otimes Y> \circ \pi = <E, E><X, Y> \circ \pi . \end{aligned}$$

Hence $<E, E> = 1$. Now let $p \in M$ and let $S \in \mathscr{T}^{r-1}(p)$ be non-zero such that $<S, E> = 0$. Then by Lemma 5 and the definition of J we have, for $X, Y \in \mathscr{T}^1(p)$,

$$\begin{aligned} <S, S><X, Y> \circ \pi &= <S \otimes X, S \otimes Y> \circ \pi \\ &= <(S \otimes X)^v, (S \otimes Y)^v> = <J(S \otimes X)^v, J(S \otimes Y)^v> \\ &= <S \otimes FX, S \otimes FY> \circ \pi = <S, S><FX, FY> \circ \pi . \end{aligned}$$

Thus at p, $<X, Y> = <FX, FY>$. Since p is arbitrary, g is Hermitian with respect to F. The sufficiency of the above conditions is easily proved by the same method.

Proposition 5. *Suppose $\bar{g}$ is Hermitian with respect to J. Then $\bar{g}$ is Kählerian with respect to J if and only if ∇ is the Riemannian connection associated with g, $R = 0$, $\nabla E = 0$ and $\nabla F = 0$.*

Proof. Let α be the field of 2-forms on T^rM defined for all $W_1, W_2 \in \mathscr{T}^1(T^rM)$ by $\alpha(W_1, W_2) = <W_1, JW_2>$. Then $\bar{g}$ is Kählerian with respect to J if and only if α is closed and J is integrable [6, Chapter VII]. As usual it is sufficient to consider the action of α and $d\alpha$ on the three distributions H, V^E and $V^\perp$ on T^rM. Then for $X, Y \in \mathscr{T}^1(M)$ and $T_1^v, T_2^v \in V^\perp$ we have

$$\begin{aligned} &\alpha(X_E^v, Y_E^v) = \alpha(X^h, Y^h) = \alpha(T_1^v, X_E^v) = \alpha(T_1^v, X^h) = 0 , \\ (29) \qquad &\alpha(X_E^v, Y^h) = <E \otimes X, E \otimes Y> \circ \pi = <X, Y> \circ \pi , \\ &\alpha(T_1^v, T_2^v) = <T_1, \bar{T}_2> \circ \pi . \end{aligned}$$

Suppose $\bar{g}$ is Kählerian with respect to J. Then by Propositions 3 and 4, $R = 0$, $\nabla_X E = 0$, and $\nabla_X e = 0$, for all $X \in \mathscr{T}^1(M)$. Let $p \in M$, $X \in \mathscr{T}^1(p)$, and choose $T \in \mathscr{T}^{r-1}(M)$ such that $< T, E > = 0$ and $< T, T > = 1$ on some neighbourhood U of p. Since $R = 0$ parallel vector fields Y and Z exist on U with arbitrary initial values at p. Then using (9), (12) and Lemma 5 we have, on $\pi^{-1}(p)$,

$$\begin{aligned} 0 &= d\alpha((T \otimes Y)^v, (T \otimes X)^v, X^h) \\ &= X < T \otimes Y, T \otimes FX > + < T \otimes FY, \nabla_X(T \otimes Z) > \\ &\qquad - < \nabla_X(T \otimes Y), T \otimes FZ > \\ &= X < Y, FZ > + 2 < T, \nabla_X T >< FY, Z > \\ &\qquad + < FY, \nabla_X Z > - < \nabla_X Y, FZ > \\ &= (\nabla_X g)(Y, FZ) - 2 < T, \nabla_X T >< Y, FZ > . \end{aligned} \tag{30}$$

Since F is non-singular it follows that

$$\nabla_X g = \alpha(X) g ,$$

for some $\alpha \in \mathscr{T}_1(p)$. Then since $\nabla_X E = 0$ and $\nabla_X e = 0$ it follows easily that for all $X \in \mathscr{T}^1(p)$,

$$0 = \nabla_X e = (r - 1)\alpha(X) e .$$

The tensor e is non-zero and so $\alpha = 0$. Thus $\nabla g = 0$ at p and hence on M since p is arbitrary. It follows that ∇, having no torsion, is the Riemannian connection associated with g.

We now prove the sufficiency of the above conditions by showing that the 2-form α is exact. Let $X \in \mathscr{T}^1(M)$, and $T^v \in V^\perp$. Define a 1-form β on $T^r M$ as follows: at each point $S \in T^r M$,

$$\beta(X^h) = < S, E \otimes X >, \quad \beta(X^v_E) = 0, \quad \beta(T^v) = \tfrac{1}{2} < S, \bar{T} > .$$

Then using (29) it follows after some calculation that $\alpha = d\beta$. Hence $d\alpha = 0$, and this together with Proposition 3 proves the sufficiency.

8. Integrability of $H + V^E$ and $H + V^\perp$

Proposition 6. *$H + V^E$ is integrable if and only if $R = 0$ and for $X \in \mathscr{T}^1(M)$, $\nabla_X E = \alpha(X) E$, where* $\alpha(X) = \dfrac{< E, \nabla_X E >}{< E, E >}$.

Proof. It follows from (12) and (13) that $H + V^E$ is an integrable distribution if and only if for $X_1, X_2 \in \mathscr{T}^1(M)$,

$$(\nabla_{X_1}(E \otimes X_2))^v \in V^E , \tag{31}$$

$$\overline{R(X_1, X_2)} \in V^E . \tag{32}$$

Let Y_1 and Y_2 be orthogonal vectors at $p \in M$, and let $<T, E> = 0$ at p. Then from (16), (32) and Corollary 6,

$$\begin{aligned} 0 &= <R(X_1, X_2)(T \otimes Y_1), T \otimes Y_2> \\ &= <T, T><R(X_1, X_2)Y_1, Y_2> . \end{aligned}$$

Hence $R(X_1, X_2)Y_1 = cY_1$ where c is some real number which depends on X_1 and X_2. Since Y_1 is arbitrary it follows that $R(X_1, X_2) = cI$ at p. Then at any point $S \in \pi^{-1}(p)$ we have $\overline{R(X_1, X_2)} = -crS^v$, and by choosing $S^v \in V^\perp$ it follows that $\overline{R(X_1, X_2)} = 0$ at S; hence $c = 0$. Since p, X_1 and X_2 are arbitrary we have $R = 0$ on M.

Using (30) and Lemma 5 we obtain $\nabla_X E = \alpha(X)E$ and α is then uniquely determined by this equation.

The proof of the sufficiency is immediate.

Proposition 7. *$H + V^\perp$ is integrable if and only if $R=0$ and for $X \in \mathscr{T}^1(M)$, $\nabla_X e = \alpha(X)e$, where $\alpha = \dfrac{<e, \nabla_X e>}{<e, e>}$.*

Proof. The proof is similar to that of Proposition 6 and we shall use the same notation. It follows from (12), (13) and Lemma 5 that $H + V^\perp$ is an integrable distribution if and only if for $S^v \in V^\perp$,

$$(\nabla_{X_1}(S \otimes X_2))^v \in V^\perp , \tag{33}$$

$$\overline{R(X_1, X_2)} \in V^\perp . \tag{34}$$

then from (16), (34) and Corollary 6,

$$\begin{aligned} 0 &= <R(X_1, X_2)(E \otimes Y_1), E \otimes Y_2> \\ &= <E, E><R(X_1, X_2)Y_1, Y_2> . \end{aligned}$$

Hence, as before, $R = 0$.

From (33) we obtain

$$0 = <\nabla_{X_1}S, E><X_2, Y>$$

for $Y \in \mathscr{T}^1(p)$. Hence

$$0 = <\nabla_{X_1}S, E> = e(\nabla_{X_1}S) = -(\nabla_{X_1}e)S .$$

It follows that $\nabla_{X_1}e = \alpha(X_1)e$ at p. Since p and X_1 are arbitrary we obtain $\nabla_{X_1}e = \alpha(X_1)e$ on M, and α is then uniquely determined.

The proof of the sufficiency is immediate.

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