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Almost conformal Ricci solituons on 3-dimensional trans-Sasakian manifold

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Abstract

In this paper we have shown that if a 3-dimensional trans-Sasakian manifold M admits conformal Ricci soliton (g, V, λ) and if the vector field V is point wise collinear with the unit vector field ξ, then V is a constant multiple of ξ. Similarly we have proved that under the same condition an almost conformal Ricci soliton becomes conformal Ricci soliton. We have also shown that if a 3-dimensional trans-Sasakian manifold admits conformal gradient shrinking Ricci soliton, then the manifold is an Einstein manifold.

Keywords: conformal Ricci soliton, almost conformal Ricci soliton, conformal gradient shrinking Ricci soliton, trans-Sasakian manifold.

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1. Introduction

In 1982 Hamilton [9] introduced the concept of Ricci flow and proved its existence. This concept was developed to answer Thurston's geometric conjecture which says that each closed three manifold admits a geometric decomposition. Hamilton also [9] classified all compact manifolds with positive curvature operator in dimension four. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

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on a compact Riemannian manifold M with Riemannian metric g.

A self-similar solution to the Ricci flow [9], [14] is called a Ricci soliton [10] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\pounds_X g + 2S = 2\lambda g, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero and negetive respectively.

A. E. Fischer developed the concept of conformal Ricci flow [7] during 2003-04 which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M where M is considered as a smooth closed connected oriented n-manifold is defined by the equation [7]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \tag{1.3}$$

and $r(g) = -1$,
where p is a scalar non-dynamical field(time dependent scalar field), $r(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.

In 2015, N. Basu and A. Bhattacharyya [2] introduced the notion of conformal Ricci soliton equation as

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g, \tag{1.4}$$

where λ is constant.
The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

The concept of Ricci almost soliton was first introduced by S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti in 2010 [12]. R. Sharma has also done excellent work in almost Ricci soliton [13]. A Riemannian manifold (M^n, g) is an almost Ricci soliton [1], if there exist a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying,

$$R_{ij} + \frac{1}{2}(X_{ij} + X_{ji}) = \lambda g_{ij},$$

where R_{ij} and $X_{ij} + X_{ji}$ stand for the Ricci tensor and the Lie derivative $\pounds_X g$ in local coordinates respectively. It will be called expanding, steady or shrinking, respectively, if $\lambda < 0, \lambda = 0$ or $\lambda > 0$.

We introduce the notion of almost conformal Ricci soliton by

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.5}$$

where $\lambda : M^n \to \mathbb{R}$ is a smooth function.

Now a gradient Ricci soliton on a Riemannian manifold (M^n, g_{ij}) is defined by [6]

$$S + \nabla\nabla f = \rho g, \tag{1.6}$$

for some constant ρ and for a smooth function f on M. f is called a potential function of the Ricci soliton and ∇ is the Levi-Civita connection on M. In particular a gradient shrinking Ricci soliton satisfies the equation,

$$S + \nabla\nabla f - \frac{1}{2\tau}g = 0,$$

where $\tau = T - t$ and T is the maximal time of the solution.

Again for conformal Ricci soliton if the vector field is the gradient of a function f, then we call it as a conformal gradient shrinking Ricci soliton [4]. For conformal gradient shrinking Ricci soliton the equation is

$$S + \nabla\nabla f = (\frac{1}{2\tau} - \frac{2}{n} - p)g. \tag{1.7}$$

where $\tau = T - t$ and T is the maximal time of the solution and f is the Ricci potential function.

2. Preliminaries:

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

for all vector field $X, Y \in \chi(M)$.

An almost contact metric structure (ϕ,ξ,η,g) on M is called a trans-Sasakian structure [11], if $(M\times R,J,G)$ belongs to the class W_4 [8], where J is the almost complex structure on $M\times R$ defined by $J(X,f\frac{d}{dt})=(\phi X-f\xi,\eta(X)\frac{d}{dt})$ for all vector fields X on M and smooth functions f on $M\times R$. It can be expressed by the condition [5],

$$(\nabla_X\phi)Y=\alpha(g(X,Y)\xi-\eta(Y)X)+\beta(g(\phi X,Y)\xi-\eta(Y)\phi X), \tag{2.5}$$

for some smooth functions α,β on M and we say that the trans-Sasakian structure is of type (α,β). From the above expression we can write

$$\nabla_X\xi=-\alpha\phi X+\beta(X-\eta(X)\xi), \tag{2.6}$$

$$(\nabla_X\eta)Y=-\alpha g(\phi X,Y)+\beta g(\phi X,\phi Y), \tag{2.7}$$

For a 3-dimensional trans-Sasakian manifold the following relations hold:

$$2\alpha\beta+\xi\alpha=0, \tag{2.8}$$

$$S(X,\xi)=(2(\alpha^2-\beta^2)-\xi\beta)\eta(X)-X\beta-(\phi X)\alpha, \tag{2.9}$$

$$\begin{aligned} S(X,Y) &= (\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2))g(X,Y)-(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Y)\\ &- (Y\beta+(\phi Y)\alpha)\eta(X)-(X\beta+(\phi X)\alpha)\eta(Y), \end{aligned} \tag{2.10}$$

where S denotes the Ricci tensor of type $(0,2)$, r is the scalar curvature of the manifold M and α,β are smooth functions on M.

For $\alpha,\beta=$ constant the following relations hold:

$$S(X,Y)=(\frac{r}{2}-(\alpha^2-\beta^2))g(X,Y)-(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\eta(Y), \tag{2.11}$$

$$S(X,\xi)=2(\alpha^2-\beta^2)\eta(X), \tag{2.12}$$

$$R(X,Y)\xi=(\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y), \tag{2.13}$$

$$QX=(\frac{r}{2}-(\alpha^2-\beta^2))X-(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\xi, \tag{2.14}$$

where Q is the Ricci operator given by $S(X,Y)=g(QX,Y)$. Again,

$$\begin{aligned} (\pounds_\xi g)(X,Y) &= (\nabla_\xi g)(X,Y)-\alpha g(\phi X,Y)+2\beta g(X,Y)-2\beta\eta(X)\eta(Y)-\alpha g(X,\phi Y)\\ &= 2\beta g(X,Y)-2\beta\eta(X)\eta(Y)\ [\because g(X,\phi Y)+g(\phi X,Y)=0]. \end{aligned}$$

Putting the above value in the conformal Ricci soliton equation (1.4) and taking $n = 3$ we get

$$\begin{aligned} S(X,Y) &= \frac{1}{2}[2\lambda - (p+\frac{2}{3})]g(X,Y) - \frac{1}{2}[2\beta g(X,Y) - 2\beta\eta(X)\eta(Y) \\ &= Ag(X,Y) - \beta g(X,Y) + \beta\eta(X)\eta(Y), \end{aligned} \tag{2.15}$$

where $A = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]$.

Hence we can state the following proposition.

Proposition 2.1 : If a 3-dimensional trans-Sasakian manifold admits conformal Ricci soliton (g, ξ, λ), then the manifolds becomes an η-Einstein manifold.

Also,

$$QX = AX - \beta X + \beta\eta(X)\xi. \tag{2.16}$$

Again for almost conformal Ricci soliton

$$\begin{aligned} S(X,Y) &= \lambda g(X,Y) - \frac{1}{2}(p+\frac{2}{3})g(X,Y) - \beta g(X,Y) + \beta\eta(X)\eta(Y) \\ &= (B + \lambda - \beta)g(X,Y) + \beta\eta(X)\eta(Y), \end{aligned}$$

where $B = -\frac{1}{2}(p + \frac{2}{3})$.

Thus we can state the following proposition.

Proposition 2.2 : A 3-dimensional trans-Sasakian manifold admitting almost conformal Ricci soliton (g, ξ, λ) is an η-Einstein manifold.

Example of a 3-dimensional trans-Sasakian manifold:

In this section we construct an example of a 3-dimensional trans-Sasakian manifold.To construct this, we consider the three dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}), e_2 = e^{-z}\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \\ g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0. \end{aligned}$$

Let η be the 1-form which satisfies the relation

$$\eta(e_3) = 1.$$

Let ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0$. Then we have

$$\phi^2(Z) = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi, (\phi, \xi, \eta, g)$ defines an almost contact metric structure on M. Now, after calculating we have

$$[e_1, e_3] = e_1, [e_1, e_2] = ye^{-z}e_2 + e^{-2z}e_3, [e_2, e_3] = e_2.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula

$$\begin{aligned} &2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ (2.17) \quad &-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By Koszul's formula we get

$$\begin{aligned} \nabla_{e_1} e_1 = -e_3, \nabla_{e_2} e_1 = -ye^{-z}e_2 - \frac{1}{2}e^{-2z}e_3, \nabla_{e_3} e_1 = -\frac{1}{2}e^{-2z}e_2, \\ \nabla_{e_1} e_2 = \frac{1}{2}e^{-2z}e_3, \nabla_{e_2} e_2 = ye^{-z}e_1 - e_3, \nabla_{e_3} e_2 = \frac{1}{2}e^{-2z}e_1, \\ \nabla_{e_1} e_3 = e_1 - \frac{1}{2}e^{-2z}e_2, \nabla_{e_2} e_3 = \frac{1}{2}e^{-2z}e_1 + e_2, \nabla_{e_3} e_3 = 0. \end{aligned}$$

From the above we have found that $\alpha = \frac{1}{2}e^{-2z}, \beta = 1$ and it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold.

3. Some results for conformal Ricci soliton and almost conformal Ricci soliton on 3-dimensional trans-Sasakian manifold

A conformal Ricci soliton equation on a Riemannian manifold M is defined by

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{3})]g,$$

where V is a vector field.

Let V be pointwise co-linear with ξ i.e. $V = \gamma\xi$ where γ is a function on 3-dimensional trans-Sasakian manifold. Then

$$(\pounds_V g + 2S - [2\lambda - (p + \frac{2}{3})]g)(X, Y) = 0,$$

which implies

$$(\pounds_{\gamma\xi} g)(X, Y) + 2S(X, Y) - [2\lambda - (p + \frac{2}{3})]g(X, Y) = 0.$$

Applying the property of Lie derivative and Levi-civita connection we have

$$\begin{aligned} \gamma g(\nabla_X \xi, Y) + (X\gamma)g(\xi, Y) + (Y\gamma)g(\xi, X) + \gamma g(\nabla_Y \xi, X) + 2S(X, Y) \\ -[2\lambda - (p + \frac{2}{3})]g(X, Y) = 0. \end{aligned}$$

Using (2.3) and (2.6) in the above equation we obtain

$$\begin{aligned} &2\beta\gamma g(X,Y) - 2\gamma\beta\eta(X)\eta(Y) + (X\gamma)\eta(Y) + (Y\gamma)\eta(X) \\ &+2S(X,Y) - [2\lambda - (p + \frac{2}{3})]g(X,Y) = 0. \end{aligned} \tag{3.1}$$

Replacing Y by ξ and using (2.12) in (3.1) we get

$$X\gamma + (\xi\gamma)\eta(X) + 2[2(\alpha^2 - \beta^2)\eta(X)] - [2\lambda - (p + \frac{2}{3})]\eta(X) = 0. \tag{3.2}$$

Again putting $X = \xi$ in (3.2) we get

$$\xi\gamma = \frac{1}{2}[2\lambda - (p + \frac{2}{3})] - 2(\alpha^2 - \beta^2). \tag{3.3}$$

Using (3.3) in (3.2) we have

$$\begin{aligned} X\gamma + (\frac{1}{2}[2\lambda - (p + \frac{2}{3})] - 2(\alpha^2 - \beta^2))\eta(X) + 2(2(\alpha^2 - \beta^2)\eta(X)) \\ -[2\lambda - (p + \frac{2}{3})]\eta(X) = 0, \end{aligned}$$

which implies

$$X\gamma = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]\eta(X) - 2(\alpha^2 - \beta^2)\eta(X). \tag{3.4}$$

Applying exterior differentiation in (3.4) and considering λ as constant we have

$$\frac{1}{2}[2\lambda - (p + \frac{2}{3})] - 2(\alpha^2 - \beta^2) = 0, \tag{3.5}$$

(since $d\eta \neq 0$).

Using (3.5) in (3.4) we have

$$X\gamma = 0$$

implies γ is constant.

Hence from (3.1) we have

$$2\beta\gamma g(X,Y) - 2\gamma\beta\eta(X)\eta(Y) + 2S(X,Y) - [2\lambda - (p + \frac{2}{3})]g(X,Y) = 0$$

i.e.

$$S(X,Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{3})]g(X,Y) - \beta\gamma g(X,Y) + \gamma\beta\eta(X)\eta(Y).$$

Putting $X = Y = e_i$ where $\{e_i\}$ is orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i we get,

$$r = \frac{3}{2}[2\lambda - (p + \frac{2}{3})] - 3\beta\gamma + \gamma\beta. \tag{3.6}$$

Now for conformal Ricci soliton $r = -1$, so putting this value in the above equation we get

$$\lambda = \frac{1}{2}p + \frac{2}{3}\beta\gamma. \tag{3.7}$$

So we can state the following theorem:

Theorem 3.1 : A 3-dimensional trans-Sasakian manifold admitting conformal Ricci soliton and if V is point-wise collinear with ξ, then V is a constant multiple of ξ. Also the value of $\lambda = \frac{1}{2}p + \frac{2}{3}\beta\gamma$ provided α, β are constants.

Again for almost conformal Ricci soliton we consider that λ is a smooth function. Then applying exterior derivative in (3.4) we get

$$\frac{1}{2}[2\lambda - (p + \frac{2}{3})] - 2(\alpha^2 - \beta^2) = 0 \tag{3.8}$$

and

$$d\lambda = 0. \tag{3.9}$$

So λ is a constant function and from (3.4) and (3.8) we get γ is constant.

Hence we can conclude the following theorem:

Theorem 3.2 : If a 3-dimensional trans-Sasakian manifold admits almost conformal Ricci soliton and if V is point-wise collinear with ξ, then V is a constant multiple of ξ as well as λ becomes a constant function i.e. almost conformal Ricci soliton becomes conformal Ricci soliton.

Now, from conformal Ricci soliton equation we have

$$(\pounds_\xi g)(X, Y) = 2\beta[g(X, Y) - \eta(X)\eta(Y)].$$

Using (2.11) in the above equation and from (1.4) we have

$$2\beta[g(X, Y) - \eta(X)\eta(Y)] + 2[(\frac{r}{2} - (\alpha^2 - \beta^2))g(X, Y) - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y)] \\ -[2\lambda - (p + \frac{2}{3})]g(X, Y) = 0.$$

For conformal Ricci soliton we have $r = -1$, so the above equation becomes

$$[2\beta + 2(\frac{-1}{2} - (\alpha^2 - \beta^2)) - (2\lambda - (p + \frac{2}{3}))]g(X, Y) \\ -[2\beta + 2(\frac{-1}{2} - 3(\alpha^2 - \beta^2))]\eta(X)\eta(Y) = 0. \tag{3.10}$$

Now taking $X = Y = \xi$ in (3.10) we get

$$2\beta + 2(\frac{-1}{2} - (\alpha^2 - \beta^2)) - (2\lambda - (p + \frac{2}{3})) - 2\beta \\ -2(\frac{-1}{2} - 3(\alpha^2 - \beta^2)) = 0,$$

which gives

$$\lambda = \frac{1}{2}[4(\alpha^2 - \beta^2) + (p + \frac{2}{3})].$$

Since $\alpha^2 \neq \beta^2$ so
(1). Suppose $\alpha^2 \geq \beta^2$, then $(\alpha + \beta)(\alpha - \beta) > 0$ which implies α always greater than β. Then $\lambda > 0$ and the conformal Ricci soliton is shrinking.
(2). Suppose $\alpha^2 < \beta^2$ and $(p + \frac{2}{3}) > 4(\alpha^2 - \beta^2)$, then $(\alpha + \beta)(\alpha - \beta) < 0$ which implies α always less than $-\beta$. Then $\lambda > 0$ and the conformal Ricci soliton becomes shrinking.
(3). Suppose $\alpha^2 < \beta^2$ and $(p + \frac{2}{3}) < 4(\alpha^2 - \beta^2)$, then $(\alpha + \beta)(\alpha - \beta) < 0$ which implies α always less than $-\beta$. Then $\lambda < 0$ and the conformal Ricci soliton becomes expanding.

Theorem 3.3 : A 3-dimensional trans-Sasakian manifold admitting a conformal Ricci soliton (g, ξ, λ) satisfies the following relations:
1. For $\alpha > \beta$, the conformal Ricci soliton is shrinking.
2. For $\alpha < -\beta$ and $(p + \frac{2}{3}) > 4(\alpha^2 - \beta^2)$ the conformal Ricci soliton becomes shrinking.
3. For $\alpha < -\beta$ and $(p + \frac{2}{3}) < 4(\alpha^2 - \beta^2)$ the conformal Ricci soliton becomes expanding.

4. Almost conformal gradient shrinking Ricci soliton on 3-dimensional trans-Sasakian manifold

A conformal gradient shrinking Ricci soliton equation is given by

$$S + \nabla\nabla f = (\frac{1}{2\tau} - \frac{2}{3} - p)g. \tag{4.1}$$

This reduces to

$$\nabla_Y Df + QY = (\frac{1}{2\tau} - \frac{2}{3} - p)Y, \tag{4.2}$$

where D is the gradient operator of g.
From (4.2) it follows that

$$\nabla_X \nabla_Y Df + \nabla_X QY = (\frac{1}{2\tau} - \frac{2}{3} - p)\nabla_X Y.$$

Now,

$$\begin{aligned} R(X,Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \\ &= (\frac{1}{2\tau} - \frac{2}{3} - p)[\nabla_X Y - \nabla_Y X - [X,Y]] - \nabla_X(QY) + \nabla_Y(QX) + Q[X,Y], \end{aligned}$$

where R is the curvature tensor.
Since ∇ is Levi-Civita connection, so from the above equation we get

$$R(X,Y)Df = -\nabla_X(QY) + \nabla_Y(QX) + Q[X,Y] = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{4.3}$$

Again differentiating equation (2.14) with respect to W and then putting $W = \xi$ we get

$$(\nabla_\xi Q)X = \frac{dr(\xi)}{2}(X - \eta(X)\xi).$$

So

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = g(\frac{dr(\xi)}{2}(X - \eta(X)\xi), \xi) = 0. \tag{4.4}$$

Putting this value in (4.3) we get

$$g(R(\xi, X)Df, \xi) = 0. \tag{4.5}$$

Again from (2.13) and (4.5) we obtain

$$(\alpha^2 - \beta^2)(g(X, Df) - \eta(X)\eta(Df)) = 0.$$

Since $\alpha^2 \neq \beta^2$, we have from the above equation

$$g(X, Df) = \eta(X)g(Df, \xi)$$

which implies

$$Df = (\xi f)\xi. \tag{4.6}$$

Now from (4.2) we have

$$g(\nabla_Y Df, X) + g(QY, X) = (\frac{1}{2\tau} - \frac{2}{3} - p)g(Y, X)$$

i.e.

$$\begin{aligned} S(X,Y) - (\frac{1}{2\tau} - \frac{2}{3} - p)g(Y,X) &= g(\nabla_Y(\xi f)\xi, X) \\ &= -\alpha(\xi f)g(\phi Y, X) + \beta(\xi f)g(X,Y) \\ &- \beta(\xi f)\eta(Y)\eta(X) + Y(\xi f)\eta(X). \end{aligned} \tag{4.7}$$

Putting $X = \xi$ in (4.7) we get

$$S(X, \xi) - (\frac{1}{2\tau} - \frac{2}{3} - p)\eta(Y) = Y(\xi f).$$

So

$$2(\alpha^2 - \beta^2)\eta(Y) - (\frac{1}{2\tau} - \frac{2}{3} - p)\eta(Y) = Y(\xi f). \tag{4.8}$$

Now from (4.7) and interchanging X, Y we obtain

$$\begin{aligned} S(X,Y) - (\frac{1}{2\tau} - \frac{2}{3} - p)g(Y,X) &= -\alpha(\xi f)g(\phi X, Y) + \beta(\xi f)g(X,Y) \\ &- \beta(\xi f)\eta(Y)\eta(X) + X(\xi f)\eta(Y). \end{aligned} \tag{4.9}$$

Adding (4.7) and (4.9) we get

$$\begin{aligned} 2S(X,Y) - 2(\frac{1}{2\tau} - \frac{2}{3} - p)g(Y,X) &= 2\beta(\xi f)g(X,Y) - 2\beta(\xi f)\eta(Y)\eta(X) \\ &+ (\xi f)(Y\eta(X) + X\eta(Y)). \end{aligned} \tag{4.10}$$

Putting the value of $Y(\xi f)$ in the above equation we get

$$\begin{aligned} QY - (\frac{1}{2\tau} - \frac{2}{3} - p)Y &= \beta(\xi f)Y - \beta(\xi f)\eta(Y)\xi + 2(\alpha^2 - \beta^2)\eta(Y)\xi \\ &- (\frac{1}{2\tau} - \frac{2}{3} - p)\eta(Y)\xi. \end{aligned}$$

Hence from (4.2) we can write

$$\nabla_Y Df = \beta(\xi f)[Y - \eta(Y)\xi] + [2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\eta(Y)\xi.$$

Now,

$$\begin{aligned} R(X,Y)Df &= \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\ &= \nabla_X(\beta(\xi f)(Y - \eta(Y)\xi) + (2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p))\eta(Y)\xi) \\ &- \nabla_Y(\beta(\xi f)(X - \eta(X)\xi) + (2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p))\eta(X)\xi) \\ &- \nabla_{[X,Y]}Df \\ &= 2(\alpha^2 - \beta^2)[\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi] - \beta(\xi f)[\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi] \\ &- (\frac{1}{2\tau} - \frac{2}{3} - p)[\eta(Y)\nabla_X\xi - \eta(X)\nabla_Y\xi] - \nabla_{[X,Y]}Df \\ &+ \beta(\xi f)[X,Y]. \end{aligned} \tag{4.11}$$

Also

$$\begin{aligned} \nabla_{[X,Y]}Df &= \beta(\xi f)([X,Y] - \eta([X,Y])\xi) + (2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p))\eta([X,Y])\xi \\ &= \beta(\xi f)[X,Y] - \beta(\xi f)\nabla_X\eta(Y)\xi + \beta(\xi f)\xi(\nabla_X\eta)Y + \beta(\xi f)\nabla_Y\eta(X)\xi \\ &- \beta(\xi f)\xi(\nabla_Y\eta)X + [2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\nabla_X\eta(Y)\xi \\ &- [2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\xi(\nabla_X\eta)Y - [2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} \\ &- \frac{2}{3} - p)]\nabla_Y\eta(X)\xi + [2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p)]\xi(\nabla_Y\eta)X. \end{aligned} \tag{4.12}$$

Putting (4.12) in (4.11) and taking inner product with ξ we have

$$2(\alpha^2 - \beta^2) - (\frac{1}{2\tau} - \frac{2}{3} - p) - \beta(\xi f) = 0.$$

From (4.8) we obtain,

$$\beta(\xi f)\eta(Y) = Y(\xi f). \tag{4.13}$$

Using (4.13) in (4.10) we have

$$S(X,Y) - (\frac{1}{2\tau} - \frac{2}{3} - p)g(X,Y) = \beta(\xi f)g(X,Y).$$

After contraction

$$(\xi f) = \frac{-1}{n\beta} - \frac{1}{\beta}(\frac{1}{2\tau} - \frac{2}{3} - p) = C,$$

where C is a constant.

So from (4.6) we get

$$Df = (\xi f)\xi = C\xi \tag{4.14}$$

Therefore

$$g(Df, X) = g(C\xi, X)$$

which gives

$$df(X) = C\eta(X).$$

Applying exterior differentiation on the above relation we get
$Cd\eta = 0$ as $d^2f(X) = 0$.

So from (4.14) we have found that f is constant as $d\eta = 0$.

Finally from (4.1) we get

$$\begin{aligned} S(X,Y) &= (\frac{1}{2\tau} - \frac{2}{3} - p)g(X,Y) \\ &= 2(\alpha^2 - \beta^2)g(X,Y). \end{aligned}$$

Hence M is an Einstein manifold.

Thus we can conclude the following theorem:

Theorem 4.1 : If a 3-dimensional trans-Sasakian manifold admits conformal gradient shrinking Ricci soliton, then the manifold is an Einstein manifold.

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