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ALMOST CONTACT MANIFOLDS WITH KILLING STRUCTURE TENSORS

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An almost Hermitian manifold whose almost complex structure is Killing is called a nearly Kaehler manifold; the usual almost complex structure on the six-sphere is a well-known example.

The purpose of this note is to introduce the study of almost contact metric manifolds whose almost contact structure tensors are Killing. In particular if such a structure is normal it is cosymplectic. Hypersurfaces of nearly Kaehler manifolds are also studied. As an example, it is shown that the five-sphere carries a nonnormal almost contact metric structure. More generally, the induced structure on a compact orientable hypersurface of a nearly Kaehler manifold of positive curvature cannot be cosymplectic.

1. Introduction. An almost Hermitian manifold whose almost complex structure is Killing is called a *nearly Kaehler manifold* by A. Gray and an *almost Tachibana space* by K. Yano. The usual almost Hermitian structure on the six-sphere is a well-known example. The reader is referred to A. Gray [3] and K. Yano [7] for a discussion of these spaces and for further references.

The purpose of this note is to introduce the study of almost contact metric manifolds whose almost contact structure tensors are Killing. In §2 we review almost contact structures and in §3 prove that if the structure tensors are Killing then, if the structure is normal it is cosymplectic. Section 4 reviews the induced almost contact metric structure on a hypersurface of an almost Hermitian manifold. Section 5 discusses hypersurfaces of nearly Kaehler manifolds and generalizes some of the results of H. Proppe [4]. Finally in §6 we show, as an example, that the five-sphere carries an almost contact metric structure with Killing structure tensors. In contrast to the canonical normal contact metric structure on an odd-dimensional sphere, this structure on the five-sphere is not normal. More generally, the induced structure on a compact orientable hyper surface of a nearly Kaehler manifold of positive curvature cannot be cosymplectic.

2. Almost contact structures. A $(2n+1)$-dimensional C^∞ manifold M^{2n+1} is said to have an *almost contact structure* if there exists on M^{2n+1} a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1 , \qquad \varphi\xi = 0 ,$$
$$\eta\circ\varphi = 0 , \qquad \varphi^2 = -I + \xi \otimes \eta ;$$

this is equivalent to a reduction of the structural group of the tangent bundle to $U(n)\times 1$ (see[5]). If M^{2n+1} has an almost contact structure (φ, ξ, η) then we can find a Riemannian metric g on M^{2n+1} such that

$$\eta(X) = g(\xi, X)$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

where X and Y are vector fields on M^{2n+1} [5].

S. Sasaki and Y. Hatakeyama [5] defined an almost complex structure J on $M^{2n+1}\times R^1$ by

$$J\Big(X, f\frac{d}{dt}\Big) = \Big(\varphi X - f\xi,\ \eta(X)\frac{d}{dt}\Big)$$

where f is a C^∞ real-valued function on $M^{2n+1}\times R^1$. Considering the Nijenhuis torsion $[J, J]$ of J, they computed $[J, J]((X, 0), (Y, 0))$ and $[J, J]((X, 0), (0, d/dt))$ which gave rise to four tensors $N^{(1)}, N^{(2)}, N^{(3)}$ $N^{(4)}$ given by

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) + d\eta(X, Y)\xi$$
$$N^{(2)}(X, Y) = (\mathfrak{L}_{\varphi X}\eta)(Y) - (\mathfrak{L}_{\varphi Y}\eta)(X)$$
$$N^{(3)}(X) = (\mathfrak{L}_\xi\varphi)X$$
$$N^{(4)}(X) = -(\mathfrak{L}_\xi\eta)(X)$$

where $\mathfrak{L}$ denotes Lie differentiation. The result is that J is integrable if and only if $N^{(1)} = 0$; in particular $N^{(1)} = 0$ implies $N^{(2)} = N^{(3)} = N^{(4)} = 0$ [5]. An almost contact structure is said to be *normal* if $N^{(1)} = 0$, that is, if the almost complex structure on $M^{2n+1}\times R^1$ is integrable.

Finally we define a fundamental 2-form Φ by

$$\Phi(X, Y) = g(X, \varphi Y) .$$

An almost contact metric structure (φ, ξ, η, g) is said to be *cosymplectic* if it is normal and both Φ and η are closed [1]. Cosymplectic manifolds are also characterized by normality and the vanishing of the Riemannian covariant derivative of φ.

The following Lemmas will be needed later.

LEMMA 2.1. *If* $N^{(4)} = 0$ (in particular if the almost contact structure is normal), *then* $d\eta(X, \xi) = 0$.

Proof. The proof is a computation using the coboundary formula. $d\eta(X, \xi) = X(\eta(\xi)) - \xi(\eta(X)) - \eta([X, \xi]) = -(\mathfrak{L}_\xi\eta)(X) = 0.$

Let ∇ denote covariant differentiation with respect to g.

LEMMA 2.2. *On an almost contact metric manifold*

$$(\nabla_X \Phi)(\varphi Y, Z) - (\nabla_X \Phi)(Y, \varphi Z) = -\eta(Z)(\nabla_X \eta)(Y) - \eta(Y)(\nabla_X \eta)(Z) .$$

Proof.

$$\begin{aligned}(\nabla_X \Phi)(\varphi Y, Z) - (\nabla_X \Phi)(Y, \varphi Z) &= -g(Z, (\nabla_X \varphi)\varphi Y) - g(Z, \varphi(\nabla_X \varphi) Y) \\ &= -g(Z, (\nabla_X \varphi^2) Y) \\ &= -g(Z, (\nabla_X (\xi \otimes \eta)) Y) \\ &= -g(Z, \nabla_X \eta(Y)\xi - \eta(\nabla_X Y)\xi) \\ &= -\eta(Z)\nabla_X \eta(Y) - \eta(Y) g(Z, \nabla_X \xi) + \eta(Z)\eta(\nabla_X Y) \\ &= -\eta(Z)(\nabla_X \eta)(Y) - \eta(Y)(\nabla_X \eta)(Z) .\end{aligned}$$

We close this section with a brief discussion of a Killing tensor of type (1, 1). Let M be a Riemannian manifold with Riemannian connexion ∇. Let φ be a tensor field of type (1, 1) on M and γ a geodesic; we denote by γ_* the tangent vector field of γ. Then we have a vector field $\varphi\gamma_*$ along γ. If $\varphi\gamma_*$ is parallel along γ we have $\nabla_{\gamma_*}\varphi\gamma_* = 0$ or $(\nabla_{\gamma_*}\varphi)\gamma_* = 0$. If this is the case for any geodesic, we have

$$(\nabla_X \varphi)X = 0 \text{ or } (\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0$$

for any vector fields X and Y. We then say φ is a *Killing tensor field*.

3. Almost contact metric manifolds with φ and η Killing.

For the moment consider an almost Hermitian structure (J, G):

$$J^2 = -I, G(JX, JY) = G(X, Y) .$$

Let $\bar{\nabla}$ denote the Riemannian connexion of G. Then J is Killing if and only if $(\bar{\nabla}_X J)\, X = 0$ for every X, equivalently

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0 \text{ or } (\bar{\nabla}_X \Omega)(Y, Z) + (\bar{\nabla}_Z \Omega)(Y, X) = 0$$

where Ω is the fundamental 2-form of the structure:

$$\Omega(X, Y) = G(X, JY) .$$

It is well-known [7] that if the almost complex structure of a nearly Kaehler manifold is integrable, then it is Kaehlerian.

In this section we prove that if an almost contact metric structure (φ, ξ, η, g) with φ and η Killing is normal then it is cosymplectic.

THEOREM 3.1. *Suppose M^{2n+1} has an almost contact metric* *structure* (φ, ξ, η, g) *such that* φ *and* η *are Killing. Then if this structure is normal it is cosymplectic.*

Proof. By normality we have

$$\begin{aligned}
0 &= g(X, [\varphi, \varphi](Z, Y)) + g(X, d\eta(Z, Y)\xi) \\
&= g(X, (\nabla_{\varphi Z}\varphi)Y - (\nabla_{\varphi Y}\varphi)Z + \varphi(\nabla_Y\varphi)Z - \varphi(\nabla_Z\varphi)Y) + g(X, d\eta(Z, Y)\xi) \\
&= (\nabla_{\varphi Z}\Phi)(X, Y) - (\nabla_{\varphi Y}\Phi)(X, Z) - (\nabla_Y\Phi)(\varphi X, Z) + (\nabla_Z\Phi)(\varphi X, Y) \\
&\quad + \eta(X)(\nabla_Z\eta)(Y) - \eta(X)(\nabla_Y\eta)(Z) \\
&= (\nabla_{\varphi Z}\Phi)(X, Y) - (\nabla_{\varphi Y}\Phi)(X, Z) - (\nabla_Y\Phi)(X, \varphi Z) + (\nabla_Z\Phi)(X, \varphi Y) \\
&\quad + \eta(Z)(\nabla_Y\eta)(X) - \eta(Y)(\nabla_Z\eta)(X) \\
&= (\nabla_X\Phi)(Y, \varphi Z) + (\nabla_X\Phi)(\varphi Y, Z) + (\nabla_X\Phi)(Y, \varphi Z) + (\nabla_X\Phi)(\varphi Y, Z) \\
&\quad + \eta(Z)(\nabla_Y\eta)(X) - \eta(Y)(\nabla_Z\eta)(X) \\
&= 4(\nabla_X\Phi)(Y, \varphi Z) - 2\eta(Z)(\nabla_X\eta)(Y) - 2\eta(Y)(\nabla_X\eta)(Z) \\
&\quad + \eta(Z)(\nabla_Y\eta)(X) - \eta(Y)(\nabla_Z\eta)(X) \\
&= 4(\nabla_X\Phi)(Y, \varphi Z) - 3\eta(Z)(\nabla_X\eta)(Y) - \eta(Y)(\nabla_X\eta)(Z) ,
\end{aligned}$$

using Lemma 2.2 several times and the fact that φ and η are Killing. On the other hand

$$\begin{aligned}
(\nabla_X\Phi)(Y, (\xi \otimes \eta)Z) &= -(\nabla_X\Phi)(\eta(Z)\xi, Y) \\
&= -\eta(Z)\eta((\nabla_X\varphi)Y) \\
&= -\eta(Z)\eta(\nabla_X\varphi Y) \\
&= \eta(Z)(\nabla_X\eta)(\varphi Y) .
\end{aligned}$$

Hence we have

$$\begin{aligned}
4(\nabla_X\Phi)(Y, (\varphi + \xi \otimes \eta)Z) &= 3\eta(Z)(\nabla_X\eta)(Y) + \eta(Y)(\nabla_X\eta)(Z) \\
&\quad + 4\eta(Z)(\nabla_X\eta)(\varphi Y) .
\end{aligned}$$

But $-\varphi + \xi \otimes \eta$ is the inverse of the nonsingular transformation $\varphi + \xi \otimes \eta$, so that

$$\begin{aligned}
4(\nabla_X\Phi)(Y, Z) &= 3\eta(Z)(\nabla_X\eta)(Y) + \eta(Y)(\nabla_X\eta)(-\varphi Z + \eta(Z)\xi) \\
&\quad + 4\eta(Z)(\nabla_X\eta)(\varphi Y) \\
&= \frac{3}{2}\eta(Z)d\eta(X, Y) - \frac{1}{2}\eta(Y)d\eta(X, \varphi Z) \qquad (3.1) \\
&\quad + 2\eta(Z)\, d\eta(X, \varphi Y)
\end{aligned}$$

by use of the fact that since η is Killing, $d\eta(Y, X) = -2(\nabla_X\eta)(Y)$ and that $d\eta(X, \xi) = 0$. The first conclusion from equation (3.1) is that $\nabla_\xi\varphi = 0$ and hence since φ is Killing

$$0 = (\nabla_\xi\varphi)X = -(\nabla_X\varphi)\xi = \varphi\nabla_X\xi .$$

Thus if X is orthogonal to ξ, $X = \varphi Z$ for some Z and we have

$$\begin{aligned} d\eta(X, Y) &= -2(\nabla_Y \eta)(X) = -2g(X, \nabla_Y \xi) \\ &= -2g(\varphi Z, \nabla_Y \xi) = 2g(Z, \varphi \nabla_Y \xi) \\ &= 0 . \end{aligned}$$

Hence, since $d\eta(\xi, Y) = 0$ also, we have $d\eta = 0$ and equation (3.1) yields $\nabla_X \Phi = 0$ and therefore $d\Phi = 0$, completing the proof.

An almost contact metric manifold whose structure tensors are Killing fields will be called a *nearly cosymplectic manifold.*

4. Hypersurfaces of almost Hermitian manifolds. Let M^{2n} be an almost Hermitian manifold with structure tensors (J, G) and M^{2n-1} a C^∞ orientable hypersurface of M^{2n}. Let B denote the differential of the imbedding and C a unit normal. The induced metric g on M^{2n-1} is given by $g(X, Y) = G(BX, BY)$ and the Gauss-Weingarten equations are

$$\bar{\nabla}_{BX} BY = B\nabla_X Y + h(X, Y)C, \; \bar{\nabla}_{BX} C = -BHX$$

where $\bar{\nabla}, \nabla$ are the Riemannian connexions of G and g respectively. h denotes the second fundamental form and H the corresponding Weingarten map.

Y. Tashiro [6] showed that the almost Hermitian structure (J, G) induces an almost contact metric structure on M^{2n-1}. We review this construction briefly.

Define a tensor field φ of type (1, 1), a vector field ξ and a 1-form η on M^{2n-1} by

$$JBX = B\varphi X + \eta(X)C, \; JC = -B\xi .$$

Then computing J^2BX we have

$$-BX = B\varphi^2 X + \eta(\varphi X)C - \eta(X)B\xi ;$$

comparing tangential and normal parts we have $\varphi^2 X = -X + \eta(X)\xi$ and $\eta(\varphi X) = 0$. Similarly computing J^2C have $-C = -B\varphi\xi - \eta(\xi)C$ which yields $\varphi\xi = 0$ and $\eta(\xi) = 1$.

Moreover $\eta(X) = G(JBX, C) = -G(BX, JC) = G(BX, B\xi) = g(X, \xi)$ and

$$\begin{aligned} g(X, Y) &= G(BX, BY) = G(JBX, JBY) \\ &= G(B\varphi X, B\varphi Y) + \eta(X)\eta(Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) . \end{aligned}$$

Thus we have

PROPOSITION 4.1 (Tashiro [6]). *A C^∞ orientable hypersurface of an* *almost Hermitian manifold carries a naturally induced almost contact metric structure.*

5. Hypersurfaces of nearly Kaehler manifolds. In this section we consider the induced almost contact metric structure (φ, ξ, η, g) on a hypersurface of a nearly Kaehler manifold. H. Proppe [4] observed that if the hypersurface is totally geodesic then φ and η are Killing. We first show that φ is Killing if and only if the second fundamental form h is proportional to $\eta \otimes \eta$. This result should be compared with a theorem of S. I. Goldberg [2] that a hypersurface of a Kaehler manifold is cosymplectic if and only if h is proportional to $\eta \otimes \eta$. We also show that in our case, if h is proportional to $\eta \otimes \eta$ then η is Killing.

THEOREM 5.1. *Let M^{2n} be a nearly Kaehler manifold and M^{2n-1} a C^∞ orientable hypersurface. Let (φ, ξ, η, g) denote the induced almost contact metric structure on M^{2n-1}. Then φ is Killing if and only if the second fundamental form h is proportional to $\eta \otimes \eta$.*

Proof. Let (J, G) denote the nearly Kaehler structure on M^{2n} and Ω its fundamental 2-form. Computing $\nabla_X \Phi$ we have

$$\begin{aligned}(\nabla_X \Phi)(Y, Z) &= Xg(Y, \varphi Z) - g(\nabla_X Y, \varphi Z) + g(\nabla_X Z, \varphi Y) \\ &= BXG(BY, JBZ) - G(\bar{\nabla}_{BX} BY, JBZ) + h(X, Y)\eta(Z) \\ &\qquad + G(\bar{\nabla}_{BX} BZ, JBY) - h(X, Z)\eta(Y) \\ &= (\bar{\nabla}_{BX}\Omega)(BY, BZ) + h(X, Y)\eta(Z) - h(X, Z)\eta(Y) .\end{aligned}$$

Now since Ω is Killing

$$\begin{aligned}(\nabla_X \Phi)(Y, Z) + (\nabla_Z \Phi)(Y, X) = &- 2\eta(Y)h(X, Z) \\ &+ \eta(Z)h(X, Y) + \eta(X)h(Z, Y) .\end{aligned} \tag{5.1}$$

Clearly then if h is proportional to $\eta \otimes \eta$, Φ is Killing. Conversely if Φ is Killing,

$$2\eta(Y)h(X, Z) - \eta(Z)h(X, Y) - \eta(X)h(Z, Y) = 0 . \tag{5.2}$$

Thus taking $Z = \xi$, we have $h(\varphi^2 X, \varphi^2 Y) = 0$ or

$$h(-X + \eta(X)\xi, \ - Y + \eta(Y)\xi) = 0$$

and hence

$$h(X, Y) = \eta(Y)h(X, \xi) + \eta(X)h(Y, \xi) - \eta(X)\eta(Y)h(\xi, \xi) . \tag{5.3}$$

On the other hand setting $X = Y = \xi$ in (5.2) gives

$$h(\xi, Z) = \eta(Z)h(\xi, \xi)$$

for all Z. Therefore (5.3) becomes

$$h(X, Y) = h(\xi, \xi)\eta(X)\eta(Y)$$

as desired.

THEOREM 5.2. *Let M^{2n} be a nearly Kaehler manifold and M^{2n-1} a C^∞ orientable hypersurface. Let η denote the induced* (almost) *contact form and suppose the second fundamental form h is proportional to $\eta \otimes \eta$. Then η is Killing, in particular M^{2n-1} is a nearly cosymplectic manifold.*

Proof. Computing $\bar{\nabla}_{BX}JC$ in two ways we have

$$\bar{\nabla}_{BX}JC = \bar{\nabla}_{BX}(-B\xi) = -B\,\nabla_X\xi - h(X, \xi)C\,,$$

$$\bar{\nabla}_{BX}JC = (\bar{\nabla}_{BX}J)C + J(-BHX) = (\bar{\nabla}_{BX}J)C - B\varphi HX - \eta(HX)C\,,$$

and hence

$$(\bar{\nabla}_{BX}J)C = -B\nabla_X\xi + B\varphi HX\,.$$

Now, since J is Killing, we have

$$\begin{aligned} 0 &= G((\bar{\nabla}_{BX}J)BY + (\bar{\nabla}_{BY}J)BX, C) \\ &= -G(BY, (\bar{\nabla}_{BX}J)C) - G(BX, (\bar{\nabla}_{BY}J)C) \\ &= G(BY, B\nabla_X\xi - B\varphi HX) + G(BX, B\nabla_Y\xi - B\varphi HY) \\ &= (\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) + h(\varphi Y, X) + h(\varphi X, Y) \\ &= (\nabla_X\eta)(Y) + (\nabla_Y\eta)(X)\,. \end{aligned}$$

6. Applications. We shall first briefly describe the well known nearly Kaehler structure on the six-sphere S^6. R^7 considered as the space of pure Cayley numbers admits a vector product $\times$. Letting N denote the outer normal to S^6 and B the differential of the imbedding, $BJX = N \times BX$ defines an almost complex structure J on S^6 which is almost Hermitian with respect to the canonically induced metric. Using the fact that the Weingarten map is $-I$ in the Gauss-Weingarten equations, a direct computation shows that J is Killing.

As an example we show that besides its structures as a normal contact metric manifold the five sphere S^5 carries a nearly cosymplectic structure which is not cosymplectic. Consider S^5 as a totally geodesic hypersurface of S^6 with the above nearly Kaehler structure, then by the results of §5 the induced almost contact metric structure (ϕ, ξ, η, g) has ϕ and η Killing. In particular since η is Killing it is coclosed. Now if this structure is normal, it is cosymplectic by Theorem 3.1 and hence η is closed. Thus η is harmonic contradicting the vanishing of the first betti number of S^5.

More generally let M^{2n} be a nearly Kaehler manifold of positive curvature and M^{2n-1} a compact orientable hypersurface, then the induced structure cannot be cosymplectic. For suppose M^{2n-1} is cosymplectic. Then $\nabla_X \Phi = 0$ so by Theorem 5.1 h is proportional to $\eta \otimes \eta$. Thus contracting the Gauss equation we see that M^{2n-1} has positive definite Ricci curvature. This implies the vanishing of the first betti number of M^{2n-1}, contradicting the fact that η is harmonic on a compact cosymplectic manifold.

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