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ALMOST CONVERGENCE OF DOUBLE SUBSEQUENCES

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Abstract

Almost-convergence of double sequences (subsequences) is equivalent to almost Cauchy condition. If the set of all almost convergent subsequences of a sequence $S = S_{nm}$ is of the second category, then S is convergent in the simple sense. For the sequence $S = S_{nm}$ which almost converges to L, Lebesgue measure of the set of all its subsequences which almost converge to L is either 1 or 0.

Introduction

The concept of almost convergence of sequences of real numbers $S = S_n$ was introduced and firstly studied by G.G. Lorentz [3].

A sequence $S = S_n$ almost converges to L if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $\left|\frac{1}{n}\sum_{i=0}^{n-1} S_{k-i} - L\right| < \varepsilon$ for $\forall n > N$ and $\forall k \in \mathbb{N}$. We write $f - \lim S = L$.

F. Moricz and B.E. Rhoades [4] have expanded the definition of almost convergence to double sequences of real numbers $S = (S_{nm})$. The sequence $S = (S_{nm})$ almost converges to L, if for $\forall\varepsilon > 0$, $\exists N \in \mathbb{N}$, such that

$$\left|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} S_{n+i,m+j} - L\right| < \varepsilon$$

for $\forall p, q > N$ and $\forall (n, m) \in \mathbb{N} \times \mathbb{N}$. We write $f - \lim S = L$.

We denote by X the set of all double sequences of 0's and 1's,namely,

$$X = \{x = (x_{nm}) : x_{nm} \in \{0, 1\},\ n, m \in \mathbb{N}\}$$

In [4] F. Moricz defined the concept of a subsequence of a double sequence.

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Let $S = (S_{nm})$ be a double sequence and $x = ((x_{nm}) \in X$. Then, by $S(x)$ we denote a double sequence defined as folows: $S_{nm}(x) = \begin{cases} S_{nm} : & x_{nm} = 1 \\ *, & x_{nm} \end{cases}$, which we call a subsequence of the sequence S.

For example, for

$$S = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ s_{31} & s_{32} & s_{33} & \cdot \\ s_{21} & s_{22} & s_{23} & \cdot \\ s_{11} & s_{12} & s_{13} & \cdot \end{pmatrix}; \ X = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & \cdot \\ 0 & 1 & 1 & \cdot \\ 1 & 1 & 0 & \cdot \end{pmatrix}$$

we have

$$S(x) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ s_{31} & * & s_{33} & \cdot \\ * & s_{22} & s_{23} & \cdot \\ s_{11} & s_{12} & * & \cdot \end{pmatrix}.$$

Mapping $x \to S(x)$ is a bijection from the set X to the set of all the subsequences of sequences $S = (S_{nm})$.

In[2], Lebesgue measure on the set of all subsequences of the given sequence is defined. Due to the bijection of the set X and the set of all subsequences of a given sequence, under Lebesgue measure on the set of all subsequences we consider Lebesgue measure on the set X.

Let $\mathcal{B}$ be the smallest σ-algebra of subsets of the set X which contains all sets of the form:

$$\{x = (x_{nm}) \in X : x_{nm} = a_1, \ldots, x_{n_k m_k} = a_k\}, \ a_1, \ldots, a_k \in \{0,1\}, \ k \in \mathbb{N}.$$

There exists the unique Lebesgue measure P on the set X, such that:

$$P\left(\{x = (x_{nm}) \in X : x_{nm} = a_1, \ldots, x_{n_k m_k} = a_k\}\right) = \frac{1}{2^k},$$

where $a_1, \ldots, a_k \in \{0,1\}, \ k \in \mathbb{N}$.

Results

We introduce the concept of almost convergence of double subsequences.

Definition 1. *Let $x \in X$ and $S = (S_{nm})$ be a double sequence of real numbers. The subsequence $S(x)$ of the sequence S almost converges to L, if for $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that:*

$$\left| \frac{\sum_{i=0}^{p-1} \sum_{j=0}^{q-1} [S_{n+i,m+j} \ : \ x_{n+i,m+j} = 1]}{\sum_{i=0}^{p-1} \sum_{j=0}^{q-1} x_{n+i,m+j}} - L \right| < \varepsilon$$

for $\forall p, q \in \mathbb{N}$ and $\forall (n,m) \in \mathbb{N} \times \mathbb{N}$ for which $x_{nm} = 1$.

Definition 2. *The subsequence $S(x)$ of the sequences S is almost Cauchy, if for*
$\forall\varepsilon > 0, \exists k \in \mathbb{N}$, *such that:*

$$\left|\frac{\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1}\left[S_{n_1+i,m_1+j} \ : \ x_{n_1+i,m_1+j}=1\right]}{\sum_{i=0}^{p_1-1}\sum_{j=0}^{q_1-1} x_{n_1+i,m_1+j}} - \frac{\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1}\left[S_{n_2+i,m_2+j} \ : \ x_{n_2+i,m_2+j}=1\right]}{\sum_{i=0}^{p_2-1}\sum_{j=0}^{q_2-1} x_{n_2+i,m_2+j}}\right| \leq \varepsilon,$$

for $\forall p_1, p_2, q_1, q_2 > k$ and $\forall (n_1, m_1), (n_2, m_2) \in \mathbb{N}\times\mathbb{N}$, for which $x_{n_1 m_1} = 1$, $x_{n_2 m_2} = 1$.

The equivalence between these two concepts is established in

Theorem 1. *Let $x \in X$ and $S = (S_{nm})$ be a double sequence. The subsequence $S(x)$ of the sequence S is almost convergent if and only if it is almost Cauchy.*

Proof. We denote

$$D_{n,m}^{p,q}(sx) = \frac{\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\left[S_{n+i,m+j} \ : \ x_{n+i,m+j} = 1\right]}{\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{n+i,m+j}}.$$

Assume that the subsequence $S(x)$ is almost convergent. Then, $\forall\varepsilon > 0$, $\exists k \in \mathbb{N}$, such that

$$\left|D_{n,m}^{p,q}(sx) - L\right| < \frac{\varepsilon}{2}$$

for $\forall p, q > K$ and $\forall (n, m) \in \mathbb{N}\times\mathbb{N}$ for which $x_{nm} = 1$. Therefore,

$$\left|D_{n_1,m_1}^{p_1,q_1}(sx) - D_{n_2,m_2}^{p_2,q_2}(sx)\right| \leq \left|D_{n_1,m_1}^{p_1,q_1}(sx) - L\right| + \left|D_{n_2,m_2}^{p_2,q_2}(sx) - L\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $\forall p_1, p_2, q_1, q_2 > k$ and $\forall (n_1, m_1), (n_2, m_2) \in \mathbb{N}\times\mathbb{N}$, for which $x_{n_1 m_1} = 1$, $x_{n_2 m_2} = 1$. Thus, the subsequence $S(x)$ is almost Cauchy.

Now assume that the subsequence $S(x)$ is almost Cauchy. Then, $\forall\varepsilon > 0$, $\exists k \in \mathbb{N}$, such that

$$\left|D_{n_1,m_1}^{p_1,q_1}(sx) - D_{n_2,m_2}^{p_2,q_2}(sx)\right| < \frac{\varepsilon}{2} \tag{1}$$

for $\forall p_1, p_2, q_1, q_2 > k$ and $\forall (n_1, m_1), (n_2, m_2) \in \mathbb{N}\times\mathbb{N}$, for which $x_{n_1 m_1} = 1$, $x_{n_2 m_2} = 1$

Taking $n_1 = n_2 = n_0$ and $m_1 = m_2 = m_0$ in relation (1), we obtain that $\left(D_{n_0,m_0}^{p,q}(sx)\right)_{p,q=1}^{\infty,\infty}$ is a Cauchy sequence, and therefore convergent.

Let $\lim_{\substack{p\to\infty \\ q\to\infty}} D_{n_0,m_0}^{p,q}(sx) = L$. Then, $\forall\varepsilon > 0$, $\exists k_1 \in \mathbb{N}$, such that $\left|D_{n_0,m_0}^{p,q}(sx) - L\right| < \frac{\varepsilon}{2}$ for $\forall p, q > K_1$. It follows that

$$\left|D_{n,m}^{p,q}(sx) - L\right| < \left|D_{n,m}^{p,q}(sx) - D_{n_0,m_0}^{p,q}(sx)\right| + \left|D_{n_0,m_0}^{p,q}(sx) - L\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $\forall p, q > \max(K, K_1)$ and $\forall (n,m) \in \mathbb{N} \times \mathbb{N}$ for which $x_{nm} = 1$ Therefore, the subsequence $S(x)$ is almost convergent.

In particular, if $x_{nm} = 1$ for $\forall (n,m) \in \mathbb{N} \times \mathbb{N}$ then the subsequence $S(x)$ is equal to the sequence S. Therefore, Theorem 1 is also true for sequences.

The set X equiped with the metric $d : X \times X \to \mathbb{R}$, $d(x,s) = \sum\limits_{n=1}^{\infty} \sum\limits_{m=1}^{\infty} \frac{|x_{nm} - y_{nm}|}{2^{n+m}}$, $x = (x_{nm}), y = (y_{nm}) \in X$, is a complete metric space. Therefore, X is of the second category.

Let $X' = \{x \in X : \exists n, m > k \text{ such that } x_{nm} = 1\}$.

The sets $\{x \in X : \exists n, m > k \text{ such that } x_{nm} = 1\}, k \in \mathbb{N}$ are dense everywhere in X. Hence, $X_k = \{x \in X : x_{nm} = 0 \text{ za } \forall n, m > k\}, k \in \mathbb{N}$, are nowhere dense in X.

Since $X \backslash X' = \bigcup\limits_{k=1}^{\infty} X_k$, $X \backslash X'$ is of the first category. Therefore, X' is of the second category.

Theorem 2. *Let a sequence $S = (S_{nm})$ be divergent in the simple sense. Then the set:*

$$U = \{x \in X' : S(x) \ \textit{is almost convergent}\}$$

is of the first category.

Proof: Theorem 1 implies $U = \{x \in X : S(x) \text{ is almost Cauchy}\}$.

The set U can be writen in the form:

$$U = \bigcap_{l=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{\substack{p_1=k \\ p_2=k \\ q_1=k \\ q_2=k}}^{\infty} \bigcap_{\substack{n_1=1 \\ n_2=1 \\ m_1=1 \\ m_2=1}}^{\infty} \left\{ x \in X' : \left| D_{n_1,m_1}^{p_1,q_1}(sx) - D_{n_2,m_2}^{p_2,q_2}(sx) \right| < \frac{1}{l} \right\} \tag{2}$$

where

$$D_{n,m}^{p,q}(sx) = \frac{\sum\limits_{i=0}^{p-1} \sum\limits_{j=0}^{q-1} [S_{n+i,m+j} \ : \ x_{n+i,m+j} = 1]}{\sum\limits_{i=0}^{p-1} \sum\limits_{j=0}^{q-1} x_{n+i,m+j}}.$$

Since the sequence S is convergent, we have two cases:

I) There exists a subsequence $\left(S_{n_g,m_g}\right)_{g=1}^{\infty}$ of the sequence S such that

$$\lim_{g\to\infty} S_{n_g,m_g} = \pm\infty, \quad \lim_{g\to\infty} n_g = \lim_{g\to\infty} m_g = \infty$$

II) There exist $a, b \in \mathbb{R}$ such that $\liminf S_{nm} < \limsup S_{nm} = b$. Let $T = \{x \in X : x_{nm} = 1 \text{ for finitely many } (n,m)\}$. The set$T$ is everywhere dense in X'.

I Let k and $l = l_0 > 3$ be fixed.

For arbitrary $y \in T$, $\exists n_1, m_1, p_1, q_1 \in \mathbb{N}$, $p_1, q_1 > k$, such that $\{(n,m) : y_{nm} = 1\} \subseteq \{(n,m) : m_1 \leq n < m_1 + p_1, \ m_1 \leq m < m_1 + q_1\}$.

Due to the structure of the subsequence $\left(S_{n_g,m_g}\right)_{g=1}^{\infty}$ of the sequence S, $\exists n_1, m_1, p_1, q_1 \in \mathbb{N}$ and $\exists y' \in T$ given by

$$y' = \begin{cases} y'_{n,m} : & n_1 \leq n < n_1 + p_1, m_1 \leq m < m_1 + q_1 \\ 1 : & (n,m) = (n_g, m_g), n_2 \leq n_g < n_2 + p_2, m_2 \leq m_g < m_2 + q_2 \\ 0 : & \text{for other } (n,m) \end{cases} \tag{3}$$

such that

$$\left|D^{p_1,q_1}_{n_1,m_1}(sy') - D^{p_2,q_2}_{n_2,m_2}(sy')\right| \geq \frac{1}{l_0}. \tag{4}$$

Let $z \in X'$ with the property $z_{nm} = x'_{nm}$ on the set

$$\{(n,m) : m_1 \leq n < m_1 + p_1,\ m_1 \leq m < m_1 + q_1\} \cup$$
$$\cup \{(n,m) : m_2 \leq n < m_2 + p_2,\ m_2 \leq m < m_2 + q_2\}.$$

Then the relation (4) is true. Therefore, the set of all such z contains open neighborhood of $y \in T$ which does not intersect the set

$$\bigcap_{\substack{p_1=k\\p_2=k\\q_1=k\\q_2=k}}^{\infty} \bigcap_{\substack{n_1=1\\n_2=1\\m_1=1\\m_2=1}}^{\infty} \left\{x \in X' : \left|D^{p_1,q_1}_{n_1,m_1}(sx) - D^{p_2,q_2}_{n_2,m_2}(sx)\right| < \frac{1}{l_0}\right\} \tag{5}$$

Therefore, the set (5) is nowhere dense. Hence, the set U is of the first category.

II Let k and $l = l_0 > \frac{5}{b-a}$ be fixed. For arbitrary $y \in T$ there exist $p_1, q_1, n_1, m_1 \in \mathbb{N}$, $p_1, q_1 > k$, such that $\{(n,m) : x_{nm} = 1\} \subseteq \{(n,m) : m_1 \leq n < m_1 + p_1,\ m_1 \leq m < m_1 + q_1\}$, $\left|D^{p_1,q_1}_{n_1,m_1}(sy) - a\right| \geq \frac{2}{l_0} \ \vee \ \left|D^{p_1,q_1}_{n_1,m_1}(sy) - b\right| \geq \frac{2}{l_0}$.

Let, for example$\left|D^{p_1,q_1}_{n_1,m_1}(sy) - a\right| \geq \frac{2}{l_0}$. Then, due to the structure of the subsequence $\left(S_{n_g,m_g}\right)_{g=1}^{\infty}$ of the sequence S, $\exists y' \in T$ given by (3) such that

$$\left|D^{p_1,q_1}_{n_1,m_1}(sy') - a\right| \leq \frac{1}{l_0}.$$

It follows that $\left|D^{p_1,q_1}_{n_1,m_1}(sy') - D^{p_2,q_2}_{n_2,m_2}(sy')\right| \geq \frac{1}{l_0}$.

With the procedure like in ***I)***, we conclude that every y from the dense set T has an open neighborhood which does not intersect the set (5). Therefore, the set (5) is nowhere dense and the set U is of the first category.

A subset E of the set X is a tail set if it is true that: $x \in E \Rightarrow x' \in E$, where $x' \in X$ with the property $x'_{nm} \neq x_{nm}$ for most finitely many (n,m).

The following lemma is well known.

Lemma *Let $E \subseteq X$ be a measurable tail set. Then, $P(E) = 1$ or 0.*

Theorem 3. *Let a sequence $S = (S_{nm})$ almost converge to L and let*

$$U_L = \{x \in X : S(x) \text{ almost converges to } L\}.$$

Then $P(U_L) = 1$ *or* 0.

Proof: Let $x \in U_L$ be arbitrarily chosen and let $y \in X$ have the property: $y_{nm} = x_{nm}$ for $n > n_0 \vee m > m_0$. Then for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that: $\left|D_{n,m}^{p,q}(sy) - D_{n,m}^{p,q}(sx)\right| < \frac{\varepsilon}{2}$ and $\left|D_{n,m}^{p,q}(sx) - L\right| < \frac{\varepsilon}{2}$ for $\forall p, q > N$ and $\forall (n,m) \in \mathbb{N} \times \mathbb{N}$ $x_{nm} = y_{nm} = 1$. Then $\left|D_{n,m}^{p,q}(sy) - L\right| < \varepsilon$ for $\forall (n,m) \in \mathbb{N} \times \mathbb{N}$, $y_{nm} = 1$. Therefore, the subsequence $S(y)$ almost converges to L. Thus, U_L isa tail set.

The set U_L has the form:

$$U_L = \bigcap_{l=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{p,q=N}^{\infty} \bigcap_{n,m=1}^{\infty} \left\{ x \in X : \left|D_{n,m}^{p,q}(sx) - L\right| < \frac{1}{l} \right\}.$$

Since $\left\{ x \in X : \left|D_{n,m}^{p,q}(sx) - L\right| < \frac{1}{l} \right\}$ are obviously measurable sets, then U_L is a measurable set. Therefore, due to Lemma, $P(U_L) = 1$ or 0.

In [1] Connor has proved that almost every sequence of 0's and 1's is not almost convergent.

The following theorem confirms the existence of almost convergent double sequences, for which almost every subsequence is not almost convergent. This is analogous to the result in [6] about single sequences.

Theorem 4. *Let the sequence* $S = (S_{nm})$ *be a sequence of 0's and 1's which almost converges to* r*. If* $0 < r < 1$ *then the set*

$$U_r = \{x \in X : S(x) \text{ almost converges to } r\}$$

is of Lebesgue measure zero.

`Proof:` Since the sequence S almost converges to r, $\exists N \in \mathbb{N}$ such that:

$$\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S_{k+i,l+j} > \frac{r}{2}$$

for $\forall n > N$ and $\forall (k,l) \in \mathbb{N} \times \mathbb{N}$.

Let

$$T_n^k = \{x \in X : x_{kn+i,kn+j} = S_{kn+i,kn+j},\ 0 \le i < n, 0 \le j < n\}.$$

Then $P\left(T_n^k\right) = \frac{1}{2^{n^2}}$ for $\forall k \in \mathbb{N}$. Since $\sum\limits_{k=1}^{\infty} P\left(T_n^k\right) = \sum\limits_{k=1}^{\infty} \frac{1}{2^{n^2}} = \infty$, the Borel-Cantelli lemma gives us:

$$P\left(\limsup_{k\to\infty} T_n^k\right) = 1 \text{ for } \forall n \in \mathbb{N}.$$

Let

$$T_n = \limsup_{k\to\infty} T_n^k \text{ and } T = \bigcap_{n=1}^{\infty} T_n,$$

Then $P(T) = 1$.

Let $x \in T$ and $n > N$. Then $x \in T_n^k$ for some $k \in \mathbb{N}$ and has more than $\frac{r}{2}n^2$ of ones in the block $[kn,(k+1)n) \times [kn,(k+1)n)$.

Therefore

$$\frac{\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}[S_{kn+i,kn+j} \quad : \quad x_{kn+i,kn+j}=1]}{\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{kn+i,kn+j}} = 1,$$

and with that subsequence $S(x)$ is not almost convergent to r for $0 < r < 1$. Hence,

$$P(U_r) = 0.$$

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