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## ALMOST DISTURBANCE DECOUPLING WITH BOUNDED PEAKING\*

HARRY L. TRENTELMAN†

**Abstract.** This paper is concerned with a generalization of the almost disturbance decoupling problem by state feedback. Apart from approximate decoupling from the external disturbances to a first to-be-controlled output, we require a second output to be uniformly bounded with respect to the accuracy of decoupling. The problem is studied using the geometric approach to linear systems. We introduce some new almost controlled invariant subspaces and study their geometric structure. Necessary and sufficient conditions for the solvability of the above problem are formulated in terms of these controlled invariant subspaces. A conceptual algorithm is introduced to calculate the feedback laws needed to achieve the design purpose.

**Key words.** almost disturbance decoupling, almost invariant subspaces, linear systems, geometric approach, high gain feedback, output stabilization

**AMS(MOS) subject classifications.** G3-B28, G3-B50, G3-C05, G3-C15, G3-C35, G3-C45, G3-C60

**1. Introduction.** In this paper we are concerned with the problem of almost disturbance decoupling by state feedback as introduced by Willems [20]. This problem deals with the situation in which we cannot achieve exact decoupling from the external disturbances to an exogenous output channel as, for example, in [22], but only *approximate* decoupling with any desired degree of accuracy. In general, the feedback gain necessary to achieve this will increase as the desired degree of accuracy increases. It may then happen however that some of the state variables tend to peak excessively. It is of considerable practical interest to know when it is possible to achieve disturbance decoupling within any desired degree of accuracy, while this peaking phenomenon will not occur.

The system that we will be considering in this paper is given by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu + Gd, \\ (1.1) \quad z_1 &= H_1x, \\ z_2 &= H_2x, \end{aligned}$$

where the control  $u(t)$ , the state  $x(t)$ , the disturbance  $d(t)$  and the outputs  $z_1(t)$  and  $z_2(t)$  are real vectors of finite dimensions. We will assume that the vector  $z_2(t)$  is an *enlargement* of  $z_1(t)$ , i.e., there is a matrix  $M$  such that  $H_1 = MH_2$ . If for any positive real number  $\varepsilon$  a feedback matrix  $F_\varepsilon$  can be chosen such that in the closed loop system with zero initial condition, for all disturbances  $d(\cdot)$  in the unit ball of  $L_p[0, \infty)$  we have

$$(1.2) \quad \|z_1\|_{L_p} \leq \varepsilon$$

then we say that for the system under consideration the  $L_p$ -almost disturbance decoupling problem from  $d$  to  $z_1$  is solvable. After choosing  $F_\varepsilon$  to achieve this approximate decoupling, the output  $z_2(t)$  of course depends on  $\varepsilon$  and, for certain disturbances  $d(\cdot)$ , it may then happen that  $\|z_2\|_{L_p} \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , i.e., as the accuracy of decoupling increases.

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As an example, consider the system (1.1) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$H_1 = (1 \ 0 \ 0), \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define a feedback matrix  $F_\varepsilon$  by

$$F_\varepsilon := \begin{pmatrix} -\frac{27}{\varepsilon^3}, & -\frac{27}{\varepsilon^2}, & -\frac{9}{\varepsilon} \end{pmatrix}.$$

It can then be verified that the impulse response from the disturbance  $d$  to  $z_1$  is given by

$$W_{1,\varepsilon}(t) := H_1 e^{(A+BF_\varepsilon)t} G = e^{-3t/\varepsilon} \left( 1 + \frac{3}{\varepsilon} t + \frac{9}{2\varepsilon^2} t^2 \right)$$

and that  $\|W_{1,\varepsilon}\|_{L_1} = \varepsilon$ . Hence, for any  $1 \leq p \leq \infty$ , the above feedback matrix  $F_\varepsilon$  achieves  $L_p$ -almost disturbance decoupling from  $d$  to  $z_1$ . On the other hand, however, the impulse response from  $d$  to  $z_2$  is calculated to be

$$W_{2,\varepsilon}(t) := H_2 e^{(A+BF_\varepsilon)t} G = e^{-3t/\varepsilon} \begin{pmatrix} 1 + \frac{3}{\varepsilon} t + \frac{9}{2\varepsilon^2} t^2 \\ -\frac{27}{2\varepsilon^3} t^2 \\ -\frac{27}{\varepsilon^3} t + \frac{81}{2\varepsilon^4} t^2 \end{pmatrix}$$

and it can be verified that  $\|W_{2,\varepsilon}\|_{L_1} = O(1/\varepsilon) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , i.e., we have obtained almost disturbance decoupling from  $d$  to  $z_1$  at the cost of highly undesired peaking behaviour of the output  $z_2(t)$ .

The question which we ask in this paper is this: *When is it possible to choose  $F_\varepsilon$  such that simultaneously (1.2) holds and there exists a constant  $C$  (independent of  $\varepsilon$ ) such that for all disturbances  $d(\cdot)$  in the unit ball of  $L_p[0, \infty)$  we have*

$$(1.3) \quad \|z_2\|_{L_p} \leq C$$

for all  $\varepsilon$ ? That is, the output  $z_2(t)$  is bounded uniformly as  $\varepsilon$  tends to zero. If this behaviour is achieved, we say that we have  $L_p$ -bounded peaking from  $d$  to  $z_2$ . Problems of this kind have been considered before. Francis and Glover [3] considered a bounded peaking problem in the context of cheap control. More recently, Kimura [9] found conditions that guarantee bounded peaking in the context of perfect regulation. We will study the above problem using the by now well known concepts of almost controlled invariant and almost controllability subspace [19], [20]. We will also use the approach of frequency domain description of geometric concepts as initiated in Hautus [5].

The outline of this paper is as follows. In § 2 we will introduce some notational conventions used in this paper and state some preliminary results and background.

Section 3 contains a description of the main problem we will be concerned with in this paper. In § 4 we will introduce the disturbance decoupling problem with output stability. This problem is an extension of the (exact) disturbance decoupling problem

as treated in [22]. Its solution will be needed to solve our main problem, but is also important in its own right. In § 5 we will derive a necessary condition for the solvability of  $(\text{ADDPBP})_p$ . This condition will be in the form of a subspace inclusion involving an almost controlled invariant subspace. Section 6 contains an investigation of the geometric structure of the almost controlled invariant subspace that was introduced in § 5. In § 7 these structural results will be used to prove that for certain classes of systems the subspace inclusion derived in § 5 in fact constitutes a necessary and sufficient condition for solvability of  $(\text{ADDPBP})_p$ . The sequences of state feedback maps that achieve the design purpose will be constructed explicitly. Section 8 contains some corollaries of our main result and some extensions. In § 9 a numerical example is worked out to illustrate the computational feasibility of our theory. Finally, the paper closes with some concluding remarks in § 10. Several technical details of proofs in this paper are deferred to Appendices A, B and C.

**2. Preliminaries and background.** In this section we will introduce some notation used in this paper and review some relevant facts on controlled invariant and almost controlled invariant subspaces. Also some basic facts on the convergence of subspaces will be given.

**2.1.** In this paper the following notation will be used: If  $\mathcal{X}$  is a normed vector space, we will write  $\|\cdot\|$  for the norm on  $\mathcal{X}$ . If  $l: [0, \infty) \rightarrow \mathcal{X}$  is a measurable function, then we will denote

$$\|l\|_{L_p} := \begin{cases} \left( \int_0^\infty \|l(t)\|^p dt \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \text{ess sup}_{t \geq 0} \|l(t)\| & \text{if } p = \infty. \end{cases}$$

If  $\|l\|_{L_p} < \infty$ , we will say that  $l \in L_p[0, \infty)$ . If  $M$  is a square matrix then  $\sigma(M)$  will denote its spectrum. If  $\Lambda_1$  and  $\Lambda_2$  are sets of complex numbers then  $\Lambda_1 \cup \Lambda_2$  will denote their disjoint union. For any positive integer  $n$ , we will denote  $\underline{n} := \{1, 2, \dots, n\}$ .

Consider the system (1.1). Let  $u(t) \in \mathcal{U} := \mathbb{R}^m$ ,  $x(t) \in \mathcal{X} := \mathbb{R}^n$ ,  $d(t) \in \mathcal{D} := \mathbb{R}^q$ ,  $z_1(t) \in \mathcal{X}_1 := \mathbb{R}^{p_1}$  and  $z_2(t) \in \mathcal{X}_2 := \mathbb{R}^{p_2}$ . Let  $A, B, G, H_1$  and  $H_2$  be real matrices of appropriate dimensions. We will write  $\mathcal{H}_i := \ker H_i$  ( $i = 1, 2$ ),  $\mathcal{B} := \text{im } B$  and  $A_F := A + BF$ . The reachable subspace will be denoted by  $\langle A | \mathcal{B} \rangle := \mathcal{B} + A\mathcal{B} + \dots + A^{n-1}\mathcal{B}$ . A collection of subspaces  $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$  will be called a chain in  $\mathcal{B}$  if  $\mathcal{B} \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \dots \supset \mathcal{B}_k$ . If  $0 \neq b \in \mathcal{B}$  we will denote  $\ell := \text{span } b$ .

If  $\mathcal{V} \subset \mathcal{X}$  is  $A_F$ -invariant, the restriction of  $A_F$  to  $\mathcal{V}$  will be denoted by  $A_F|_{\mathcal{V}}$ . We will write  $A_F|_{\mathcal{X}/\mathcal{V}}$  or  $\bar{A}_F$  for the quotient map induced by  $A_F$  on the factor space  $\mathcal{X}/\mathcal{V}$  (see [22]). If  $\mathcal{V}$  and  $\mathcal{W}$  are both  $A_F$ -invariant and  $\mathcal{W} \subset \mathcal{V}$ , then  $A_F|_{\mathcal{V}/\mathcal{W}}$  will denote the map induced by  $A_F|_{\mathcal{V}}$  on the factor space  $\mathcal{V}/\mathcal{W}$ . We define the canonical projection  $P: \mathcal{X} \rightarrow \mathcal{X}/\mathcal{V}$  by  $Px := x + \mathcal{V}$ . If  $\bar{B} := PB$ , then  $(\bar{A}_F, \bar{B})$  will be called the system induced in  $\mathcal{X}/\mathcal{V}$ . If  $H: \mathcal{X} \rightarrow \mathcal{Z}$  is a linear map and  $\mathcal{V} \subset \ker H$ , then  $\bar{H}: \mathcal{X}/\mathcal{V} \rightarrow \mathcal{Z}$  is defined by  $\bar{H}P = H$ . A distribution  $f \in \mathcal{D}'_+$  (i.e., the space of finite-dimensional valued distributions with support on  $[0, \infty)$ ) will be called a Bohl distribution if there exist vectors  $f_i$  and matrices  $K, L, M$  such that  $f = \sum_{i=0}^N f_i \delta^{(i)} + f_{-1}$ . Here  $f_{-1}(t) := Ke^{Lt}M$ ,  $\delta^{(0)}$  denotes Dirac's delta and  $\delta^{(i)}$  its  $i$ th distributional derivative.  $f$  will be called regular if  $f_i = 0$  ( $i = 0, \dots, N$ ) and impulsive if  $f_{-1} \equiv 0$ .

**2.2.** We will now review some basic facts from geometric control theory. If  $\mathcal{H} \subset \mathcal{X}$  is a subspace, then  $\mathcal{V}^*(\mathcal{H})$  will denote the largest  $(A, B)$ - or controlled invariant subspace in  $\mathcal{H}$  and  $\mathcal{R}^*(\mathcal{H})$  will denote the largest controllability subspace in  $\mathcal{H}$  [22].

If  $\mathbb{C}_g$  is a symmetric subset of the complex plane  $\mathbb{C}$  (i.e.,  $\lambda \in \mathbb{C}_g \Leftrightarrow \bar{\lambda} \in \mathbb{C}_g$  and  $\mathbb{C}_g$  contains at least one point of the real axis), then  $\mathcal{V}_g^*(\mathcal{X})$  will denote the largest stabilizability subspace in  $\mathcal{X}$  ([5] or [11]).

A subspace  $\mathcal{V}_a \subset \mathcal{X}$  will be called almost controlled invariant if for all  $x_0 \in \mathcal{V}_a$  and for all  $\varepsilon > 0$  there is a state trajectory  $x_\varepsilon(\cdot)$  such that  $x_\varepsilon(0) = x_0$  and  $d(\mathcal{V}_a, x_\varepsilon(t)) \leq \varepsilon$  for all  $t$ . A subspace  $\mathcal{R}_a \subset \mathcal{X}$  will be called an almost controllability subspace if for all  $x_0, x_1 \in \mathcal{R}_a$  there is a  $T > 0$  such that for all  $\varepsilon > 0$  there is a state trajectory  $x_\varepsilon(\cdot)$  such that  $x_\varepsilon(0) = x_0$ ,  $x_\varepsilon(T) = x_1$  and  $d(\mathcal{R}_a, x_\varepsilon(t)) \leq \varepsilon$  for all  $t$ . Basic facts on these classes of subspaces can be found in [19] or [20] (see also [17]). A subspace  $\mathcal{V}_a \subset \mathcal{X}$  is almost controlled invariant if and only if  $\mathcal{V}_a = \mathcal{V} + \mathcal{R}_a$ , where  $\mathcal{V}$  is controlled invariant and  $\mathcal{R}_a$  is an almost controllability subspace. A subspace  $\mathcal{R}_a$  is an almost controllability subspace if and only if there is a map  $F: \mathcal{X} \rightarrow \mathcal{U}$  and a chain  $\{\mathcal{B}_i\}_{i=1}^k$  in  $\mathcal{B}$  such that  $\mathcal{R}_a = \mathcal{B}_1 + A_F \mathcal{B}_2 + \dots + A_F^{k-1} \mathcal{B}_k$ . We will say that  $\mathcal{R}_a$  is a singly generated almost controllability subspace if there is a map  $F: \mathcal{X} \rightarrow \mathcal{U}$ , a vector  $b \in \mathcal{B}$  and an integer  $k > 0$  such that  $\mathcal{R}_a = b \oplus A_F b \oplus \dots \oplus A_F^{k-1} b$ .

Again, for  $\mathcal{X} \subset \mathcal{X}$ ,  $\mathcal{V}_a^*(\mathcal{X})$  will denote the largest almost controlled invariant subspace in  $\mathcal{X}$  and  $\mathcal{R}_a^*(\mathcal{X})$  the largest almost controllability subspace in  $\mathcal{X}$ . We will denote  $\mathcal{R}_b^*(\mathcal{X}) := \mathcal{B} + A \mathcal{R}_a^*(\mathcal{X})$  and  $\mathcal{V}_b^*(\mathcal{X}) := \mathcal{V}^*(\mathcal{X}) + \mathcal{R}_b^*(\mathcal{X})$ . The subspace  $\mathcal{V}_b^*(\mathcal{X})$  plays an important role in the problem of almost disturbance decoupling. In fact, in [20] the following result was obtained:

**PROPOSITION 2.1.** *Consider the system  $\dot{x} = Ax + Bu$ ,  $z = Hx$ . Then for all  $\varepsilon > 0$  there exists a map  $F_\varepsilon: \mathcal{X} \rightarrow \mathcal{U}$  such that  $\|H \exp[t(A + BF_\varepsilon)]G\|_{L_1} \leq \varepsilon$  if and only if  $\text{im } G \subset \mathcal{V}_b^*(\ker H)$ .*

Let  $\mathcal{X} := \ker H$ . The space  $\mathcal{V}_b^*(\mathcal{X})$  will be called the space of distributionally weakly unobservable states with respect to the output  $z$ .  $\mathcal{R}_b^*(\mathcal{X})$  will be called the space of strongly controllable states with respect to the output  $z$ . For this terminology see [6].

A proof of the following result can be found in [1, Lemma 1].

**LEMMA 2.2.** *Let  $\mathcal{X} \subset \mathcal{X}$ . Then the following equalities hold:*

- (i)  $\mathcal{R}_b^*(\mathcal{X}) \cap \mathcal{X} = \mathcal{R}_a^*(\mathcal{X})$ ,
- (ii)  $\mathcal{R}_a^*(\mathcal{X}) \cap \mathcal{V}^*(\mathcal{X}) = \mathcal{R}^*(\mathcal{X})$ ,
- (iii)  $\mathcal{R}_b^*(\mathcal{X}) \cap \mathcal{V}^*(\mathcal{X}) = \mathcal{R}^*(\mathcal{X})$ . □

This paper will sometimes deal with a new system  $(A, BW)$ , obtained by deleting the part of the input matrix  $B$  lying in  $\mathcal{V}^*(\mathcal{X})$ . This system is obtained by taking  $\tilde{\mathcal{B}} \subset \mathcal{B}$  such that  $\tilde{\mathcal{B}} \oplus (\mathcal{B} \cap \mathcal{V}^*(\mathcal{X})) = \mathcal{B}$  and by letting  $W$  be a map such that  $\tilde{\mathcal{B}} = \text{im } BW$  (see also [1]). The supremal almost controllability subspace contained in  $\mathcal{X}$  with respect to this new system  $(A, BW)$  will be denoted by  $\tilde{\mathcal{R}}_a^*(\mathcal{X})$ . We will correspondingly denote  $\tilde{\mathcal{B}} + A\tilde{\mathcal{R}}_a^*(\mathcal{X})$  by  $\tilde{\mathcal{R}}_b^*(\mathcal{X})$ . The following result follows from [1, Lemma 2]:

**LEMMA 2.3.**

$$\mathcal{V}^*(\mathcal{X}) \cap \tilde{\mathcal{R}}_b^*(\mathcal{X}) = \{0\}. \quad \square$$

Assume now that  $\mathcal{V} \subset \mathcal{X}$  is  $(A, B)$ -invariant. Let  $F$  be such that  $(A + BF)\mathcal{V} \subset \mathcal{V}$ . Let  $(\tilde{A}_F, \tilde{B})$  be the system induced in  $\mathcal{X}/\mathcal{V}$  and  $P: \mathcal{X} \rightarrow \mathcal{X}/\mathcal{V}$  the canonical projection. We then have the following result:

**LEMMA 2.4.** *If  $\mathcal{R}_a$  is an almost controllability subspace with respect to  $(A, B)$ , then  $P\mathcal{R}_a$  is an almost controllability subspace with respect to  $(\tilde{A}_F, \tilde{B})$ .*

*Proof.* Let  $Px_0$  and  $Px_1$  be in  $P\mathcal{R}_a$ , with  $x_0, x_1 \in \mathcal{R}_a$ . There is a  $T > 0$  and, for all  $\varepsilon > 0$ , a trajectory  $x_\varepsilon(\cdot)$  such that  $x_\varepsilon(0) = x_0$ ,  $x_\varepsilon(T) = x_1$  and  $d(\mathcal{R}_a, x_\varepsilon(t)) \leq \varepsilon$  for all  $t$ . It can be seen immediately that  $z_\varepsilon(t) := Px_\varepsilon(t)$  is a trajectory of the system  $(\tilde{A}_F, \tilde{B})$ . Moreover,  $z_\varepsilon(0) = Px_0$ ,  $z_\varepsilon(T) = Px_1$  and  $d(P\mathcal{R}_a, z_\varepsilon(t)) = \inf_{r \in \mathcal{R}_a} \|Pr - Px_\varepsilon(t)\| \leq \|P\|d(\mathcal{R}_a, x_\varepsilon(t)) \leq \varepsilon \|P\|$ . □

We will also need the following proposition, which is proven in [17, Thm. 2.39] (see also [15] or [16]).

**PROPOSITION 2.5.** *Consider the system  $\dot{x} = Ax + Bu$ . Let  $\mathcal{R}_a$  be an almost controllability subspace. Suppose  $\Lambda$  is a symmetric set of  $\dim \langle A | \mathcal{B} \rangle - \dim \mathcal{R}_a$  complex numbers. Then there is an  $(A, B)$ -invariant subspace  $\mathcal{V}$  and a map  $F: \mathcal{X} \rightarrow \mathcal{U}$  such that  $\mathcal{V} \oplus \mathcal{R}_a = \langle A | \mathcal{B} \rangle$  and  $\sigma(A_F | \mathcal{V}) = \Lambda$ .*

To conclude this section, we shall recall some facts on left-invertibility of linear systems. Again consider the system  $\dot{x} = Ax + Bu, z = Hx$ . Assume that the map  $B$  is injective. We will say that the system  $(A, B, H)$  is left-invertible if the transfer matrix  $H(Is - A)^{-1}B$  is an injective rational matrix. The following result was proven in [22, Ex. 4.4] (see also [6, Thm. 3.26]).

**LEMMA 2.6.**  *$(A, B, H)$  is a left-invertible system if and only if  $\mathcal{R}^*(\ker H) = 0$ .  $\square$*

**2.3.** In the following, we will review some basic facts on the frequency domain approach to the geometric concepts of this paper. We will denote  $\mathcal{X}[s]$  (respectively,  $\mathcal{X}(s), \mathcal{X}_+(s)$ ) for the set of all  $n$ -vectors whose components are polynomials (respectively, rational functions, strictly proper rational functions) with coefficients in  $\mathbb{R}$ . If  $\mathcal{X} \subset \mathcal{X} = \mathbb{R}^n$ , then  $\mathcal{X}[s]$  (respectively,  $\mathcal{X}(s), \mathcal{X}_+(s)$ ) will denote the set of all  $\xi(s) \in \mathcal{X}[s]$  (respectively,  $\mathcal{X}(s), \mathcal{X}_+(s)$ ) with the property that  $\xi(s) \in \mathcal{X}$  for all  $s$ . Slightly generalizing a definition by Hautus [5], if for a given  $x \in \mathcal{X}$  there are rational functions  $\xi(s) \in \mathcal{X}(s)$  and  $\omega(s) \in \mathcal{U}(s)$  such that  $x = (Is - A)\xi(s) + B\omega(s)$  for all  $s$ , we will say that  $x$  has a  $(\xi, \omega)$ -representation.

For a description of (almost) controlled invariant subspaces in terms of  $(\xi, \omega)$ -representations, we refer to [5], [12], [13] and [17]. We shall need the following fact:

**LEMMA 2.7.** *Let  $\mathcal{X} \subset \mathcal{X}$ . Then we have:  $x \in \mathcal{R}_b^*(\mathcal{X})$  if and only if  $x$  has a  $(\xi, \omega)$ -representation with  $\xi(s) \in \mathcal{X}[s]$  and  $\omega(s) \in \mathcal{U}[s]$ .  $\square$*

**2.4.** Finally, we will recall some facts on the convergence of subspaces. In this paper we will use the common notion of convergence of subspaces in the sense of Grassmanian topology. Let  $\{\mathcal{V}_\varepsilon; \varepsilon > 0\}$  be a sequence of subspaces of  $\mathcal{X}$  of fixed dimension. It can be proven that  $\mathcal{V}_\varepsilon \rightarrow \mathcal{V}(\varepsilon \rightarrow 0)$  if and only if there is a basis  $\{v_1, \dots, v_q\}$  for  $\mathcal{V}$  and there are bases  $\{v_1(\varepsilon), \dots, v_q(\varepsilon)\}$  of  $\mathcal{V}_\varepsilon$  such that, for all  $i, v_i(\varepsilon) \rightarrow v_i$  as  $\varepsilon \rightarrow 0$  (convergence in  $\mathcal{X}$ ). We will need the following lemma, which can be proven by standard means:

**LEMMA 2.8.** *Suppose  $v_1, \dots, v_q$  are independent vectors and  $v_i(\varepsilon) \rightarrow v_i$  for all  $i$ . Then for  $\varepsilon$  sufficiently small,  $v_1(\varepsilon), \dots, v_q(\varepsilon)$  are linearly independent. Consequently, if  $\mathcal{V}_\varepsilon \rightarrow \mathcal{V}$  and  $\mathcal{W}_\varepsilon \rightarrow \mathcal{W}$ , where  $\mathcal{V} \cap \mathcal{W} = \{0\}$ , then for  $\varepsilon$  sufficiently small  $\mathcal{V}_\varepsilon \cap \mathcal{W}_\varepsilon = \{0\}$  and  $\mathcal{V}_\varepsilon \oplus \mathcal{W}_\varepsilon \rightarrow \mathcal{V} \oplus \mathcal{W}$ .  $\square$*

**3. Mathematical problem formulation.** Consider the system (1.1). We will assume that  $z_2(t)$  is an enlargement of  $z_1(t)$ , that is, there is a matrix  $M$  such that  $H_1 = MH_2$  or, equivalently,

$$(3.1) \quad \ker H_2 =: \mathcal{K}_2 \subset \mathcal{K}_1 := \ker H_1.$$

From now on, (3.1) will be a standing assumption. Throughout this paper we will also assume that  $B$  is injective.

Consider the following synthesis problem. Fix  $1 \leq p \leq \infty$ . We will say that the  $L_p$ -almost disturbance decoupling problem with bounded peaking (ADDPBP) $_p$  is solvable if there is a constant  $C$  and for all  $\varepsilon > 0$  there is a feedback map  $F_\varepsilon: \mathcal{X} \rightarrow \mathcal{U}$  such that, with the feedback law  $u = F_\varepsilon x$  in the closed loop system for  $x(0) = 0$  for all  $d \in L_p[0, \infty)$ , the following inequalities hold:

$$(3.2) \quad \|z_1\|_{L_p} \leq \varepsilon \|d\|_{L_p},$$

$$(3.3) \quad \|z_2\|_{L_p} \leq C \|d\|_{L_p}.$$

Note that if we take  $H_1 = H_2$ , we obtain the original  $L_p$ -almost disturbance decoupling problem, (ADDP) $_p$ , without the requirement of bounded peaking (see [20] or [17]). Another interesting special case is to take  $H_2 = I$ , which corresponds to the requirement of bounded peaking of the entire state vector.

In the present paper, necessary and sufficient geometric conditions for the solvability of the above problem will be derived for the cases  $p = 1$ ,  $p = 2$  and  $p = \infty$ . We will first show how the solvability of (ADDPBP) $_p$  can be expressed in terms of the closed loop impulse response matrices from the disturbance  $d$  to the outputs  $z_1$  and  $z_2$ , respectively. If  $F_\varepsilon: \mathcal{X} \rightarrow \mathcal{U}$  is a state feedback map, then denote the closed loop transition matrix by

$$(3.4) \quad T_\varepsilon(t) := e^{(A+BF_\varepsilon)t}$$

and let

$$(3.5) \quad \hat{T}_\varepsilon(s) := (Is - A - BF_\varepsilon)^{-1}$$

denote its Laplace transform. We then have the following:

LEMMA 3.1. Fix  $p \in \{1, \infty\}$ . Then (ADDPBP) $_p$  is solvable if and only if there is a constant  $C$  and for all  $\varepsilon > 0$  a feedback map  $F_\varepsilon: \mathcal{X} \rightarrow \mathcal{U}$  such that  $\|H_1 T_\varepsilon G\|_{L_1} \leq \varepsilon$  and  $\|H_2 T_\varepsilon G\|_{L_1} \leq C$ .

(ADDPBP) $_2$  is solvable if and only if there is a constant  $C$  and for all  $\varepsilon > 0$  a feedback map  $F_\varepsilon$  such that  $H_1 \hat{T}_\varepsilon(s)G$  and  $H_2 \hat{T}_\varepsilon(s)G$  are asymptotically stable and such that  $\sup_{\omega \in \mathbb{R}} \|H_1 \hat{T}_\varepsilon(i\omega)G\| \leq \varepsilon$  and  $\sup_{\omega \in \mathbb{R}} \|H_2 \hat{T}_\varepsilon(i\omega)G\| \leq C$ .

*Proof.* The proof follows immediately from the fact that for  $p = 1$  and for  $p = \infty$  the  $L_p$ -included norm of the closed loop convolution operator from  $d$  to  $z_i$  equals exactly the  $L_1$ -norm of its kernel, i.e.,  $\|H_i T_\varepsilon G\|_{L_1}$ . The second statement follows from the fact that the  $L_2$ -induced norm of the convolution operator from  $d$  to  $z_i$  equals the  $H^\infty$ -norm  $\sup_{\omega \in \mathbb{R}} \|H_i \hat{T}_\varepsilon(i\omega)G\|$  (see, for example, [2]).  $\square$

**4. Disturbance decoupling with stability constraints.** Prior to considerations involving the peaking behaviour of the enlarged output  $z_2$ , we should make sure that the output  $z_2$  is in  $L_p[0, \infty)$  at all. Hence, an important part of the solution of the problem posed in § 3 is to construct the required feedback maps  $F_\varepsilon$  in such a way that, for any  $d \in L_p[0, \infty)$ , in the closed loop system with  $x(0) = 0$  we have  $z_2 \in L_p[0, \infty)$ . Therefore, in this section the following variation on the well known (exact) disturbance decoupling problem [22] will be considered. Again, consider the system given by (1.1). The usual disturbance decoupling problem is concerned with the determination of a feedback map  $F: \mathcal{X} \rightarrow \mathcal{U}$  such that in the closed loop system the external disturbance  $d$  does not influence a specified output  $z_1$ . We will consider the more general situation in which simultaneously we demand stability of the second, larger, output  $z_2$ .

In this section,  $\mathbb{C}_g$ , the stability set, will be a given subset of the complex plane  $\mathbb{C}$  which is symmetric. Asymptotic stability is thus obtained by taking  $\mathbb{C}_g = \{\lambda \in \mathbb{C}: \operatorname{Re} \lambda < 0\}$ . A rational matrix or rational vector is called stable if all its poles are in  $\mathbb{C}_g$ . We will consider the following problem: (DDPOS) the disturbance decoupling problem with output stabilization is said to be solvable if there is a feedback map  $F$  such that  $H_1(Is - A_F)^{-1}G = 0$  and  $H_2(Is - A_F)^{-1}G$  is stable.

In order to be able to formulate conditions for the solvability of the above problem, introduce the following subspace:

DEFINITION 4.1.  $\mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2)$  will denote the subspace of all points  $x \in \mathcal{X}_1$  for which there is a  $(\xi, \omega)$ -representation with  $\xi(s) \in \mathcal{K}_{1,+}(s)$ ,  $\omega(s) \in \mathcal{U}_+(s)$  and  $H_2 \xi(s)$  is stable.

Thus, interpreted in the time domain,  $\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$  is the subspace consisting of all points in which a regular Bohl state trajectory starts that lies entirely in  $\mathcal{H}_1$ . The components of this trajectory modulo  $\mathcal{H}_2$  are stable. It follows immediately from the definition that  $\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$  is contained in  $\mathcal{V}^*(\mathcal{H}_1)$ . By the assumption (3.1), if a trajectory lies in  $\mathcal{H}_2$ , the same is true for  $\mathcal{H}_1$ . Consequently we also have the inclusion  $\mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$ .

We note that Definition 4.1 is a generalization of a definition by Hautus [5]. His space  $S_{\Sigma}^-$  (see [5, p. 706]) coincides with  $\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$  if  $\mathcal{H}_1$  is taken to be  $\mathcal{X}$ . The following theorem can be proven to be completely analogous to [5, Thm. 4.3]:

THEOREM 4.2.

$$\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{V}_g^*(\mathcal{H}_1) + \mathcal{V}^*(\mathcal{H}_2). \quad \square$$

Note that it follows from the above theorem that  $\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$  is controlled invariant. The next theorem provides the key step in the solution of DDPOS. The result states that what can be done in Definition 4.1 by open loop control can in fact be done by state feedback:

THEOREM 4.3. *There exists a map  $F: \mathcal{X} \rightarrow \mathcal{U}$  such that*

$$(4.1) \quad A_F \mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2) \subset \mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2),$$

$$(4.2) \quad A_F \mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}^*(\mathcal{H}_2),$$

$$(4.3) \quad \sigma(A_F | \mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2) / \mathcal{V}^*(\mathcal{H}_2)) \subset \mathbb{C}_g.$$

*Proof.* During this proof, denote  $\mathcal{V}_g := \mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$ . Since  $\mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}_g$  and since both spaces are controlled invariant, they are compatible (see [22, Ex. 9.1]). Hence, there is a map  $F_0: \mathcal{X} \rightarrow \mathcal{U}$  such that  $A_{F_0} \mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}^*(\mathcal{H}_2)$  and  $A_{F_0} \mathcal{V}_g \subset \mathcal{V}_g$ . Let  $\bar{\mathcal{B}} := \mathcal{B} \cap \mathcal{V}_g$  and let  $V$  be any matrix such that  $\bar{\mathcal{B}} = \text{im } BV$ .

Consider the controllability subspace  $\langle A_{F_0} | \bar{\mathcal{B}} \rangle$ . By the facts that  $\bar{\mathcal{B}} \subset \mathcal{V}_g$  and  $A_{F_0} \mathcal{V}_g \subset \mathcal{V}_g$ , this controllability subspace is contained in  $\mathcal{H}_1$ . Since any controllability subspace is also a stabilizability subspace, it must be contained in the largest stabilizability subspace  $\mathcal{V}_g^*(\mathcal{H}_1)$  in  $\mathcal{H}_1$ . It then follows that  $\bar{\mathcal{B}} \subset \mathcal{V}_g^*(\mathcal{H}_1)$ , so

$$(4.4) \quad \mathcal{B} \cap \mathcal{V}_g^*(\mathcal{H}_1) = \bar{\mathcal{B}}.$$

Next, we claim that  $\mathcal{V}_g^*(\mathcal{H}_1)$  is  $A_{F_0}$ -invariant. First, since it is  $(A, B)$ -invariant, we have  $A_{F_0} \mathcal{V}_g^*(\mathcal{H}_1) \subset \mathcal{V}_g^*(\mathcal{H}_1) + \mathcal{B}$ . On the other hand,  $A_{F_0} \mathcal{V}_g^*(\mathcal{H}_1) \subset A_{F_0} \mathcal{V}_g \subset \mathcal{V}_g$ . Hence, again using  $\mathcal{V}_g^*(\mathcal{H}_1) \subset \mathcal{V}_g$ , we obtain  $A_{F_0} \mathcal{V}_g^*(\mathcal{H}_1) \subset (\mathcal{V}_g^*(\mathcal{H}_1) + \mathcal{B}) \cap \mathcal{V}_g \subset \mathcal{V}_g^*(\mathcal{H}_1) + (\mathcal{B} \cap \mathcal{V}_g) = \mathcal{V}_g^*(\mathcal{H}_1)$ . The last equality follows from (4.4).

Using (4.4) and [5, Prop. 2.16], we deduce that the pair  $(A_{F_0} | \mathcal{V}_g^*(\mathcal{H}_1), BV)$  is stabilizable.

Let  $P_1: \mathcal{V}_g \rightarrow \mathcal{V}_g / \mathcal{V}^*(\mathcal{H}_2)$  be the canonical projection. Let  $(\bar{A}_{F_0}, \bar{B}V)$  be the system induced in  $\mathcal{V}_g / \mathcal{V}^*(\mathcal{H}_2)$ . It can easily be seen, for example, by using a rank test (see [4] or [5, Thm. 2.13]), that the latter system is stabilizable. Hence, there is a map  $\bar{F}_1$  on the factor space such that  $\sigma(\bar{A}_{F_0} + \bar{B}V\bar{F}_1) \subset \mathbb{C}_g$ . Now, let  $F_1$  be any map on  $\mathcal{V}_g$  such that  $F_1 = \bar{F}_1 P_1$  and define  $F_1$  arbitrary on a complement of  $\mathcal{V}_g$ . Define  $F := F_0 + VF_1$ . Since  $F | \mathcal{V}^*(\mathcal{H}_2) = F_0 | \mathcal{V}^*(\mathcal{H}_2)$  (“|” denotes “restriction to”), we then have  $A_F \mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}^*(\mathcal{H}_2)$  and it can be verified that Fig. 1 commutes.

We are now in a position to prove the main result of this section.

THEOREM 4.4. *DDPOS is solvable iff  $\text{im } G \subset \mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$ .*

*Proof.* ( $\Leftarrow$ ) Choose  $F$  as in Theorem 4.3. Then  $\langle A_F | \text{im } G \rangle \subset \mathcal{H}_1$ , which yields the decoupling from  $d$  to  $z_1$ .



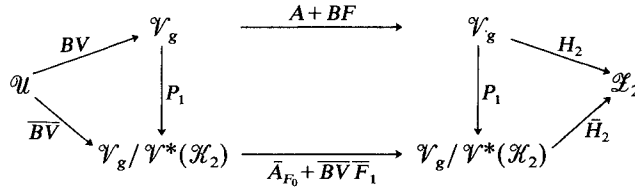


FIG. 1

Let  $\bar{H}_2$  be as in the Fig. 1 and let  $\bar{A}_F := A_F | \mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2) / \mathcal{V}^*(\mathcal{K}_2)$ . Then  $H_2(Is - A_F)^{-1}G = \bar{H}_2(Is - \bar{A}_F)^{-1}P_1G$ , which is stable since  $\sigma(\bar{A}_F) \subset \mathbb{C}_g$ .

( $\Rightarrow$ ) If  $F$  is such that  $H_1(Is - A_F)^{-1}G = 0$  and  $H_2(Is - A_F)^{-1}G$  is stable then for  $d \in \mathcal{D}$  let  $\xi(s) := (Is - A_F)^{-1}Gd$  and  $\omega(s) := F\xi(s)$ . Then clearly  $Gd = (Is - A)\xi(s) + B\omega(s)$  and  $H_2\xi(s)$  is stable.  $\square$

*Remark 4.5.* If in the above problem we take  $H_1 = H_2 = H$ , DDPOS reduces to the ordinary disturbance decoupling problem DDP (see [22]). In this case we have, denoting  $\mathcal{K} := \ker H$ ,  $\mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2) = \mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}^*(\mathcal{K}) = \mathcal{V}^*(\mathcal{K})$ . If we take  $H_1 = 0$  and  $H_2 = H$ , we arrive at OSDP as studied in Hautus [5]. The solvability of this problem requires the existence of a state feedback  $F$  such that  $H(Is - A_F)^{-1}G$  is stable. Necessary and sufficient conditions can be found by noting that  $\mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2) = \mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}^*(\mathcal{K})$ . As also noted in [5], if we take  $H_1 = 0$ ,  $H_2 = H$  and  $\text{im } G = \mathcal{X}$ , the above theorem provides necessary and sufficient conditions for the solvability of the output stabilization problem, OSP.

**5. A necessary geometric condition for the solvability of (ADDPBP)<sub>p</sub>.** In this section we shall establish a necessary condition for the solvability of (ADDPBP)<sub>p</sub>. This condition will be in the form of a subspace inclusion. The proof is rather technical and some of the details are deferred to Appendix A. In the rest of this paper, the stability set will be taken to be  $\mathbb{C}_g = \{\lambda \in \mathbb{C} \mid \text{Re } \lambda < 0\}$ .

Consider the system  $\dot{x} = Ax + Bu$ ,  $z_1 = H_1x$ ,  $z_2 = H_2x$  and assume that (3.1) is satisfied. The following subspace will play an important role in the sequel:

**DEFINITION 5.1.**  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  will denote the subspace of all  $x \in X$  that have a  $(\xi, \omega)$ -representation with  $\xi(\omega) \in \mathcal{K}_1(s)$ ,  $\omega(s) \in \mathcal{U}(s)$  and  $H_2\xi(s)$  is proper and stable.

Interpreted in the time domain,  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  consists exactly of those points in  $\mathcal{X}$  that can serve as an initial condition for some Bohl distributional trajectory that lies entirely in  $\mathcal{K}_1$ , while the vector of components of the trajectory modulo  $\mathcal{K}_2$  is the sum of a stable regular Bohl function and a Dirac delta.

It follows immediately from the definition and [12, Thm. 4.1] that  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  is contained in  $\mathcal{V}_b^*(\mathcal{K}_1)$ , the subspace of distributionally weakly unobservable states with respect to the output  $z_1$ . It is also immediate that  $\mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2)$  is contained in  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ . We are now in a position to state the main result of this section:

**THEOREM 5.2.** Fix  $p \in \{1, 2, \infty\}$ . Then the following holds:

$$\{(\text{ADDPBP})_p \text{ is solvable}\} \Rightarrow \{\text{im } G \subset \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)\}.$$

In the remainder of this section we will establish a proof of the above theorem. Again, consider the system  $\dot{x} = Ax + Bu$ ,  $z_1 = H_1x$ ,  $z_2 = H_2x$ . Assume that for  $\varepsilon > 0$ ,  $u_\varepsilon(t)$  is a regular Bohl input. Let  $x_0 \in X$ . Let  $z_{1,\varepsilon}(t)$  and  $z_{2,\varepsilon}(t)$  be the outputs corresponding to the above input and initial condition  $x(0) = x_0$ . Denote  $\hat{z}_{i,\varepsilon}(s)$  for the corresponding Laplace transforms of  $z_{i,\varepsilon}(t)$ . We then have the following lemma:

**LEMMA 5.3.** Suppose that either of the following conditions is satisfied:

- (i)  $\|z_{1,\varepsilon}\|_{L_1} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and there exists a constant  $C$  such that  $\|z_{2,\varepsilon}\|_{L_1} \leq C$  for all  $\varepsilon$ .

(ii)  $\hat{z}_{1,\varepsilon}(s)$  and  $\hat{z}_{2,\varepsilon}(s)$  are stable for all  $\varepsilon$ ,  $\sup_{\omega \in \mathbb{R}} \|\hat{z}_{1,\varepsilon}(i\omega)\| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and there exists a constant  $C$  such that  $\sup_{\omega \in \mathbb{R}} \|\hat{z}_{2,\varepsilon}(i\omega)\| \leq C$  for all  $\varepsilon$ .

Then  $x_0 \in \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ .  $\square$

A detailed proof of Lemma 5.3 can be found in Appendix A. The idea of the proof is the following. First we note that the initial condition  $x_0$  above has for each  $\varepsilon > 0$  a  $(\xi, \omega)$ -representation  $x_0 = (Is - A)\xi_\varepsilon(s) + B\omega_\varepsilon(s)$ . Here  $\omega_\varepsilon(s)$  is the (rational) Laplace transform of  $u_\varepsilon(t)$ . Using the asymptotic behaviour as described by the condition (i) or (ii) above, we then analyse the limiting behaviour for  $\varepsilon \rightarrow 0$  of the sequences of rational vectors  $\xi_\varepsilon(s)$  and  $\omega_\varepsilon(s)$ . This will lead to a  $(\xi, \omega)$ -representation for  $x_0$  with the properties described in Definition 5.1. To conclude this section we apply Lemma 5.3 to obtain the following:

*Proof of Theorem 5.2.* Assume that  $(\text{ADDPBP})_p$  is solvable. Let  $x_0 \in \text{im } G$ . Let  $F_\varepsilon$  be as in Lemma 3.1 and define  $u_\varepsilon(t) := F_\varepsilon T_\varepsilon(t)x_0$ . Then, depending on  $p$ , one of the conditions (i) or (ii) in Lemma 5.3 is satisfied. It follows that  $x_0 \in \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ .  $\square$

**6. The geometric structure of  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ .** In the sequel, it will turn out that under certain assumptions on the system (1.1) the subspace inclusion in Theorem 5.2 is also a sufficient condition for the solvability of  $(\text{ADDPBP})_p$ . In order to prove this and to be able to construct the required feedback maps, we need more detailed information on the geometric structure of the subspace  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  as introduced in the previous section. In the present section, we will first show that the subspace  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  can always be written as the sum of the subspace  $\mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2)$  (see § 4) together with an almost controllability subspace depending on  $\mathcal{K}_1$  and  $\mathcal{K}_2$ . Using this result, we will show that if either  $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$  or  $\mathcal{K}_2 = \{0\}$ , then  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  admits a decomposition into the direct sum of  $\mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2)$  together with a number of singly generated almost controllability subspaces, with a particular position with respect to the subspaces  $\mathcal{K}_1$  and  $\mathcal{K}_2$ . The main result of this section will be the following theorem:

**THEOREM 6.1.** *Assume that  $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$  or that  $\mathcal{K}_2 = \{0\}$ . Then there is an integer  $m' \in \mathbb{N}$ , there are integers  $r_1, \dots, r_{m'} \in \mathbb{N}$  and vectors  $b_1, \dots, b_{m'} \in \mathcal{B}$  and there is a map  $F: \mathcal{X} \rightarrow \mathcal{U}$  such that*

$$(6.1) \quad \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2) = \mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2) \oplus \bigoplus_{i=1}^{m'} \mathcal{L}_i$$

where

$$\mathcal{L}_i := \bigoplus_{j=1}^{r_i} A_F^{j-1} \ell_i$$

with

$$(6.2) \quad \bigoplus_{j=1}^{r_i-1} A_F^{j-1} \ell_i \subset \mathcal{K}_1$$

and

$$(6.3) \quad \bigoplus_{j=1}^{r_i-2} A_F^{j-1} \ell_i \subset \mathcal{K}_2.$$

If in the statement of the above theorem one of the integers  $r_i$  is such that  $r_i - 1 < 1$  or  $r_i - 2 < 1$ , then the corresponding sums in (6.2) or (6.3) are understood to be equal to  $\{0\}$ . It will turn out in the proof of Theorem 6.1 that in the case that  $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$  the integer  $m'$  may be chosen to be equal to  $m$  ( $= \dim \mathcal{B}$ ). In the case that  $\mathcal{K}_2 = \{0\}$  it will appear that  $m'$  may be chosen to be equal to  $m - \dim [\mathcal{V}^*(\mathcal{K}_1) \cap \mathcal{B}]$  and also that

in this case the integers  $r_i$  may be taken to be either 1 or 2. Since  $\mathcal{V}_g(\mathcal{K}_1, \{0\}) = \mathcal{V}_g^*(\mathcal{K}_1)$  (see Theorem 4.2) the theorem thus states that  $\mathcal{V}_b(\mathcal{K}_1, \{0\})$  is equal to the direct sum of  $\mathcal{V}_g^*(\mathcal{K}_1)$  together with a number of singly generated almost controllability subspaces which are equal to either  $\text{span}\{b_i\}$  (with  $0 \neq b_i \in \mathcal{B}$ ) or  $\text{span}\{b_i, A_F b_i\}$ , with  $\{b_i, A_F b_i\}$  linearly independent and  $b_i \in \mathcal{K}_1 \cap \mathcal{B}$ .

The result of Theorem 6.1 will be instrumental in the next section, where we will consider the sufficiency of the subspace inclusion  $\text{im } G \subset \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  for solvability of  $(\text{ADDPBP})_p$  and propose a “scheme” for calculation of the required feedback maps. In the remainder of the present section we will establish a proof of Theorem 6.1.

We introduce the following subspace:

DEFINITION 6.2.  $\mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2)$  will denote the subspace of all  $x \in \mathcal{X}$  that have a  $(\xi, \omega)$ -representation with  $\xi(s) \in \mathcal{K}_1[s]$ ,  $\omega(s) \in \mathcal{U}[s]$  and  $H_2 \xi(s)$  is constant (i.e., if  $\xi(s) = \sum_{i=0}^N x_i s^i$  then  $H_2 x_i = 0$  for  $i \geq 1$ ).

Interpreted in the time domain,  $\mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2)$  consists exactly of those points in  $\mathcal{X}$  that can be driven to 0 along a purely distributional Bohl trajectory that lies entirely in  $\mathcal{K}_1$ , while the vector of components of this trajectory modulo  $\mathcal{K}_2$  is a Dirac delta.

It follows immediately from the definition and Lemma 2.7 that every point in  $\mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2)$  is strongly controllable with respect to the output  $z_1$ . Moreover, it is also immediate that every point  $x$  that is strongly controllable with respect to the output  $z_2$ , is an element of  $\mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2)$ . Hence, the inclusion  $\mathcal{R}_b^*(\mathcal{K}_2) \subset \mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2) \subset \mathcal{R}_b^*(\mathcal{K}_1)$  holds. In fact, we have the following nice result:

THEOREM 6.3.

(i)  $\mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2) = \mathcal{B} + A(\mathcal{R}_b^*(\mathcal{K}_2) \cap \mathcal{K}_1)$ ,

(ii)  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2) = \mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2) + \mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2)$ .

Proof. (i) Suppose that  $x = (Is - A)\xi(s) + B\omega(s)$ , with  $\xi(s) \in \mathcal{K}_1[s]$ ,  $\omega(s) \in \mathcal{U}[s]$  and  $H_2 \xi(s)$  is constant. Let  $\xi(s) = \sum_{i=0}^N x_i s^i$  and  $\omega(s) = \sum_{i=0}^{N+1} u_i s^i$ . Obviously,  $\xi(s) = x_0 + s\xi_1(s)$  and  $\omega(s) = u_0 + s\omega_1(s)$ , where  $\xi_1(s) \in \mathcal{K}_2[s]$  and  $\omega_1(s) \in \mathcal{U}[s]$ . Hence,  $x = Bu_0 - Ax_0 + sx_0 + s^2 \xi_1(s) - As\xi_1(s) + Bs\omega_1(s)$  and by equating powers it follows that

$$(6.4) \quad x = -Ax_0 + Bu_0,$$

$$(6.5) \quad -x_0 = (Is - A)\xi_1(s) + B\omega_1(s).$$

Therefore,  $x_0 \in \mathcal{R}_b^*(\mathcal{K}_2)$  (see Lemma 2.7). Since also  $x_0 \in \mathcal{K}_1$ , we obtain  $x \in \mathcal{B} + A(\mathcal{R}_b^*(\mathcal{K}_2) \cap \mathcal{K}_1)$ . Conversely, if  $x = Bu_0 - Ax_0$  with  $x_0 \in \mathcal{R}_b^*(\mathcal{K}_2) \cap \mathcal{K}_1$ , there is  $\xi_1(s) \in \mathcal{K}_2[s]$  and  $\omega_1(s) \in \mathcal{U}[s]$  such that  $-x_0 = (Is - A)\xi_1(s) + B\omega_1(s)$ . Defining then  $\xi(s) := x_0 + s\xi_1(s)$  and  $\omega(s) := u_0 + s\omega_1(s)$ , we obtain a  $(\xi, \omega)$ -representation of  $x$  with  $\xi(s) \in \mathcal{K}_1[s]$ ,  $\omega(s) \in \mathcal{U}[s]$  and  $H_2 \xi(s) = H_2 x_0$  is constant.

(ii) Assume that  $x \in \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ . There is a  $(\xi, \omega)$ -representation for  $x$  with  $\xi(s) \in \mathcal{K}_1(s)$ ,  $\omega(s) \in \mathcal{U}(s)$  and  $H_2 \xi(s)$  proper and stable. Decompose  $\xi(s) = \xi_1(s) + \xi_2(s)$  and  $\omega(s) = \omega_1(s) + \omega_2(s)$ , where  $\xi_1(s)$  and  $\omega_1(s)$  are polynomial vectors and  $\xi_2(s)$  and  $\omega_2(s)$  are strictly proper. Obviously,  $\xi_1(s) \in \mathcal{K}_1[s]$ ,  $\xi_2(s) \in \mathcal{K}_{1,+}(s)$ ,  $\omega_1(s) \in \mathcal{U}[s]$  and  $\omega_2(s) \in \mathcal{U}_+(s)$ . Moreover,  $H_2 \xi_1(s)$  is constant and  $H_2 \xi_2(s)$  is strictly proper and stable. Now, since the left-hand side of this equation is proper and the right-hand side is a polynomial vector, both sides must, in fact, be constant. Thus, there is a vector  $x_1 \in \mathcal{X}$  such that  $x_1 = (Is - A)\xi_1(s) + B\omega_1(s) = x - (Is - A)\xi_2(s) - B\omega_2(s)$ . It follows that  $x_1 \in \mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2)$  and  $x - x_1 \in \mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2)$ . Since  $x = x_1 + (x - x_1)$ , we obtain that  $x \in \mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2) + \mathcal{R}_b(\mathcal{K}_1, \mathcal{K}_2)$ . The converse inclusion follows immediately from the definitions.  $\square$

The importance of the above theorem is that it shows, together with Theorem 4.2, that

$$(6.6) \quad \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2) = \mathcal{V}_g^*(\mathcal{K}_1) + \mathcal{V}_g^*(\mathcal{K}_2) + \mathcal{B} + A(\mathcal{R}_b^*(\mathcal{K}_2) \cap \mathcal{K}_1).$$



It may immediately be verified that the above list of vectors can be rearranged to obtain  $m$  subspaces  $\mathcal{L}_i := \text{span} \{b_i, A_F b_i, \dots, A_F^{r_i-1} b_i\}$ , with

$$\text{span} \{b_i, A_F b_i, \dots, A_F^{r_i-2} b_i\} \subset \mathcal{H}_2$$

and

$$\text{span} \{b_i, A_F b_i, \dots, A_F^{r_i-3} b_i\} \subset \mathcal{H}_2.$$

This completes the proof of Theorem 6.1 for the case that  $\mathcal{R}^*(\mathcal{H}_1) = \{0\}$ .  $\square$

In the remainder of this section, we will set up a proof of Theorem 6.1, the case that  $\mathcal{H}_2 = \{0\}$ . In the following, let  $\tilde{\mathcal{B}}$  be a subspace of  $\mathcal{B}$  such that  $\tilde{\mathcal{B}} \oplus [\mathcal{B} \cap \mathcal{V}^*(\mathcal{H}_1)] = \mathcal{B}$ . Let  $W$  be a map such that  $\tilde{\mathcal{B}} = \text{im } BW$  and let  $\tilde{\mathcal{H}}_b^*(\mathcal{H}_1) := \tilde{\mathcal{B}} + A\tilde{\mathcal{H}}_a^*(\mathcal{H}_1)$ , where  $\tilde{\mathcal{H}}_a^*(\mathcal{H}_1)$  denotes the supremal almost controllability subspace contained in  $\mathcal{H}_1$  with respect to the system  $(A, BW)$  (see also Lemma 2.3). Define

$$(6.14) \quad \tilde{\mathcal{W}}(\mathcal{H}_1) := \tilde{\mathcal{B}} + A(\tilde{\mathcal{B}} \cap \mathcal{H}_1).$$

We will show that if  $\mathcal{H}_2 = \{0\}$ , then  $\mathcal{V}_b(\mathcal{H}_1, \mathcal{H}_2)$  has a decomposition into the direct sum of  $\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$  (which, in that case, is equal to  $\mathcal{V}_g^*(\mathcal{H}_1)$ ) and the subspace  $\tilde{\mathcal{W}}(\mathcal{H}_1)$ :

LEMMA 6.6. *Let  $\mathcal{H}_1$  be a subspace of  $\mathcal{X}$ . Then*

$$(6.15) \quad \mathcal{V}_b(\mathcal{H}_1, \{0\}) = \mathcal{V}_g(\mathcal{H}_1, \{0\}) \oplus \tilde{\mathcal{W}}(\mathcal{H}_1).$$

*Proof.* In this proof, denote  $\mathcal{V}_g := \mathcal{V}_g(\mathcal{H}_1, \{0\})$ . Also, let  $\mathcal{B}_1 := \mathcal{B} \cap \mathcal{V}^*(\mathcal{H}_1)$ . Since  $\mathcal{R}_b^*(\{0\}) = \mathcal{B}$ , it follows from Theorem 6.3 that

$$\begin{aligned} \mathcal{V}_b(\mathcal{H}_1, \{0\}) &= \mathcal{V}_g + \mathcal{B} + A[\mathcal{B} \cap \mathcal{H}_1] \\ &= \mathcal{V}_g + \mathcal{B} + A[(\mathcal{B}_1 \oplus \tilde{\mathcal{B}}) \cap \mathcal{H}_1] \\ &= \mathcal{V}_g + \mathcal{B} + A[\mathcal{B}_1 + (\tilde{\mathcal{B}} \cap \mathcal{H}_1)]. \end{aligned}$$

Now, note that  $\mathcal{B}_1 \subset \mathcal{R}^*(\mathcal{H}_1)$  (see [22, Thm. 5.5]). Consequently,  $A\mathcal{B}_1 \subset \mathcal{R}^*(\mathcal{H}_1) + \mathcal{B} \subset \mathcal{V}_g + \mathcal{B}$ . Hence we find

$$\begin{aligned} \mathcal{V}_b(\mathcal{H}_1, \{0\}) &= \mathcal{V}_g + \mathcal{B} + A(\tilde{\mathcal{B}} \cap \mathcal{H}_1) \\ &= \mathcal{V}_g + \mathcal{B}_1 + \tilde{\mathcal{B}} + A(\tilde{\mathcal{B}} \cap \mathcal{H}_1). \end{aligned}$$

Again, by the fact that  $\mathcal{B}_1 \subset \mathcal{R}^*(\mathcal{H}_1) \subset \mathcal{V}_g$ , we have

$$\mathcal{V}_b(\mathcal{H}_1, \{0\}) = \mathcal{V}_g + \tilde{\mathcal{B}} + A(\tilde{\mathcal{B}} \cap \mathcal{H}_1).$$

Finally, since  $\mathcal{V}_g \subset \mathcal{V}^*(\mathcal{H}_1)$  and  $\tilde{\mathcal{W}}(\mathcal{H}_1) \subset \tilde{\mathcal{H}}_b^*(\mathcal{H}_1)$ , it follows from Lemma 2.3 that the sum in (6.15) is direct.  $\square$

Using the above lemma we may now obtain the following proof of Theorem 6.1, the case that  $\mathcal{H}_2 = \{0\}$ ;

*Proof of Theorem 6.1 (Case 2:  $\mathcal{H}_2 = \{0\}$ ).* We claim that  $\tilde{\mathcal{W}}(\mathcal{H}_1) = \tilde{\mathcal{B}} \oplus A(\tilde{\mathcal{B}} \cap \mathcal{H}_1)$ . To prove this, assume that there is a vector  $0 \neq x \in \tilde{\mathcal{B}}$  such that  $x = A\bar{x}$ , with  $\bar{x}$  a vector in  $\tilde{\mathcal{B}} \cap \mathcal{H}_1$ . Define  $\mathcal{V} := \text{span} \{\bar{x}\}$ . Since  $A\mathcal{V} \subset \mathcal{V} + \mathcal{B}$ ,  $\mathcal{V}$  is controlled invariant. Since also  $\mathcal{V} \subset \mathcal{H}_1$ , we find that  $\mathcal{V} \subset \mathcal{V}^*(\mathcal{H}_1)$ . It follows that  $\bar{x} \in \mathcal{V}^*(\mathcal{H}_1) \cap \tilde{\mathcal{B}} = \{0\}$  and hence that  $x = 0$ . This yields a contradiction. Next, we claim that  $\dim \tilde{\mathcal{B}} \cap \mathcal{H}_1 = \dim A(\tilde{\mathcal{B}} \cap \mathcal{H}_1)$ . Assume the contrary. Then we may find a vector  $0 \neq x \in \tilde{\mathcal{B}} \cap \mathcal{H}_1$  such that  $Ax = 0$ . It follows that  $\text{span} \{x\}$  is a controlled invariant subspace contained in  $\mathcal{H}_1$  and hence that  $x \in \mathcal{V}^*(\mathcal{H}_1) \cap \tilde{\mathcal{B}} = \{0\}$ . Again, this is a contradiction. Now, choose a basis for  $\tilde{\mathcal{W}}(\mathcal{H}_1)$  as follows: first choose a basis  $b_1, \dots, b_r$  of  $\tilde{\mathcal{B}} \cap \mathcal{H}_1$ . Extend this to a basis  $\{b_1, \dots, b_r, b_{r+1}, \dots, b_m\}$  of  $\tilde{\mathcal{B}}$ . By the above, the vectors  $\{b_1, \dots, b_r, Ab_1, \dots, Ab_r, b_{r+1}, \dots, b_m\}$  form a basis for  $\tilde{\mathcal{W}}(\mathcal{H}_1)$ . These vectors can be rearranged

into one- and two-dimensional singly generated almost controllability subspaces with the properties (6.2) and (6.3). This completes the proof of Theorem 6.1.  $\square$

**7. The main result.** In the present section we will combine our results of the previous sections to show that if the system (1.1) is such that it satisfies at least one of the following two properties:

(7.1) the system  $(A, B, H_1)$  is left-invertible,

(7.2) the mapping  $H_2$  is injective,

then the subspace inclusion  $\text{im } G \subset \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  is both a necessary and sufficient condition for solvability of the  $L_p$ -almost disturbance decoupling problem with bounded peaking (ADDPBP) $_p$  for the values  $p = 1, p = 2$  and  $p = \infty$ .

Recall from § 5 that for these values of  $p$  the latter subspace inclusion was already shown to be a necessary condition without the extra assumptions (7.1), (7.2). Here we shall, in fact, prove that if either (7.1) or (7.2) holds then  $\text{im } G \subset \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  is a sufficient condition for solvability of (ADDPBP) $_p$  for all  $1 \leq p \leq \infty$ .

The following result is the main result of this paper:

**THEOREM 7.1.** *Assume that at least one of the two conditions (7.1), (7.2) is satisfied. Let  $p \in \{1, 2, \infty\}$ . Then (ADDPBP) $_p$  is solvable if and only if  $\text{im } G \subset \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ .  $\square$*

In order to obtain a proof of the latter statement, we will prove the following:

**LEMMA 7.2.** *Assume that at least one of the two conditions (7.1), (7.2) is satisfied. Let  $T_\varepsilon(t)$  and  $\hat{T}_\varepsilon(s)$  be defined by (3.4) and (3.5). Then the following statements are equivalent:*

- (i) *There exists a constant  $C$  and a sequence  $\{F_\varepsilon; \varepsilon > 0\}$  such that  $\|H_1 T_\varepsilon G\|_{L_1} \rightarrow 0$  ( $\varepsilon \rightarrow 0$ ) and  $\|H_2 T_\varepsilon G\|_{L_1} \leq C \forall \varepsilon$ .*
- (ii) *There exists a constant  $C$  and a sequence  $\{F_\varepsilon; \varepsilon > 0\}$  such that, for all  $\varepsilon$ ,  $H_1 \hat{T}_\varepsilon G$  and  $H_2 \hat{T}_\varepsilon G$  are stable and  $\sup_{\omega \in \mathbb{R}} \|H_1 \hat{T}_\varepsilon(i\omega)G\| \rightarrow 0$  ( $\varepsilon \rightarrow 0$ ) and  $\sup_{\omega \in \mathbb{R}} \|H_2 \hat{T}_\varepsilon(i\omega)G\| \leq C, \forall \varepsilon$ .*
- (iii)  $\text{im } G \subset \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ .

Note that the implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) follow immediately from Lemma 5.3. Also note that once we have proven the above lemma, a proof of our main result Theorem 7.1 may be obtained by combining Theorem 5.2 and Lemma 3.1. We stress that the implications (iii) $\Rightarrow$ (i) in the above, in fact, yields sufficiency of the subspace inclusion  $\text{im } G \subset \mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  for solvability of (ADDPBP) $_p$  for all  $1 \leq p \leq \infty$ .

The idea of the proof of the implication (iii) $\Rightarrow$ (i) of Lemma 7.2 is as follows. First we note that left-invertibility of the system  $(A, B, H_1)$  is equivalent to the condition  $\mathcal{R}^*(\mathcal{K}_1) = \{0\}$  (Lemma 2.6), while injectivity of the map  $H_2$  is equivalent to  $\mathcal{K}_2 = \{0\}$ . Thus, under the assumptions of Lemma 7.2,  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$  may be decomposed according to (6.1), (6.2) and (6.3). Each of the singly generated almost controllability subspaces  $L_i$  appearing in this decomposition will then be approximated by sequences of controlled invariant subspaces  $\{\mathcal{L}_{j\varepsilon}; \varepsilon > 0\}$ . If we then define  $\mathcal{V}_\varepsilon := \mathcal{V}_g \oplus \bigoplus_{j=1}^{m'} \mathcal{L}_{j\varepsilon}$ , the sequence  $\{\mathcal{V}_\varepsilon; \varepsilon > 0\}$  will converge to  $\mathcal{V}_b(\mathcal{K}_1, \mathcal{K}_2)$ . In this sense,  $\text{im } G$  is “almost contained” in the controlled invariant subspace  $\mathcal{V}_\varepsilon$ . The subspace  $\mathcal{V}_\varepsilon$  in turn is “almost contained” in  $\mathcal{K}_1$  (where the latter “almost” should be interpreted in the  $L_1$ -sense, see also [20]), while its distance from  $\mathcal{K}_2$  is uniformly bounded with respect to  $\varepsilon$ . Using the structure of the  $\mathcal{L}_{j\varepsilon}$  above, we will construct a particular sequence of feedback maps  $\{F_\varepsilon; \varepsilon > 0\}$  such that  $(A + BF_\varepsilon)\mathcal{V}_\varepsilon \subset \mathcal{V}_\varepsilon$ . Finally it will be shown that this sequence has the properties required by (i) and (ii) in Lemma 7.2. To start with, we will show how a singly generated almost controllability subspace can be approximated by controlled invariant subspaces. Let  $b \in \mathcal{B}$  and let  $\mathcal{L} := \ell \oplus \dots \oplus A_F^{k-1} \ell$ . For  $i \in k$  and  $\varepsilon > 0$ ,

define vectors in  $\mathcal{X}$  by

$$(7.3) \quad x_1(\varepsilon) := (I + \varepsilon A_F)^{-1}b, \quad x_i(\varepsilon) := (I + \varepsilon A_F)^{-1}A_F x_{i-1}(\varepsilon).$$

Note that the matrix inversions in the above expressions are defined for  $\varepsilon$  sufficiently small. Moreover, it can be seen immediately that  $x_i(\varepsilon) \rightarrow A_F^{i-1}b$  ( $\varepsilon \rightarrow 0$ ). Thus it follows from Lemma 2.8 that for  $\varepsilon$  sufficiently small, the vectors  $\{x_i(\varepsilon), i \in \underline{k}\}$  are linearly independent. For each  $\varepsilon$ , define a subspace  $\mathcal{L}_\varepsilon$  by

$$(7.4) \quad \mathcal{L}_\varepsilon := \text{span} \{x_1(\varepsilon), \dots, x_k(\varepsilon)\}.$$

Assume  $u \in \mathcal{U}$  is such that  $b = Bu$  and define a map  $F_\varepsilon : \mathcal{L}_\varepsilon \rightarrow U$  by

$$(7.5) \quad F_\varepsilon x_i(\varepsilon) := -\varepsilon^{-i}u, \quad (i \in \underline{k}).$$

The main properties of the sequences  $\{\mathcal{L}_\varepsilon; \varepsilon > 0\}$  and  $\{F_\varepsilon; \varepsilon > 0\}$  are summarized in the following lemma:

LEMMA 7.3. *For  $i \in \underline{k}$  we have  $x_i(\varepsilon) \rightarrow A_F^{i-1}b$  as  $\varepsilon \rightarrow 0$ . Consequently,  $\mathcal{L}_\varepsilon \rightarrow \mathcal{L}$ . Each  $\mathcal{L}_\varepsilon$  is controlled invariant and, with  $F_\varepsilon$  defined by (7.5),  $(A_F + BF_\varepsilon)\mathcal{L}_\varepsilon \subset \mathcal{L}_\varepsilon$ . Moreover, a matrix of  $(A_F + BF_\varepsilon)|_{\mathcal{L}_\varepsilon}$  is given by*

$$(7.6) \quad M_\varepsilon := - \begin{pmatrix} \varepsilon^{-1} & \varepsilon^{-2} & \dots & \varepsilon^{-k} \\ 0 & \varepsilon^{-1} & \dots & \\ \vdots & \dots & \ddots & \varepsilon^{-2} \\ 0 & \dots & 0 & \varepsilon^{-1} \end{pmatrix}.$$

Finally, for each  $\varepsilon$ ,  $\mathcal{L}_\varepsilon \subset \langle A | \mathcal{B} \rangle$ , the reachable subspace of  $(A, B)$ .

*Proof.* The claim  $x_i(\varepsilon) \rightarrow A_F^{i-1}b$  is immediate. Since the vectors  $A_F^{i-1}b$  are a basis for  $\mathcal{L}$ , it follows from §§ 2 and 4 that  $\mathcal{L}_\varepsilon \rightarrow \mathcal{L}$ . Using (7.3) and (7.5), it may be verified by straightforward calculation that  $(A_F + BF_\varepsilon)x_i(\varepsilon) = -\sum_{j=1}^i \varepsilon^{j-i-1}x_j(\varepsilon)$ . It follows that  $\mathcal{L}_\varepsilon$  is indeed  $A_F + BF_\varepsilon$ -invariant and that a matrix of the map restricted to  $\mathcal{L}_\varepsilon$  is given by (7.6). Finally, to prove that  $\mathcal{L}_\varepsilon$  is contained in the reachable subspace, make a Taylor expansion to find that  $(I + \varepsilon A_F)^{-1} = \sum_{m=0}^\infty (-\varepsilon)^m A_F^m$ . It then follows immediately that  $x_1(\varepsilon) \in \langle A_F | \mathcal{B} \rangle$  for all  $\varepsilon$ . The same follows for  $x_2(\varepsilon), x_3(\varepsilon)$  etc.

We note that a slightly different construction leading to an approximating sequence for a singly generated controllability subspace was described in [13]. The construction described by us however exhibits an important property which will be formulated in the following lemma. The proof of this result is straightforward but rather technical and will be deferred to Appendix C.

LEMMA 7.4. *Let  $\mathcal{L} := \bigoplus_{i=1}^k A_F^{i-1}\ell$  be such that  $\bigoplus_{i=2}^k A_F^{i-2}\ell \subset \mathcal{H}_1$  and  $\bigoplus_{i=3}^k A_F^{i-3}\ell \subset \mathcal{H}_2$ . Let  $x_i(\varepsilon)$  and  $F_\varepsilon$  be as defined above. Then the following holds: there is constant  $C$  such that for all  $i \in \underline{k}$ :*

$$(7.7) \quad \|H_1 e^{(A_F + BF_\varepsilon)t} x_i(\varepsilon)\|_{L_1} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

$$(7.8) \quad \|H_2 e^{(A_F + BF_\varepsilon)t} x_i(\varepsilon)\|_{L_1} \leq C \quad \text{for all } \varepsilon. \quad \square$$

Now, in order to complete a proof of Lemma 7.2, we need one more preliminary result. Up to now we have constructed a sequence of controlled invariant subspaces converging to a singly generated almost controllability subspace and defined a feedback map on each of these controlled invariant subspaces. By applying the decomposition theorem, Theorem 6.1, and applying the above construction to each  $\mathcal{L}_j$  appearing in (6.1), we can find a sequence of controlled invariant subspaces  $\mathcal{R}_\varepsilon$  converging to  $\bigoplus_{i=1}^m \mathcal{L}_i$ . In the obvious way we can define a map  $F_\varepsilon$  on  $\mathcal{R}_\varepsilon$ . Now the question is, can we define  $F_\varepsilon$  appropriately on a subspace complementary to  $\mathcal{R}_\varepsilon$ ? The next construction

theorem states that, indeed, we can. It is here that we will use the results on exact disturbance decoupling with stability constraints from § 4. In the following,  $\mathcal{V}_b := \mathcal{V}_b(\mathcal{H}_1, \mathcal{H}_2)$ ,  $\mathcal{V}_g := \mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$  and  $\mathcal{R}_b := \mathcal{R}_b(\mathcal{H}_1, \mathcal{H}_2)$  are denoted:

**THEOREM 7.5.** *Consider the system (1.1). Let  $\Lambda$  be a symmetric set of  $\dim [(\langle A | \mathcal{B} \rangle + \mathcal{V}_g) / \mathcal{V}_b]$  complex numbers. Then there is a map  $F_1: \mathcal{X} \rightarrow \mathcal{U}$  and a subspace  $\mathcal{Z} \subset \mathcal{X}$  such that the following conditions are satisfied:*

- (7.9)  $(A + BF_1)\mathcal{V}_g \subset \mathcal{V}_g,$
- (7.10)  $(A + BF_1)\mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}^*(\mathcal{H}_2),$
- (7.11)  $\sigma(A + BF_1 | \mathcal{V}_g / \mathcal{V}^*(\mathcal{H}_2)) \subset \mathbb{C}_g,$
- (7.12)  $\mathcal{V}_b \oplus \mathcal{Z} = \mathcal{V}_g + \langle A | \mathcal{B} \rangle,$
- (7.13)  $(A + BF_1)(\mathcal{V}_g \oplus \mathcal{Z}) \subset \mathcal{V}_g \oplus \mathcal{Z},$
- (7.14)  $\sigma(A + BF_1 | (\mathcal{V}_g \oplus \mathcal{Z}) / \mathcal{V}_g) = \Lambda.$

*Proof.* Let  $F_0: \mathcal{X} \rightarrow \mathcal{U}$  be a map that satisfies the conditions (4.1), (4.2) and (4.3) of Theorem 4.3. Let  $P: \mathcal{X} \rightarrow \mathcal{X} / \mathcal{V}_g$  be the canonical projection. Let  $(\bar{A}_{F_0}, \bar{B})$  be the system induced by  $(A_{F_0}, B)$  in the factor space  $\mathcal{X} / \mathcal{V}_g$ . Since  $\mathcal{V}_b = \mathcal{V}_g + \mathcal{R}_b$  and  $\ker P = \mathcal{V}_g$ , we have  $P\mathcal{V}_b = P\mathcal{R}_b$ . By Lemma 2.4 and the fact that  $\mathcal{R}_b$  is an almost controllability subspace, it follows that  $P\mathcal{R}_b$  is an almost controllability subspace with respect to the system  $(\bar{A}_{F_0}, \bar{B})$ . By [22, Prop. 1.2],  $P\langle A | \mathcal{B} \rangle = \langle \bar{A}_{F_0} | \text{im } \bar{B} \rangle$ . Let  $\Lambda$  be as above. It can easily be verified that  $\# \Lambda = \dim [(\bar{A}_{F_0} | \text{im } \bar{B}) / P\mathcal{R}_b]$ . Thus, we may apply Proposition 2.5 to find an  $(\bar{A}_{F_0}, \bar{B})$ -invariant subspace  $\bar{\mathcal{Z}} \subset \mathcal{X} / \mathcal{V}_g$  and a map  $\bar{F}: \mathcal{X} / \mathcal{V}_g \rightarrow \mathcal{U}$  such that

- (7.15)  $P\mathcal{R}_b \oplus \bar{\mathcal{Z}} = \langle \bar{A}_{F_0} | \text{im } \bar{B} \rangle,$
- (7.16)  $(\bar{A}_{F_0} + \bar{B}\bar{F})\bar{\mathcal{Z}} \subset \bar{\mathcal{Z}},$
- (7.17)  $\sigma(\bar{A}_{F_0} + \bar{B}\bar{F} | \bar{\mathcal{Z}}) = \Lambda.$

Now let  $\mathcal{Z} \subset \mathcal{X}$  be any subspace such that  $P\mathcal{Z} = \bar{\mathcal{Z}}$  and  $\mathcal{Z} \cap \mathcal{V}_g = \{0\}$ . Define a map  $F_1: \mathcal{X} \rightarrow \mathcal{U}$  by  $F_1 := F_0 + \bar{F}P$ . We contend that the subspace  $\mathcal{Z}$  and the map  $F_1$  satisfy the claims of the theorem. To prove (7.9) to (7.11), note that  $F_1 | \mathcal{V}_g = F_0 | \mathcal{V}_g$ . The claim (7.12) can be proven as follows: From (7.15) we have  $P(\mathcal{V}_b + \mathcal{Z}) = P\langle A | \mathcal{B} \rangle$ . Hence, since  $\mathcal{V}_g \subset \mathcal{V}_b$ ,  $\mathcal{V}_b + \mathcal{Z} = \mathcal{V}_g + \langle A | \mathcal{B} \rangle$ . Assume  $x \in \mathcal{V}_b \cap \mathcal{Z}$ . Then  $Px \in P\mathcal{V}_b \cap \bar{\mathcal{Z}} = \{0\}$ . Thus,  $x \in \ker P \cap \mathcal{Z} = \mathcal{V}_g \cap \mathcal{Z} = \{0\}$ . It follows that  $\mathcal{V}_b + \mathcal{Z} = \mathcal{V}_b \oplus \mathcal{Z}$ .

To prove (7.13), note by using (7.16) that  $P(A + BF_1)(\mathcal{V}_g \oplus \mathcal{Z}) = P(A_{F_0} + B\bar{F}P)(\mathcal{V}_g \oplus \mathcal{Z}) = (\bar{A}_{F_0} + \bar{B}\bar{F})\bar{\mathcal{Z}} \subset \bar{\mathcal{Z}} = P(\mathcal{V}_g \oplus \mathcal{Z})$ . Finally, (7.14) follows immediately from (7.17).  $\square$

We are now in a position to complete the proof of Lemma 7.2:

*Proof of Lemma 7.2.* (i)  $\Rightarrow$  (ii). This follows immediately from the fact that the  $L_2$ -induced norm of a convolution operator is bounded from above by the  $L_1$ -norm of its kernel (see, for example, [2]).

(iii)  $\Rightarrow$  (i). In this part we will construct a sequence of feedback maps  $\{F_\varepsilon; \varepsilon > 0\}$  such that, for each  $x \in \mathcal{V}_b$ ,  $\|H_1 T_\varepsilon x\|_{L_1} \rightarrow 0$  and  $\|H_2 T_\varepsilon x\|_{L_1} \leq C$  for all  $\varepsilon$ , for some constant  $C$ . The construction is divided into five steps:

1. *Decomposition.* Apply Theorem 6.1 to find a decomposition  $\mathcal{V}_b = \mathcal{V}_g \oplus \bigoplus_{j=1}^{m'} \mathcal{L}_j$ , with  $\mathcal{L}_j = \bigoplus_{i=1}^{r_j} A_F^{i-1} \ell_{j,i}$ , such that (6.2) and (6.3) hold.

2. *Approximation of singly generated controllability subspaces.* For each  $\mathcal{L}_j$ , apply the construction (7.3) to (7.6). Thus we find vectors  $x_i^{(j)}(\varepsilon)$  ( $i \in r_j$ ), subspaces  $\mathcal{L}_{j\varepsilon} := \text{span} \{x_i^{(j)}(\varepsilon); i \in r_j\}$  and maps  $F_{j\varepsilon}: \mathcal{L}_{j\varepsilon} \rightarrow \mathcal{U}$  such that

$$(7.18) \quad x_i^{(j)}(\varepsilon) \rightarrow A_F^{i-1} b_j; \quad \mathcal{L}_{j\varepsilon} \rightarrow \mathcal{L}_j(\varepsilon \rightarrow 0).$$



Moreover, by applying Lemma 7.4, there are constants  $C_j$  such that

$$(7.19) \quad \|H_1 e^{(A_F + BF_{je})^t} x_i^{(j)}(\varepsilon)\|_{L_1} \rightarrow 0 (\varepsilon \rightarrow 0),$$

$$(7.20) \quad \|H_2 e^{(A_F + BF_{je})^t} x_i^{(j)}(\varepsilon)\|_{L_1} \leq C \quad \text{for all } \varepsilon.$$

3. *Composition.* Since the  $\mathcal{L}_j$  are independent, it follows from Lemma 2.8 that for  $\varepsilon$  sufficiently small the  $\mathcal{L}_{j\varepsilon}$  ( $j \in \underline{m}'$ ) are independent. Define  $\mathcal{R}_\varepsilon := \mathcal{L}_{1\varepsilon} \oplus \dots \oplus \mathcal{L}_{m'\varepsilon}$ . It follows that  $\mathcal{R}_\varepsilon \rightarrow \bigoplus_j \mathcal{L}_j$ . Define now  $F_\varepsilon^0: \mathcal{R}_\varepsilon \rightarrow \mathcal{U}$  by defining  $F_\varepsilon^0|_{\mathcal{L}_{j\varepsilon}} := (F + F_{j\varepsilon})|_{\mathcal{L}_{j\varepsilon}}$  ( $j \in \underline{m}'$ ).

4. *Construction of feedback outside  $\mathcal{R}_\varepsilon$ .* To define a map on a complement of  $\mathcal{R}_\varepsilon$ , let  $\Lambda \subset \mathbb{C}_g$  be a symmetric set of  $\dim [(\langle A|\mathcal{B} \rangle + \mathcal{V}_g)/\mathcal{V}_b]$  complex numbers and apply the construction theorem Theorem 7.5 to find a subspace  $\mathcal{Z} \subset \mathcal{X}$  and a map  $F_1: \mathcal{X} \rightarrow \mathcal{U}$  such that (7.9) to (7.14) are satisfied. In the remainder of this proof, denote  $\bigoplus_{j=1}^{m'} \mathcal{L}_j$  by  $\tilde{\mathcal{R}}_b$ . We may then prove the following:

LEMMA 7.6. *For all  $\varepsilon$  sufficiently small the following holds:*

$$(7.21) \quad \mathcal{V}_g \oplus \tilde{\mathcal{R}}_b \oplus \mathcal{Z} = \mathcal{V}_g \oplus \mathcal{R}_\varepsilon \oplus \mathcal{Z}.$$

*Proof.* By Lemma 6.6,  $\mathcal{V}_b \oplus \mathcal{Z} = \mathcal{V}_g \oplus \tilde{\mathcal{R}}_b \oplus \mathcal{Z}$ . Since, for each  $\varepsilon$ ,  $\mathcal{R}_\varepsilon \subset \langle A|\mathcal{B} \rangle$  (Lemma 7.3), it follows from (7.12) that  $\mathcal{R}_\varepsilon \subset \mathcal{V}_g \oplus \tilde{\mathcal{R}}_b \oplus \mathcal{Z}$ . Since  $\mathcal{R}_\varepsilon \rightarrow \tilde{\mathcal{R}}_b$ , we obtain from Lemma 2.8 that  $\mathcal{R}_\varepsilon \cap (\mathcal{V}_g \oplus \mathcal{Z}) = \{0\}$  for  $\varepsilon$  sufficiently small. The equality (7.21) now follows immediately by noting that for  $\varepsilon$  sufficiently small  $\dim \mathcal{R}_\varepsilon = \dim \tilde{\mathcal{R}}_b$ .

5. *Definition of the sequence  $\{F_\varepsilon; \varepsilon > 0\}$ .* Let  $\mathcal{W}$  be an arbitrary subspace of  $\mathcal{X}$  such that  $\mathcal{X} = \mathcal{V}_g \oplus \mathcal{R}_\varepsilon \oplus \mathcal{Z} \oplus \mathcal{W}$ . In this ( $\varepsilon$ -dependent) decomposition of  $\mathcal{X}$  define  $F_\varepsilon: \mathcal{X} \rightarrow \mathcal{U}$  by  $F_\varepsilon|_{\mathcal{V}_g \oplus \mathcal{Z}} := F_1|_{\mathcal{V}_g \oplus \mathcal{Z}}$ ,  $F_\varepsilon|_{\mathcal{R}_\varepsilon} = F_\varepsilon^0|_{\mathcal{R}_\varepsilon}$  and  $F_\varepsilon$  arbitrary on  $\mathcal{W}$ .

We contend that the sequence  $\{F_\varepsilon; \varepsilon > 0\}$  defined in this way satisfies the condition (i) of Lemma 7.2. To prove this, first let  $x \in \mathcal{V}_g$ . Since  $F_\varepsilon|_{\mathcal{V}_g} = F_1|_{\mathcal{V}_g}$ , we have by (7.9) and the fact that  $\mathcal{V}_g \subset \mathcal{H}_1$  that  $x \in \langle A_{F_\varepsilon}|\mathcal{V}_g \rangle \subset \mathcal{H}_1$  for all  $\varepsilon$ . Thus, for all  $\varepsilon$ ,  $H_1 T_\varepsilon(t)x = 0$  for all  $t$ . Let  $\bar{A}_{F_1}$  and  $\bar{H}_2$  be defined by the following commutative diagram (Fig. 2), in which  $P_1$  is the canonical projection:

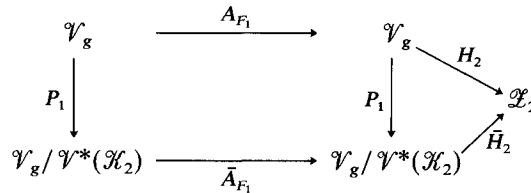


FIG. 2

We then have  $H_2 T_\varepsilon(t)x = \bar{H}_2 e^{\bar{A}_{F_1} t} x$  for all  $\varepsilon$ . It follows from (7.11) that  $H_2 T_\varepsilon x$  is in  $L_1[0, \infty)$  with, obviously,  $L_1$ -norm independent of  $\varepsilon$ . To complete the proof it now suffices to show that for all  $x \in \tilde{\mathcal{R}}_b$ ,  $\|H_1 T_\varepsilon x\|_{L_1} \rightarrow 0$  and  $\|H_2 T_\varepsilon x\|_{L_1}$  is uniformly bounded with respect to  $\varepsilon$ . Since  $\tilde{\mathcal{R}}_b$  is spanned by the vectors  $A_F^{s-1} b_b$ , it suffices to take  $x = A_F^{s-1} b_b$ . By (7.21), we have  $\tilde{\mathcal{R}}_b \subset \mathcal{V}_g \oplus \mathcal{R}_\varepsilon \oplus \mathcal{Z}$ . Thus, there are vectors  $v(\varepsilon) \in \mathcal{V}_g \oplus \mathcal{Z}$  and coefficients  $\tau_{ij}(\varepsilon) \in \mathbb{R}$  such that

$$(7.22) \quad A_F^{s-1} b_b = v(\varepsilon) + \sum_{j=1}^{m'} \sum_{i=1}^{I_j} \tau_{ij}(\varepsilon) x_i^{(j)}(\varepsilon).$$

Since  $x_i^{(j)}(\varepsilon) \rightarrow A_F^{i-1} b_j$ , it can be proven by standard means that  $v(\varepsilon) \rightarrow 0$ ,  $\tau_{ij}(\varepsilon) \rightarrow 0$

$(i, j) \neq (s, l)$  and that  $\tau_{st}(\varepsilon) \rightarrow 1 (\varepsilon \rightarrow 0)$ . Now, for  $\alpha = 1, 2$  we have

$$(7.23) \quad \begin{aligned} \|H_\alpha T_\varepsilon A_F^{-1} b_l\|_{L_1} &\leq \|H_\alpha e^{(A+BF_1)t} v(\varepsilon)\|_{L_1} \\ &+ \sum_{j=1}^{m'} \sum_{i=1}^{r_j} |\tau_{ij}(\varepsilon)| \|H_\alpha e^{(A_F+BF_{je})t} x_i^{(j)}(\varepsilon)\|_{L_1}. \end{aligned}$$

By (7.19), note that for  $\alpha = 1$  the last term converges to 0 as  $\varepsilon \rightarrow 0$ . Using (7.20), it follows that for  $\alpha = 2$  the last term is bounded from above, independent of  $\varepsilon$ .

Finally, we will show that for both  $\alpha = 1, 2$  the first term on the right in (7.23) tends to 0 as  $\varepsilon \rightarrow 0$ . For this, let  $\tilde{A}_{F_1}$  and  $\tilde{H}_1$  be defined by the commutative diagram (Fig. 3) ( $P_2$  is the canonical projection):

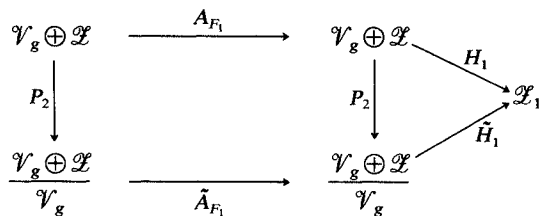


FIG. 3

By (7.14), note that  $\sigma(\tilde{A}_{F_1}) = \Lambda \subset \mathbb{C}_g$ . Moreover,

$$\|H_1 e^{A_{F_1}t} v(\varepsilon)\|_{L_1} = \|\tilde{H}_1 e^{\tilde{A}_{F_1}t} P_2 v(\varepsilon)\|_{L_1} \leq \|\tilde{H}_1 e^{\tilde{A}_{F_1}t} P_2\|_{L_1} \|v(\varepsilon)\|.$$

Since  $v(\varepsilon) \rightarrow 0$ , this expression tends to 0 as  $(\varepsilon \rightarrow 0)$ .

Let  $\hat{A}_{F_1}$  and  $\hat{H}_2$  be defined by the commutative diagram (Fig. 4) ( $P_3$  is the canonical projection):

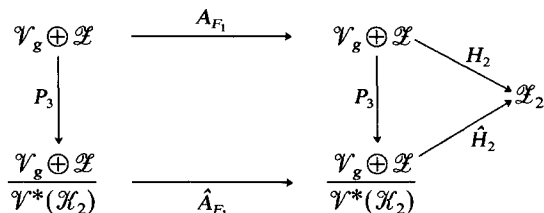


FIG. 4

It can be verified that  $\sigma(\hat{A}_{F_1}) = \sigma(\tilde{A}_{F_1}) \cup \sigma(A_{F_1} | \mathcal{V}_g / \mathcal{V}^*(\mathcal{K}_2))$ , which, by (7.11) and (7.14) is contained in  $\mathbb{C}_g$ . It follows that

$$\|H_2 e^{A_{F_1}t} v(\varepsilon)\|_{L_1} = \|\hat{H}_2 e^{\hat{A}_{F_1}t} P_3 v(\varepsilon)\|_{L_1} \leq \|\hat{H}_2 e^{\hat{A}_{F_1}t} P_3\|_{L_1} \|v(\varepsilon)\|,$$

which again converges to 0 as  $\varepsilon \rightarrow 0$ . This completes the proof of Lemma 7.2.  $\square$

*Remark 7.7.* It is worthwhile to point out which freedom in the spectrum assignment we have in  $A + BF_\varepsilon$  when we use the construction of the sequence  $\{F_\varepsilon; \varepsilon > 0\}$  as in the proof of Lemma 7.2.

The lattice diagram (Fig. 5) shows the hierarchy of the relevant subspaces in combination with the freedom in the spectrum of  $A + BF_\varepsilon$ .

Denote  $\mathcal{V}_\varepsilon := \mathcal{V}_\varepsilon(\mathcal{K}_1, \mathcal{K}_2) \oplus \mathcal{R}_\varepsilon$ .

Note by Lemma 7.3 that the spectrum of the map  $A + BF_\varepsilon | \mathcal{R}_\varepsilon$  consists of an eigenvalue in  $-\varepsilon^{-1}$  with multiplicity equal to  $\dim[\mathcal{V}_\varepsilon / \mathcal{V}_g]$ .

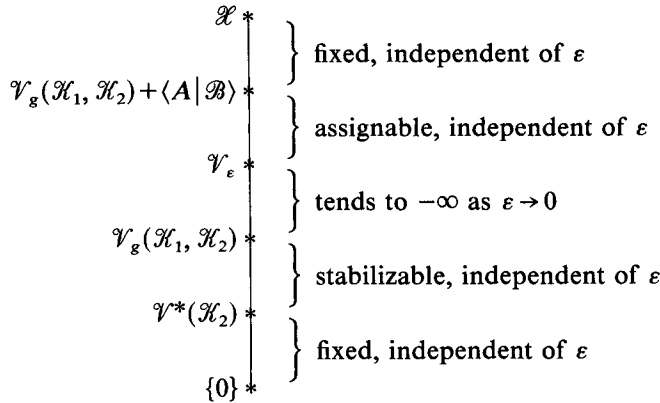


FIG. 5

**8. Some special cases and extensions.** In this section we will consider some special cases of the main theorem, Theorem 7.1, and spend a few words on some extensions of this result. One interesting special case of (ADDPBP)<sub>p</sub> is the situation that we take  $H_1 = H$  and  $H_2 = I$ . This corresponds to the almost disturbance decoupling problem with bounded peaking of the entire state vector. Denote  $\mathcal{K} := \ker H$ . Since, by Theorem 4.2,  $\mathcal{V}_g(\mathcal{K}, \{0\}) = \mathcal{V}_g^*(\mathcal{K}) + \mathcal{V}^*(\{0\}) = \mathcal{V}_g^*(\mathcal{K})$  and since, by Theorem 6.3,  $\mathcal{R}_b(\mathcal{K}, \{0\}) = \mathcal{B} + A(\mathcal{R}_b^*(\{0\}) \cap \mathcal{K}) = \mathcal{B} + A(\mathcal{B} \cap \mathcal{K})$ , we have the following corollary of Theorem 7.1:

**COROLLARY 8.1.** *Fix  $p \in \{1, 2, \infty\}$ . Then the  $L_p$ -almost disturbance decoupling problem with bounded peaking of the entire state vector is solvable if and only if  $\text{im } G \subset \mathcal{V}_g^*(\mathcal{K}) + \mathcal{B} + A(\mathcal{B} \cap \mathcal{K})$ .*

Next, we will spend some words on possible extensions of the results of this paper.

First we would like to point out that, while (ADDPBP)<sub>p</sub> is a nontrivial extension of (ADDP)<sub>p</sub>, we might also consider an extension of the  $L_p - L_q$  almost disturbance decoupling problem (ADDP)', see [20] or [17]. This would lead to the following synthesis problem:

We will say that the  $L_p - L_q$  almost disturbance decoupling problem with bounded peaking (ADDPBP)' is solvable if there is a constant  $C$  and, for all  $\epsilon > 0$ , a feedback map  $F_\epsilon : \mathcal{X} \rightarrow \mathcal{U}$  such that with the feedback law  $u = F_\epsilon x$ , in the closed loop system for  $x(0) = 0$  there holds, for all  $1 \leq p \leq q \leq \infty$ , for all  $d \in L_q[0, \infty)$

$$\|z_1\|_{L_p} \leq \epsilon \|d\|_{L_q}, \quad \|z_2\|_{L_p} \leq C \|d\|_{L_q}$$

It may be shown that the solvability of the above problem is equivalent to the existence of a sequence of feedback maps  $\{F_\epsilon; \epsilon > 0\}$  and a constant  $C$  such that for both  $p = 1$  and  $p = \infty$   $\|H_1 T_\epsilon G\|_{L_p} \rightarrow 0 (\epsilon \rightarrow 0)$  and  $\|H_2 T_\epsilon G\|_{L_p} \leq C$  for all  $\epsilon$ .

A theory analogous to the one above may be developed around this problem. It can be shown that, again under the assumption that either  $(A, B, H_1)$  is left-invertible or that  $H_2$  is injective, a necessary and sufficient condition for the solvability of this problem is that

$$\text{im } G \subset \mathcal{V}_g(\mathcal{K}_1, \mathcal{K}_2) + (\mathcal{R}_b^*(\mathcal{K}_2) \cap \mathcal{K}_1).$$

For more details, the reader is referred to [17].

A final extension of the results of the present paper is the situation in which we require internal stability of the closed loop system. This would lead to the following synthesis problem: We will say that the  $L_p$ -almost disturbance decoupling problem with

*bounded peaking and strong stabilization*  $(\text{ADDPBPSS})_p$  is solvable if the following is true. There is a constant  $C$  and for all  $\varepsilon > 0$  and all real numbers  $S$  a feedback map  $F_{\varepsilon,S}: \mathcal{X} \rightarrow \mathcal{U}$  such that, with the feedback law  $u = F_{\varepsilon,S}x$ , in the closed loop system for  $x(0) = 0$  for all  $d \in L_p[0, \infty)$  the inequalities (3.2) and (3.3) hold and such that  $\text{Re } \sigma(A + BF_{\varepsilon,S}) \leq S$ .

Thus, we require that the spectrum of the closed loop matrix can be located to the left of any vertical line  $\text{Re } s = S$  in the complex plane. It may be proven that if at least one of the conditions (7.1), (7.2) hold, then for  $p \in \{1, 2, \infty\}$  the latter problem is solvable if and only if  $(A, B)$  is controllable and

$$(8.1) \quad \text{im } G \subset \mathcal{R}^*(\mathcal{H}_1) + \mathcal{R}_b(\mathcal{H}_1, \mathcal{H}_2).$$

We note that if  $(A, B, H_1)$  is a left-invertible system then the inclusion (8.1) becomes

$$\text{im } G \subset \mathcal{B} + A[\mathcal{R}_b^*(\mathcal{H}_2) \cap \mathcal{H}_1],$$

(see Theorem 6.3). If  $H_2$  is injective then (8.1) becomes

$$\text{im } G \subset \mathcal{R}^*(\mathcal{H}_1) + \mathcal{B} + A[\mathcal{B} \cap \mathcal{H}_1].$$

Again, for details the reader is referred to [17].

**9. A worked example.** To illustrate the theory developed in this paper and to demonstrate its computational feasibility, in this section we will present a worked example. We will consider a linear system with two outputs and check whether  $(\text{ADDPBP})_p$  is solvable for this system. Next, we will actually compute a sequence of feedback mappings that achieves our design purpose. The system that will be considered is given by  $\dot{x}(t) = Ax(t) + Bu(t) + Gd(t)$ ,  $z_1(t) = H_1x(t)$ ,  $z_2(t) = H_2x(t)$ , with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_1 = (0 \ 0 \ 0 \ 1 \ 0), \quad H_2 = I_{5 \times 5}$$

and

$$G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus,  $\mathcal{X} = \mathbb{R}^5$  and  $\mathcal{U} = \mathbb{R}^2$ . Denote  $\mathcal{H}_i = \ker H_i$ . The route that we will take is as follows. First, we will check whether the subspace inclusion  $\text{im } G \subset \mathcal{V}_b(\mathcal{H}_1, \mathcal{H}_2)$  holds to see if  $(\text{ADDPBP})_p$  is solvable. It turns out that this is indeed true. After this, we will follow closely the lines of the development in § 7 and construct a required sequence  $\{F_n\}$ . As before,  $\mathbb{C}_g = \{\lambda \in \mathbb{C} \mid \text{Re } \lambda < 0\}$  and the subspaces  $\mathcal{V}_g^*(\mathcal{H}_1)$  and  $\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2)$  are taken with respect to this stability set. Let the standard basis vectors in  $\mathbb{R}^5$  be denoted by  $e_i$ .

Using the algorithm ISA (see [22, p. 91]) and a construction as in [22, p. 114], we calculate that  $\mathcal{V}_g(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{V}_g^*(\mathcal{H}_1) = \text{span}\{e_1, e_2\}$  (since  $\mathcal{V}^*(\mathcal{H}_2) = \{0\}$ ). Thus, by Theorem 4.4, DDPOS is not solvable for the above system. Since  $\mathcal{H}_2 = \{0\}$ , by Theorem 6.3 we should check if the subspace inclusion  $\text{im } G \subset \mathcal{V}_g^*(\mathcal{H}_1) + \mathcal{B} + A(\mathcal{B} \cap \mathcal{H}_1)$  holds. It may be calculated that  $\mathcal{V}_g^*(\mathcal{H}_1) + \mathcal{B} + A(\mathcal{B} \cap \mathcal{H}_1) = \text{span}\{e_1, e_2, e_4, e_5\}$ . Since  $\text{im } G$  is

indeed contained in this subspace,  $(\text{ADDPBP})_p$  is solvable for all  $1 \leq p \leq \infty$ . Unfortunately,  $(\text{ADDPBSS})_p$  is *not* solvable because  $(A, B)$  is an uncontrollable pair. We will now construct a required sequence of feedback mappings:

*Step 1: decomposition.* We decompose  $\mathcal{V}_b = \mathcal{V}_g \oplus \tilde{\mathcal{W}}$ , with  $\tilde{\mathcal{W}} := \tilde{\mathcal{B}} + A(\tilde{\mathcal{B}} \cap \mathcal{K}_1)$  and  $\tilde{\mathcal{B}}$  such that  $\tilde{\mathcal{B}} \oplus (\mathcal{B} \cap \mathcal{V}^*(\mathcal{K}_1)) = \mathcal{B}$ . Then  $\tilde{\mathcal{W}} = \text{span}\{e_4, e_5\}$ . Since  $e_5 \in \mathcal{B}$  and  $e_4 = Ae_5$ ,  $\tilde{\mathcal{W}}$  is equal to the two-dimensional singly generated almost controllability subspace  $\ell \oplus A\ell$ , where  $b = e_5$ . Note that indeed (6.2) and (6.3) are satisfied.

*Step 2: approximation.* Approximate  $\ell \oplus A\ell$  by  $(A, B)$ -invariant subspaces  $\mathcal{R}_\varepsilon = \text{span}\{x_1(\varepsilon), x_2(\varepsilon)\}$  according to (7.3). In our case it can be calculated that  $x_1(\varepsilon) = (0 \ 0 \ -\varepsilon^2 \ -\varepsilon \ 1)^T$  and  $x_2(\varepsilon) = Ab = (0 \ 0 \ 0 \ 1 \ 0)^T$ . Note that  $b = Bu$  with  $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Following (7.5) for  $\varepsilon > 0$  define  $F_\varepsilon^0: \mathcal{R}_\varepsilon \rightarrow \mathcal{U}$  by  $F_\varepsilon^0 x_1(\varepsilon) = -\varepsilon^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and  $F_\varepsilon^0 x_2(\varepsilon) = -\varepsilon^{-2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

*Step 3: a feedback mapping outside  $\mathcal{R}_\varepsilon$ .* Note that for our system  $\langle A | \mathcal{B} \rangle + \mathcal{V}_g = \mathcal{X}$ . It can be verified that the conditions (7.9) to (7.14) are satisfied with  $\Lambda = \{-3\}$ ;  $\mathcal{Z} = \text{span}\{(0 \ 0 \ 1 \ -3 \ 9)^T\}$  and  $F_1: \mathcal{X} \rightarrow \mathcal{U}$  given by

$$F_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

*Step 4: definition of the required sequence  $\{F_\varepsilon; \varepsilon > 0\}$ .* Note that  $\mathcal{X} = \mathcal{V}_g \oplus \mathcal{R}_\varepsilon \oplus \mathcal{Z}$ . In this decomposition define  $F_\varepsilon | (\mathcal{V}_g \oplus \mathcal{Z}) := F_1 | (\mathcal{V}_g \oplus \mathcal{Z})$  and  $F_\varepsilon | \mathcal{R}_\varepsilon := F_\varepsilon^0 | \mathcal{R}_\varepsilon$ . It can be verified that the matrix of  $F_\varepsilon$  with respect to the standard bases in  $\mathcal{X} = \mathcal{R}^5$  and  $\mathcal{U} = \mathcal{R}^2$  is given by

$$F_\varepsilon := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{23}(\varepsilon) & -1/\varepsilon^2 & f_{25}(\varepsilon) \end{pmatrix},$$

where

$$f_{23}(\varepsilon) = \frac{-27\varepsilon^2 + 18\varepsilon - 3}{\varepsilon^2 + 9\varepsilon^4} \quad \text{and} \quad f_{25}(\varepsilon) = \frac{27\varepsilon^3 - 3\varepsilon - 2}{9\varepsilon^3 + \varepsilon}.$$

Finally, by valuating  $A + BF_\varepsilon$  in the basis suggested by the decomposition  $\mathcal{X} = \mathcal{V}_g \oplus \mathcal{R}_\varepsilon \oplus \mathcal{Z}$ , we can calculate the closed loop impulse response matrices from  $d$  to  $z_1$  and  $z_2$ , respectively:

$$W_{1\varepsilon}(t) := H_1 e^{(A+BF_\varepsilon)t} G = \left(\frac{t}{\varepsilon} + 1\right) e^{-t/\varepsilon},$$

$$W_{2\varepsilon}(t) := H_2 e^{(A+BF_\varepsilon)t} G = \begin{pmatrix} 0 \\ 0 \\ t \\ t/\varepsilon + 1 \\ -t/\varepsilon^2 \end{pmatrix} e^{-t/\varepsilon}.$$

A standard calculation shows that  $\|W_{1\varepsilon}\|_{L_1} = 2\varepsilon \rightarrow 0$  and that

$$\|W_{2\varepsilon}\|_{L_1} = \int_0^\infty \|W_{2\varepsilon}(t)\| dt \leq 1 + 2\varepsilon + \varepsilon^2.$$

Here,  $\|\cdot\|$  denotes the Euclidean norm. From this it can be seen that indeed for every  $1 \leq p \leq \infty$  the  $L_p - L_p$  induced norm of the closed loop operator from  $d$  to  $z_1$  tends to zero as  $\varepsilon \rightarrow 0$  and that the induced norm of the operator from  $d$  to  $z_2$  is bounded with respect to  $\varepsilon$ . Note that  $\|F_\varepsilon\| \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .

**10. Conclusions.** In this paper we have developed a theory around the almost disturbance decoupling problem with bounded peaking. Necessary and sufficient conditions for the solvability of this synthesis problem were formulated in terms of a subspace inclusion involving a certain almost controlled invariant subspace. We showed that this almost controlled invariant subspace can be calculated using existing algorithms. We also provided a conceptual algorithm to calculate the sequence of feedback maps that achieve the design purpose. The calculations involved were illustrated in a numerical example.

Several questions remain to be answered. As a first direction for future research we mention the extension of the above results to the case that the system under consideration does not satisfy one of the conditions (7.1), (7.2), i.e., the system  $(A, B, H_1)$  is not left-invertible and the map  $H_2$  is not injective. Another interesting issue would be to extend this theory to the more general situation that we allow only output feedback instead of state feedback. In this context we mention [21] and recent results in [18]. Finally, connections between this work and results on bounded peaking in the context of the nearly singular optimal control problem [3] remain to be worked out.

**Appendix A.** In this appendix we will establish a proof of Lemma 5.3. The proof will proceed through a series of lemmas. The first lemma is concerned with the convergence of sequence of rational functions and was proven in [7]. In the following, if  $f(s)$  is a strictly proper rational function, then  $\text{deg } f$  will denote its McMillan degree. We have:

**LEMMA A.1.** *Let  $\{f_\varepsilon\}$  be a sequence of strictly proper rational functions. Suppose that there exists  $r \in \mathbb{N}$  such that  $\text{def } f_\varepsilon \leq r$  for all  $\varepsilon$ . Assume that  $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(s)$  exists for infinitely many  $s \in \mathbb{C}$ . Then there exists a rational function  $f$  such that  $f_\varepsilon(s) \rightarrow f(s)$  ( $\varepsilon \rightarrow 0$ ) for all but finitely many  $s \in \mathbb{C}$ .  $\square$*

We can then prove the following:

**LEMMA A.2.** *Suppose that either the condition (i) or (ii) in Lemma 5.3 is satisfied. Then there are a rational vector  $\hat{z}(s)$ , proper and stable and, for  $i = 1, 2$ , subsequences  $\{\hat{z}_{i,\varepsilon}(s)\}$  such that  $\hat{z}_{1,\varepsilon}(s) \rightarrow 0$  ( $\varepsilon' \rightarrow 0$ ) and  $\hat{z}_{2,\varepsilon}(s) \rightarrow \hat{z}(s)$  ( $\varepsilon' \rightarrow 0$ ) for all but finitely many  $s$ .*

*Proof.* If the condition (i) of Lemma 5.3 holds, then for  $\sigma := \text{Re } s \geq 0$ :

$$(A.1) \quad \|\hat{z}_{i,\varepsilon}(s)\| \leq \int_0^\infty e^{-\sigma t} \|z_{i,\varepsilon}(t)\| dt \leq \|z_{i,\varepsilon}\|_{L_1}.$$

If the condition (ii) of Lemma 5.3 holds, then by the fact that  $\hat{z}_{i,\varepsilon}(s)$  is strictly proper and has no poles in  $\text{Re } s \geq 0$ , applying the maximum modulus principle [8] gives, for all  $\text{Re } s \geq 0$ ,

$$(A.2) \quad \|\hat{z}_{i,\varepsilon}(s)\| \leq \sup_{\omega \in \mathbb{R}} \|\hat{z}_{i,\varepsilon}(i\omega)\|.$$

Hence, in both cases we have  $\hat{z}_{1,\varepsilon}(s) \rightarrow 0$  ( $\varepsilon \rightarrow 0$ ) and  $\|\hat{z}_{2,\varepsilon}(s)\| \leq C\forall\varepsilon$ . Since, for all  $\varepsilon$ ,  $\hat{z}_{2,\varepsilon}(s)$  is analytic in  $\text{Re } s > 0$  and since the sequence  $\{\hat{z}_{2,\varepsilon}(s)\}$  is uniformly bounded there, by Vitali's theorem [8] there exists a function  $\hat{z}(s)$ , analytic in  $\text{Re } s > 0$ , and a subsequence  $\{\hat{z}_{2,\varepsilon'}(s)\}$  such that  $\hat{z}_{2,\varepsilon'}(s) \rightarrow \hat{z}(s)$  ( $\varepsilon' \rightarrow 0$ ) uniformly on each compact set  $K$  in the open right half plane. Therefore,  $\hat{z}_{2,\varepsilon'}(s) \rightarrow \hat{z}(s)$  ( $\varepsilon' \rightarrow 0$ ) pointwise in  $\text{Re } s > 0$ . By Lemma A.1, we may assume that  $\hat{z}_{2,\varepsilon'}(s) \rightarrow \hat{z}(s)$  ( $\varepsilon' \rightarrow 0$ ) for all but finitely many  $s$  and  $\hat{z}(s)$  is rational. We contend that  $\hat{z}(s)$  cannot have poles in  $\text{Re } s = 0$ , for define  $J := \{s \mid \text{Re } s = 0, s \text{ is not a pole of } \hat{z}(s) \text{ and } \hat{z}_{2,\varepsilon'}(s) \rightarrow \hat{z}(s) \text{ ( $\varepsilon' \rightarrow 0$ )}\}$ . Then the complement of  $J$  in  $\text{Re } s = 0$  is a finite set. Suppose  $s_0 \in J$ . For  $\varepsilon'$  sufficiently small,  $\|\hat{z}_{2,\varepsilon'}(s_0) - \hat{z}(s_0)\| \leq 1$ . Hence we have

$$(A.3) \quad \|\hat{z}(s_0)\| \leq \|\hat{z}(s_0) - \hat{z}_{2,\varepsilon'}(s_0)\| + \|\hat{z}_{2,\varepsilon'}(s_0)\| \leq 1 + C.$$

Therefore,  $\hat{z}(s)$  is bounded on  $J$  and hence bounded on the entire imaginary axis. Also from (5.3) there follows that  $\hat{z}(s)$  is *proper*. Finally,  $\hat{z}_{1,\varepsilon}(s) \rightarrow 0$  ( $\varepsilon' \rightarrow 0$ ) in  $\text{Re } s \geq 0$  and hence, again by Lemma A.1, for all but finitely many  $s$ .  $\square$

We will now complete the proof of Lemma 5.3. Recall that  $u_\varepsilon(t)$  is a regular Bohl input and  $z_{1,\varepsilon}(t) = H_1x_\varepsilon(t)$ ,  $z_{2,\varepsilon}(t) = H_2x_\varepsilon(t)$ , where  $\dot{x}_\varepsilon = A_Fx_\varepsilon + Bu_\varepsilon$ ,  $x_\varepsilon(0) = x_0$ . We will prove that if either the condition (i) or (ii) in Lemma 5.3 is satisfied, then  $x_0 \in \mathcal{V}_b(\mathcal{H}_1, \mathcal{H}_2)$ :

*Proof of Lemma 5.3.* Let  $F: \mathcal{X} \rightarrow \mathcal{U}$  be such that  $A_F\mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}^*(\mathcal{H}_2)$ . Denote  $v_\varepsilon(t) := u_\varepsilon(t) - Fx_\varepsilon(t)$ . Then  $\dot{x}_\varepsilon = A_Fx_\varepsilon + Bv_\varepsilon$ . Note that  $x_\varepsilon$  and  $v_\varepsilon$  have rational Laplace transforms. Let  $P: \mathcal{X} \rightarrow \mathcal{X}/\mathcal{V}^*(\mathcal{H}_2)$  be the canonical projection and let  $\bar{A}_F$  be the quotient map of  $A_F$  modulo  $\mathcal{V}^*(\mathcal{H}_2)$ . Let  $\bar{B} := PB$  and let  $\bar{H}_1$  and  $\bar{H}_2$  be mappings such that  $\bar{H}_1P = H_1$  and  $\bar{H}_2P = H_2$ . Decompose  $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$  with  $\mathcal{U}_1 = \ker \bar{B}$  and  $\mathcal{U}_2$  an arbitrary complement. Accordingly, partition  $\bar{B} = (0, \bar{B}_2)$ . Then  $\bar{B}_2$  is injective. Let  $\bar{G}(s) := \bar{H}_2(Is - \bar{A}_F)^{-1}\bar{B}_2$ . Let  $\bar{\mathcal{R}}^*(\mathcal{H}_2)$  be the supremal controllability subspace in  $\mathcal{H}_2$  with respect to  $(\bar{A}_F, \bar{B})$ . By [22, Ex. 5.8], we have  $\bar{\mathcal{R}}^*(\mathcal{H}_2) = \{0\}$ . Hence, by [22, Ex. 4.4],  $\bar{G}(s)$  is a left-invertible rational matrix, with left-inverse  $\bar{G}^+(s)$ . Now, let  $\xi_\varepsilon(s)$  and  $\omega_\varepsilon(s)$  be the Laplace transforms of  $x_\varepsilon$  and  $v_\varepsilon$  respectively. Let  $\bar{\xi}_\varepsilon(s) := P\xi_\varepsilon(s)$  and  $\bar{x}_0 := Px_0$ . Partition

$$\omega_\varepsilon(s) = \begin{pmatrix} \omega_{1,\varepsilon}(s) \\ \omega_{2,\varepsilon}(s) \end{pmatrix},$$

and conform the decomposition  $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$ . The following relations then hold

$$(A.4) \quad x_0 = (Is - A_F)\xi_\varepsilon(s) - B\omega_\varepsilon(s),$$

$$(A.5) \quad \bar{x}_0 = (Is - \bar{A}_F)\bar{\xi}_\varepsilon(s) - \bar{B}_2\omega_{2,\varepsilon}(s),$$

and hence

$$(A.6) \quad \omega_{2,\varepsilon}(s) = \bar{G}^+(s)[\hat{z}_{2,\varepsilon}(s) - \bar{H}_2(Is - \bar{A}_F)^{-1}\bar{x}_0],$$

$$(A.7) \quad \bar{\xi}_\varepsilon(s) = (Is - \bar{A}_F)^{-1}(\bar{x}_0 + \bar{B}_2\omega_{2,\varepsilon}(s)).$$

Apply Lemma A.2 to obtain  $\hat{z}(s)$  such that  $\hat{z}_{2,\varepsilon}(s) \rightarrow \hat{z}(s)$  ( $\varepsilon \rightarrow 0$ ) for all but finitely many  $s$  (write  $\varepsilon$  in the subsequence for which this holds). It follows that there are rational vectors  $\omega_2(s)$  and  $\bar{\xi}(s)$  such that  $\omega_{2,\varepsilon}(s) \rightarrow \omega_2(s)$  and  $\bar{\xi}_\varepsilon(s) \rightarrow \bar{\xi}(s)$  for all but finitely many  $s$ . Define now

$$\omega(s) := \begin{pmatrix} 0 \\ \omega_2(s) \end{pmatrix}.$$

Then we have  $\bar{x}_0 = (Is - \bar{A}_F)\bar{\xi}(s) - \bar{B}\omega(s)$ . Moreover, since  $\bar{H}_1\bar{\xi}_\varepsilon(s) = \hat{z}_{1,\varepsilon}(s) \rightarrow 0$  ( $\varepsilon \rightarrow 0$ ), we have  $\bar{H}_1\bar{\xi}(s) = 0$ . Also, since  $\bar{H}_2\bar{\xi}_\varepsilon(s) = \hat{z}_{2,\varepsilon}(s) \rightarrow \hat{z}(s)$ ,  $\bar{H}_2\bar{\xi}(s)$  is proper and stable.

Finally, let  $\xi(s)$  be any rational vector such that  $\bar{\xi}(s) = P\xi(s)$ . Then  $H_1\xi(s) = 0$ ,  $H_2\xi(s)$  is proper and stable and, for some vector  $x_1 \in \mathcal{V}^*(\mathcal{H}_2)$ ,  $x_0 = (Is - A_F)\xi(s) - B\omega(s) + x_1$ . It follows that  $x_0 - x_1 \in \mathcal{V}_b(\mathcal{H}_1, \mathcal{H}_2)$ . Since  $\mathcal{V}^*(\mathcal{H}_2) \subset \mathcal{V}_b(\mathcal{H}_1, \mathcal{H}_2)$  we thus obtain  $x_0 \in \mathcal{V}_b(\mathcal{H}_1, \mathcal{H}_2)$ . This completes the proof of Lemma 5.3.  $\square$

**Appendix B.** In this appendix we will state and prove a result on the geometrical structure of the sequence of subspaces  $\mathcal{F}^i(\mathcal{H}_1, \mathcal{H}_2)$ , given by (6.8). Our result is a generalization of [19, Thm. 7.1] and deals with the representation of subspaces in terms of chains in the input space  $\mathcal{B}$ . Related results can be found in [14] and [10]. Further, in this appendix we will prove Lemma 6.5.

LEMMA B.1. Given the system  $\dot{x} = Ax + Bu$  and subspaces  $\mathcal{K}_2 \subset \mathcal{K}_1 \subset \mathcal{X}$ , let  $\mathcal{F}^i(\mathcal{K}_1, \mathcal{K}_2)$  be defined by (6.7) and (6.8). Then the following holds: for all  $i \in \mathfrak{n}$ , there are a chain  $\{\mathcal{B}_i\}_{i=1}^i$  and a map  $F: \mathcal{X} \rightarrow \mathcal{U}$  such that

$$(B.1) \quad \mathcal{F}^i(\mathcal{K}_1, \mathcal{K}_2) = \bigoplus_{l=1}^i A_F^{l-1} \mathcal{B}_l,$$

$$(B.2) \quad \bigoplus_{l=2}^i A_F^{l-2} \mathcal{B}_l \subset \mathcal{K}_2,$$

$$(B.3) \quad \dim \mathcal{B}_i = \dim A_F^{l-1} \mathcal{B}_i = \dim [\mathcal{F}^l(\mathcal{K}_1, \mathcal{K}_2) / \mathcal{F}^{l-1}(\mathcal{K}_1, \mathcal{K}_2)] \quad (l \in \underline{i}).$$

*Remark.* In the above, for consistence define  $\bigoplus_{l=2}^i A_F^{l-2} \mathcal{B}_l = \{0\}$  if  $i = 1$ . In the following we will denote  $\mathcal{F}^i := \mathcal{F}^i(\mathcal{K}_1, \mathcal{K}_2)$ .

*Proof.* The proof is by induction. For  $i = 1$ , the lemma is trivially true: take  $\mathcal{B}_1 := \mathcal{B} \cap \mathcal{K}_1$ . Suppose now the lemma is true for  $i$ . Let  $\{\mathcal{B}_i\}_{i=1}^i$  be a chain in  $\mathcal{B}$  and  $F$  be a map such that the conditions (B.1), (B.2) and (B.3) are satisfied. We will show that  $\mathcal{F}^{i+1}$  can also be represented as above. This will be done by constructing an *extra term*  $\mathcal{B}_{i+1}$  for the chain  $\{\mathcal{B}_i\}_{i=1}^i$  and by defining a *new feedback map*  $F_{\text{new}}$ . First, let  $\mathcal{B}'_l$ ,  $l \in \underline{i}$ , be subspaces such that  $\mathcal{B}'_l \oplus \mathcal{B}_l = \mathcal{B}_{l-1}$  (define  $\mathcal{B}_0 := \mathcal{B}$ ). Using (B.1) we then have:

$$\begin{aligned} \mathcal{F}^i \cap \mathcal{K}_2 &= \left( \sum_{l=1}^i A_F^{l-1} (\mathcal{B}_{l+1} + \mathcal{B}'_{l+1}) + A_F^{i-1} \mathcal{B}_i \right) \cap \mathcal{K}_2 \\ &= \left( \sum_{l=1}^{i-1} A_F^{l-1} \mathcal{B}_{l+1} + \sum_{l=1}^{i-1} A_F^{l-1} \mathcal{B}'_{l+1} + A_F^{i-1} \mathcal{B}_i \right) \cap \mathcal{K}_2. \end{aligned}$$

Using the modular distributive rule [22, p. 4] and (B.2), it follows that

$$(B.4) \quad \mathcal{F}^i \cap \mathcal{K}_2 = \sum_{l=1}^i A_F^{l-1} \mathcal{B}_{l+1} + \left( \sum_{l=1}^{i-1} A_F^{l-1} \mathcal{B}'_{l+1} + A_F^{i-1} \mathcal{B}_i \right) \cap \mathcal{K}_2.$$

On the other hand, by (6.8),

$$(B.5) \quad \mathcal{F}^{i+1} = \mathcal{F}^{i+1}(\mathcal{K}_2) \cap \mathcal{K}_1 = (\mathcal{B} + A_F(\mathcal{F}^i(\mathcal{K}_2) \cap \mathcal{K}_2)) \cap \mathcal{K}_1.$$

Since, by the fact that  $\mathcal{K}_2 \subset \mathcal{K}_1$ ,  $\mathcal{F}^i(\mathcal{K}_2) \cap \mathcal{K}_2 = \mathcal{F}^i \cap \mathcal{K}_2$ , it follows by combining (B.4) and (B.5)

$$(B.6) \quad \begin{aligned} \mathcal{F}^{i+1} &= \left( \mathcal{B} + \sum_{l=1}^i A_F^l \mathcal{B}_{l+1} + A_F \left[ \left( \sum_{l=1}^{i-1} A_F^{l-1} \mathcal{B}'_{l+1} + A_F^{i-1} \mathcal{B}_i \right) \cap \mathcal{K}_2 \right] \right) \cap \mathcal{K}_1 \\ &= (\mathcal{F}^i + \mathcal{G}) \cap \mathcal{K}_1. \end{aligned}$$

Here, we defined

$$\mathcal{G} := \mathcal{B}'_1 + A_F \left[ \left( \sum_{l=1}^i A_F^{l-1} \mathcal{B}'_{l+1} + A_F^{i-1} \mathcal{B}_i \right) \cap \mathcal{K}_2 \right].$$

Again using the modular distributive rule and  $\mathcal{F}^i \subset \mathcal{K}_1$ , we obtain

$$(B.7) \quad \mathcal{F}^{i+1} = \mathcal{F}^i + (\mathcal{G} \cap \mathcal{K}_1).$$

Let  $\hat{\mathcal{G}} \subset \mathcal{G} \cap \mathcal{K}_1$  be a subspace such that  $\mathcal{F}^{i+1} = \mathcal{F}^i \oplus \hat{\mathcal{G}}$  and let  $\{v_1, \dots, v_r\}$  be a basis for  $\hat{\mathcal{G}}$ . By definition of  $\mathcal{G}$ , each  $v_j$  can be represented as  $v_j = \sum_{l=1}^i A_F^{l-1} b'_{l,j} + A_F^i b_j$ , with

$$(B.8) \quad \sum_{l=2}^i A_F^{l-2} b'_{l,j} + A_F^{i-1} b_j \in \mathcal{K}_2.$$



Here,  $b_j \in \mathcal{B}_i$  and  $b'_{l,j} \in \mathcal{B}'_l$  ( $j \in \mathcal{I}, l \in \underline{i}$ ). By the assumption that  $\hat{\mathcal{G}} \cap \mathcal{F}^i = \{0\}$ , it can be verified that for fixed  $l \in \{0, \dots, i\}$ , the vectors  $\{A_F^l b_1, \dots, A_F^l b_r\}$  are linearly independent. Define now

$$(B.9) \quad \mathcal{B}_{i+1} := \text{span} \{b_1, \dots, b_r\}.$$

Note that  $\mathcal{B}_{i+1} \subset \mathcal{B}_i$ . Also, for  $l \in \underline{i}$ , define vectors  $x_{j,l}$  by

$$(B.10) \quad x_{j,i} := b_j, \quad x_{j,l} := \sum_{k=1}^{l-1} A_F^{l-k-1} b'_{i-k+1,j} + A_F^{l-1} b_j.$$

From (B.10), for  $l=2, \dots, i$  we have  $x_{j,l} = A_F x_{j,l-1} + b'_{i-l+2,j}$  and  $v_j = A_F x_{j,i} + b'_{i,j}$ . Moreover, by the independency of the vectors  $\{A_F^l b_j; j \in \mathcal{I}\}$  and by the fact that the spaces  $A_F^{l-1} \mathcal{B}_l$  ( $l \in \underline{i}$ ) are independent (the sum in (B.1) is direct), it can be shown that the vectors  $\{x_{j,l}; j \in \mathcal{I}, l \in \underline{i}\}$  are linearly independent. Extend this system to a basis for  $\mathcal{X}$ . Let  $u_{j,k} \in \mathcal{U}$  be such that  $b'_{i-k+1,j} = B u_{j,k}$  and define a map  $F'' : \mathcal{X} \rightarrow \mathcal{U}$  by defining it in the above basis by  $F'' x_{j,l} := u_{j,l}$  and  $F''$  arbitrary on the extension. It can then be seen that for  $l \in \underline{i}$

$$(B.11) \quad \begin{aligned} \text{span} \{x_{1,l}, \dots, x_{r,l}\} &= (A_F + B F'')^{l-1} \mathcal{B}_{i+1}, \\ \hat{\mathcal{G}} &= \text{span} \{v_1, \dots, v_r\} = (A_F + B F'')^i \mathcal{B}_{i+1} \end{aligned}$$

and, for  $l=1, \dots, i-1$

$$(B.12) \quad B F'' (A_F + B F'')^{l-1} \mathcal{B}_{i+1} \subset \mathcal{B}'_{i+1-l} \subset \mathcal{B}_1.$$

Let  $\{\mathcal{B}''_l\}_{l=1}^i$  be a chain in  $\mathcal{B}$  such that  $\mathcal{B}_{i+1} \oplus \mathcal{B}''_i = \mathcal{B}_i$ . Since  $\mathcal{F}^{i+1} = \mathcal{F}^i \oplus \hat{\mathcal{G}}$ , by (B.1) and (B.11) we obtain

$$\begin{aligned} \mathcal{F}^{i+1} &= \sum_{l=1}^i A_F^{l-1} \mathcal{B}_l + (A_F + B F'')^i \mathcal{B}_{i+1} \\ &= \sum_{l=1}^i A_F^{l-1} \mathcal{B}_{i+1} + (A_F + B F'')^i \mathcal{B}_{i+1} + \sum_{l=1}^i A_F^{l-1} \mathcal{B}''_l. \end{aligned}$$

Thus, by (B.12):

$$(B.13) \quad \mathcal{F}^{i+1} = \sum_{l=1}^{i+1} (A_F + B F'')^{l-1} \mathcal{B}_{i+1} + \sum_{l=1}^i A_F^{l-1} \mathcal{B}''_l.$$

We contend that *all* sums in (B.13) are, in fact, direct sums. To prove this, assume the contrary. Then the following strict inequality must hold

$$\begin{aligned} \dim \mathcal{F}^{i+1} &< \sum_{l=1}^{i+1} \dim (A_F + B F'')^{l-1} \mathcal{B}_{i+1} + \sum_{l=1}^i \dim A_F^{l-1} \mathcal{B}''_l \\ &\equiv \sum_{l=1}^i \dim \mathcal{B}_{i+1} + \dim \hat{\mathcal{G}} + \sum_{l=1}^i \dim \mathcal{B}''_l = \sum_{l=1}^i \dim \mathcal{B}_l + \dim \hat{\mathcal{G}}, \end{aligned}$$

where the last equality follows from the fact that  $\mathcal{B}_{i+1} \oplus \mathcal{B}''_i = \mathcal{B}_i$ . On the other hand, however,  $\dim \mathcal{F}^{i+1} = \dim \mathcal{F}^i + \dim \hat{\mathcal{G}}$ , which by (B.3) and (B.1) equals  $\sum_{l=1}^i \dim \mathcal{B}_l + \dim \hat{\mathcal{G}}$ . Hence we obtain a contradiction. It follows that

$$(B.14) \quad \mathcal{F}^{i+1} = \bigoplus_{l=1}^{i+1} (A_F + B F'')^{l-1} \mathcal{B}_{i+1} \oplus \bigoplus_{l=1}^i A_F^{l-1} \mathcal{B}''_l.$$

Define  $\mathcal{V} := \bigoplus_{i=1}^i (A_F + BF^n)^{i-1} \mathcal{B}_{i+1}$  and  $\mathcal{W} := \bigoplus_{i=1}^i A_F^{i-1} \mathcal{B}_i''$ . By (B.14) we have that  $\mathcal{V} \cap \mathcal{W} = \{0\}$ . Decompose  $\mathcal{X} = \mathcal{V} \oplus \mathcal{W} \oplus \mathcal{R}$ , where  $\mathcal{R}$  is arbitrary. In this decomposition, define  $F_{\text{new}}: \mathcal{X} \rightarrow \mathcal{U}$  as follows

$$(B.15) \quad F_{\text{new}}|_{\mathcal{V}} := (F + F^n)|_{\mathcal{V}}, \quad F_{\text{new}}|_{\mathcal{W}} := F|_{\mathcal{W}}$$

and  $F_{\text{new}}$  arbitrary on  $\mathcal{R}$ . It can now be seen immediately from (B.14) that

$$(B.16) \quad \mathcal{F}^{i+1} = \bigoplus_{l=1}^{i+1} (A + BF_{\text{new}})^{l-1} \mathcal{B}_{i+1} \oplus \bigoplus_{l=1}^i (A + BF_{\text{new}})^{l-1} \mathcal{B}_l'' = \bigoplus_{l=1}^{i+1} (A + BF_{\text{new}})^{l-1} \mathcal{B}_l$$

This already verifies (B.1). Next, we will verify (B.3).

It is claimed that for  $l = 1, 2, \dots, i + 1$ ,  $\dim (A + BF_{\text{new}})^{l-1} \mathcal{B}_l = \dim \mathcal{B}_l$ . Suppose the contrary. Then we have

$$\begin{aligned} \sum_{l=1}^i \dim \mathcal{B}_l + \dim \hat{\mathcal{G}} &= \dim \mathcal{F}^{i+1} = \sum_{l=1}^{i+1} \dim (A + BF_{\text{new}})^{l-1} \mathcal{B}_l \\ &= \sum_{l=1}^i \dim (A + BF_{\text{new}})^{l-1} \mathcal{B}_l + \dim \hat{\mathcal{G}} \\ &< \sum_{l=1}^i \dim \mathcal{B}_l + \dim \hat{\mathcal{G}}, \end{aligned}$$

which is a contradiction. Equation (B.3) then follows immediately by noting that  $\dim (A + BF_{\text{new}})^i \mathcal{B}_{i+1} = \dim \hat{\mathcal{G}} = \dim [\mathcal{F}^{i+1} / \mathcal{F}^i]$ . Finally, it can be verified using (B.2), (B.8), (B.11) and (B.15) that  $\bigoplus_{l=2}^{i+1} (A + BF_{\text{new}})^{l-2} \mathcal{B}_l \subset \mathcal{H}_2$ .  $\square$

*Remark B.2.* Note that the proof of the above lemma is straightforward but notationally rather involved. An alternative proof could be given using the concept of train basis, see [14].

The above lemma is needed in the following:

*Proof of Lemma 6.5.* By Lemma B.1, there is a chain  $\{\mathcal{B}_i\}_{i=1}^n$  in  $\mathcal{B}$  and a map  $F: \mathcal{X} \rightarrow \mathcal{U}$  such that  $\mathcal{F}^n = \bigoplus_{i=1}^n A_F^{i-1} \mathcal{B}_i$  and such that (6.12) holds. Since  $\mathcal{F}^n \subset \mathcal{H}_1$ , also (6.11) holds. By (B.3), to prove (6.13) it is sufficient to show that, for  $i \in \mathbb{n}$ ,  $\dim A_F^{i-1} \mathcal{B}_i = \dim A_F^i \mathcal{B}_i$ . Suppose the contrary, i.e., suppose that  $\dim A_F^{i-1} \mathcal{B}_i > \dim A_F^i \mathcal{B}_i$  for some  $i$ . Then there is a vector  $v \neq 0$  in  $A_F^{i-1} \mathcal{B}_i$  such that  $A_F v = 0$ . Since a subspace  $\mathcal{V} \subset \mathcal{X}$  is controlled invariant iff  $A_F \mathcal{V} \subset \mathcal{V} + \mathcal{B}$  [22, Lemma 4.2] it follows that  $\text{span}\{v\}$  is controlled invariant. Since also  $v \in \mathcal{H}_1$ ,  $v$  must be contained in  $\mathcal{V}^*(\mathcal{H}_1)$ , the largest controlled invariant subspace in  $\mathcal{H}_1$ . On the other hand,  $v \in \mathcal{F}^n = \mathcal{R}_b^*(\mathcal{H}_2) \cap \mathcal{H}_1 \subset \mathcal{R}_b^*(\mathcal{H}_1) \cap \mathcal{H}_1$ . By Lemma 2.2 it follows that  $b \in \mathcal{R}_a^*(\mathcal{H}_1)$ . Thus,  $v \in \mathcal{R}_a^*(\mathcal{H}_1) \cap \mathcal{V}^*(\mathcal{H}_1)$ , which by Lemma 2.2 contradicts the assumption that  $\mathcal{R}^*(\mathcal{H}_1) = \{0\}$ . Thus, we have proved formula (6.13).

Finally, to prove (6.10), note from (6.9) that  $\mathcal{R}_b(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{B} + A_F \mathcal{F}^n = \mathcal{B} + A_F \mathcal{B}_1 + \dots + A_F^n \mathcal{B}_n$ . It will be shown that this is, in fact, a direct sum. Suppose it is not. Then there are vectors  $b_i \in \mathcal{B}_i$  and  $b_0 \in \mathcal{B}$ , not all zero, such that  $\sum_{i=0}^n A_F^i b_i = 0$ . Define  $w := \sum_{i=1}^n A_F^{i-1} b_i$ . Then we have  $A_F w = -b_0 \in \mathcal{B}$ . Since also  $w \in \mathcal{H}_1$ ,  $w$  must be contained in  $\mathcal{V}^*(\mathcal{H}_1)$ . On the other hand,  $w \in \mathcal{R}_a^*(\mathcal{H}_1)$ , and it follows as above that  $w = 0$ . Hence,  $b_0 = 0$ . Repeating this argument with  $w = \sum_{i=1}^n A_F^{i-1} b_i$  replaced by  $w = \sum_{i=2}^n A_F^{i-2} b_i$  then yields  $A_F w = -b_1$ , and thus  $b_1 = 0$ , etc. In this way we find  $b_i = 0$  ( $i \in \mathbb{n}$ ), which is a contradiction.  $\square$

**Appendix C.** This appendix will be devoted to a proof of Lemma 7.4. The proof will be given through a series of smaller lemmas. For  $\varepsilon > 0$ , let  $x_i(\varepsilon)$ ,  $i \in \underline{k}$ , a subspace  $\mathcal{L}_\varepsilon$  and a map  $F_\varepsilon: \mathcal{L}_\varepsilon \rightarrow \mathcal{U}$  be given by (7.3) to (7.5). Recall from Lemma 7.3 that a

matrix of the map  $(A_F + BF_\varepsilon)|_{\mathcal{L}_\varepsilon}$  with respect to the basis  $X_\varepsilon := \{x_1(\varepsilon), \dots, x_k(\varepsilon)\}$  is given by (7.6). Now, let  $D_\varepsilon: \mathcal{L}_\varepsilon \rightarrow \mathcal{L}_\varepsilon$  be the linear map with matrix  $\text{diag}(-1/\varepsilon, \dots, -1/\varepsilon)$  with respect to  $X_\varepsilon$ . Define a nilpotent map  $N_\varepsilon: \mathcal{L}_\varepsilon \rightarrow \mathcal{L}_\varepsilon$  by  $N_\varepsilon := (A_F + BF_\varepsilon)|_{\mathcal{L}_\varepsilon} - D_\varepsilon$ . Obviously, the matrix of  $N_\varepsilon$  with respect to  $X_\varepsilon$  is given by

$$(C.1) \quad \text{mat } N_\varepsilon = \begin{pmatrix} 0 & -\varepsilon^{-2} & \cdots & -\varepsilon^{-k} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -\varepsilon^{-2} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The following lemma is then immediate:

LEMMA C.1. *Let  $i \in \underline{k}$ . Then for  $j = i, i + 1, \dots, k$  we have  $N_\varepsilon^j x_i(\varepsilon) = 0$ . On the other hand, for  $j = 1, 2, \dots, i - 1$  the following holds*

$$(C.2) \quad N_\varepsilon^j x_i(\varepsilon) = \sum_{l_1=1}^{i-1} \sum_{l_2=1}^{l_1-1} \cdots \sum_{l_j=1}^{l_{j-1}-1} (-1)^j \varepsilon^{l_j-i-j} x_{l_j}(\varepsilon).$$

(For consistency, define  $l_0 := i$ .)

*Proof.* Use (C.1) to obtain an expression for  $N_\varepsilon x_i(\varepsilon)$ . Apply  $N_\varepsilon$  to the result, etc.  $\square$

Another technical ingredient we will need in our proof is:

LEMMA C.2. *Let  $i \in \underline{k}$ . Then we have*

$$(C.3) \quad x_i(\varepsilon) = A_F^{i-1} b - \varepsilon \sum_{l=1}^i A_F^{i-l+1} x_l(\varepsilon).$$

*Proof.* This follows immediately from (7.3), using induction.  $\square$

Finally, we will need the following result:

LEMMA C.3. *Under the assumptions of Lemma 7.4, the following holds for all  $i \in \underline{k}$ :  $H_1 x_i(\varepsilon) = O(\varepsilon^{k-i})$  and  $H_2 x_i(\varepsilon) = O(\varepsilon^{k-i-1})$ .*

*Proof.* By iterating formula (C.3), we obtain

$$(C.4) \quad \begin{aligned} x_i(\varepsilon) &= A_F^{i-1} b - \varepsilon \sum_{l_1=1}^i A_F^i b + \varepsilon^2 \sum_{l_1=1}^i \sum_{l_2=1}^{l_1} A_F^{i+1} b + \cdots \\ &+ (-\varepsilon)^{k-i-1} \left( \sum_{l_1=1}^i \sum_{l_2=1}^{l_1} \cdots \sum_{l_{k-i-1}=1}^{l_{k-i-2}} A_F^{k-2} b \right) \\ &+ (-\varepsilon)^{k-i} \left( \sum_{l_1=1}^i \sum_{l_2=1}^{l_1} \cdots \sum_{l_{k-i}=1}^{l_{k-i-1}} A_F^{k-l_{k-i}} x_{l_{k-i}}(\varepsilon) \right) \end{aligned}$$

(assume that  $1 \leq i \leq k - 2$ ). Under the assumptions of Lemma 7.4 we have  $A_F^{i-1} b, \dots, A_F^{k-3} b \in \mathcal{X}_2 \subset \mathcal{X}_1$  and  $A_F^{k-2} b \in \mathcal{X}_1$ . Thus, in (C.4) all terms but the last are in  $\mathcal{X}_1$  and all terms but the last two are in  $\mathcal{X}_2$ . It follows then that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{i-k} \cdot H_1 x_i(\varepsilon)$  exists and that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{i+1-k} \cdot H_2 x_i(\varepsilon)$  exists.

For  $i = k - 1$ , the existence of the former limit follows again from (C.4), while the existence of the latter is obvious. For  $i = k$ , the existence of both limits is obvious.  $\square$

*Proof of Lemma 7.4.* By the nilpotency of  $N_\varepsilon$ , note that for  $i \in \underline{k}$

$$e^{(A_F + BF_\varepsilon)t} x_i(\varepsilon) = e^{N_\varepsilon t} e^{D_\varepsilon t} x_i(\varepsilon) = \left( \sum_{j=0}^{k-1} \frac{t^j N_\varepsilon^j}{j!} \right) e^{-(1/\varepsilon)t} x_i(\varepsilon).$$

By the triangle inequality it therefore suffices to prove the following: for  $j = 0, 1, \dots, k - 1$ , the sequence  $\|t^j e^{-(1/\varepsilon)t} H_\alpha N_\varepsilon^j x_i(\varepsilon)\|_{L_1}$  tends to 0 as  $\varepsilon \rightarrow 0$  for  $\alpha = 1$  and is uniformly bounded with respect to  $\varepsilon$  if  $\alpha = 2$ . Since  $\int_0^\infty t^j e^{-(1/\varepsilon)t} dt = j! \varepsilon^{j+1}$ , it suffices to prove this asymptotic behaviour for  $\|\varepsilon^{j+1} H_\alpha N_\varepsilon^j x_i(\varepsilon)\|$  (Euclidean norm!). Apply now Lemma

C.1 to obtain a representation of  $N_\varepsilon^j x_i(\varepsilon)$ . Again by the triangular inequality, it is then sufficient to prove that  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{j+1} H_\alpha \varepsilon^{l-i-j} x_l(\varepsilon)$  is 0 for  $\alpha = 1$  and exists for  $\alpha = 2$ . ( $l$  is some index ranging between 1 and  $k$ .) Now, by Lemma C.3  $H_1 x_l(\varepsilon) = O(\varepsilon^{k-l})$  and  $H_2 x_l(\varepsilon) = O(\varepsilon^{k-l-1})$ . Thus, indeed  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{l-i-1} H_1 x_l(\varepsilon) = 0$  for all  $l \in \underline{k}$  and  $i \in \underline{k}$  and  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{l-i-1} H_2 x_l(\varepsilon)$  exists for all  $l \in \underline{k}$  and  $i \in \underline{k}$ . This completes the proof of Lemma 7.4.  $\square$

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