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Almost Everywhere Convergence of a Subsequence of the Logarithmic Means of Vilenkin-Fourier Series

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Abstract: The main aim of this paper is to prove that the maximal operator of a subsequence of the (one-dimensional) logarithmic means of Vilenkin-Fourier series is of weak type $(1,1)$. Moreover, we prove that the maximal operator of the logarithmic means of quadratical partial sums of double Vilenkin-Fourier series is of weak type $(1,1)$, provided that the supremum in the maximal operator is taken over special indices. The set of Vilenkin polynomials is dense in L^1, so by the well-known density argument the logarithmic means $t_{2^n}(f)$ converge a.e. to f for all integrable function f.

Keywords: Vilenkin group, Vilenkin system, double Vilenkin-Fourier series, logarithmic means, a.e. convergence.

1 Introduction

THE n-th Riesz's logarithmic means of a Fourier series is defined by

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k},$$

where $l_n := \sum_{k=1}^{n-1}\frac{1}{k}$. The Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors, e.g. Szász [1] and Yabuta [2], with respect to Walsh, Vilenkin system by Simon [3] and Gát [4].

Let $\{q_k : k \geq 1\}$ be a sequence of nonnegative numbers, the n-th Nörlund means of an integrable function f is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n-1}q_{n-k}S_k(f),$$

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where $Q_n := \sum_{k=1}^{n-1} q_k$. This Nörlund means of Walsh-Fourier series was investigated by Móricz and Siddiqi [5]. The case, when $q_k = \frac{1}{k}$ is excluded, since the method of Móricz and Siddiqi do not work in this case.

If $q_k := \frac{1}{k}$, then we get the (Nörlund) logarithmic means:

$$t_n(f) := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f)}{n-k}.$$

From now, we will write simply logarithmic means $t_n(f)$. Recently, for the Walsh system Gát and Goginava [6] proved some convergence and divergence properties of this logarithmic means of functions in the class of continuous functions, and in the Lebesgue space. They proved that the maximal norm convergence function space of this logarithmic means is $L\log^+ L$.

The a.e. convergence of a subsequence of logarithmic means of Walsh-Fourier series of integrable functions was discussed by Gát and Goginava [7, 8]. We will generalize the results of Gát and Goginava for Vilenkin systems.

More results on this logarithmic means with respect to unbounded Vilenkin system can be found in [9].

First, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were defined by N.Ja. Vilenkin in 1947 [10, 11] as follows.

Let $\mathbf{P}$ denote the set of positiv integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. Let $m = (m_0, m_1, ..., m_k, ...)$ $(2 \le m_k \in \mathbf{N}, k \in \mathbf{N})$ be a sequence of natural numbers and denote by $\mathbf{Z}_{m_k}$ the m_k-th cyclic group $(k \in \mathbf{N})$. That is $\mathbf{Z}_{m_k}$ can be represented by the set $\{0, 1, ..., m_k - 1\}$,where the group operation is the mod m_k addition and every subset is open. Haar measure on $\mathbf{Z}_{m_k}$ is given in the way that $\mu_k(\{j\}) := \frac{1}{m_k}$ $(j \in \mathbf{Z}_{m_k}, k \in \mathbf{N})$. Let G_m be the complete direct product of the compact groups $\mathbf{Z}_{m_k}$ $(k \in \mathbf{N})$. G_m is a compact Abelian group and called Vilenkin group. The elements of G_m are of the form $x = (x_0, x_1, ..., x_k, ...)$ with $0 \le x_k < m_k$ $(k \in \mathbf{N})$. The group operation on G_m is the coordinate-wise addition, the normalised Haar measure μ is the product measure. The topology on G_m is the product topology, a base for which can be given in the following way:

$$I_0(x) := G_m,\ I_n(x) := \{y \in G_m : y = (x_0, ..., x_{n-1}, y_n, ...)\}\ (x \in G_m, n \in \mathbf{P}),$$

$I_n := I_n(0)$ $(n \in \mathbf{N})$. Furthermore, let $L^p(G_m)$ denote the usual Lebesgue spaces on G_m (with the corresponding norm $\|.\|_p$), $\mathscr{A}_n$ the σ-algebra generated by the sets $I_n(x) (x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathscr{A}_n (n \in \mathbf{N})$.

Let $\Gamma(m) := \{\psi_n : n \in \mathbf{N}\}$ denote the character group of G_m. We enumerate the elements of $\Gamma(m)$ as follows. For $k \in \mathbf{N}$ and $x \in G_m$ denote r_k the k-th generalized

Rademacher function:

$$r_k(x) := \exp\left(\frac{2\pi i x_k}{m_k}\right) \ (i := \sqrt{-1}).$$

If we define the sequence $(M_k : k \in \mathbf{N})$ by $M_0 := 1$ and $M_k := m_0 m_1 \ldots m_{k-1}$ $(k \in \mathbf{P})$ then each $n \in \mathbf{N}$ has a uniqe representation of the form $n = \sum_{k=0}^{\infty} n_k M_k$, where $0 \le n_k < m_k$ $(n_k \in \mathbf{N})$. Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. That is, $M_{|n|} \le n < M_{|n|+1}$.

Now, we define the Vilenkin functions ψ_n by

$$\psi_n := \prod_{k=0}^{\infty} (r_k)^{n_k}.$$

We remark that $\Gamma(m)$ is a complete orthonormal system related to the normalized Haar measure on G_m.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, the Fejér kernels, the logarithmic means and logarithmic kernels by

$$\hat{f}^{\psi}(n) := \int_{G_m} f\bar{\psi}_n, \ S_n f := \sum_{k=0}^{n-1} \hat{f}^{\psi}(k)\psi_k, \ \ D_n := \sum_{k=0}^{n-1} \psi_k,$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n} S_k f, \ \ K_n := \frac{1}{n}\sum_{k=0}^{n} D_k,$$

$$t_n(f) := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{S_k f}{n-k}, \ \ F_n := \frac{1}{l_n}\sum_{k=1}^{n-1} \frac{D_k}{n-k},$$

where $n \in \mathbf{P}$ and $D_0 := 0, K_0 := 0$.

It is known [11, 12] that

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

and $E_n f = S_{M_n} f$ $(n \in \mathbf{N})$.

Next, we introduce some notation with respect to the theory of two-dimensional system. Let the two-dimensional Vilenkin group be $G_m \times G_m$ and the two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, Dirichlet kernels, the Marcinkiewicz means and Marcinkiewicz kernels be defined as:

$$\hat{f}^{\psi}(n_1,n_2) := \int_{G_m\times G_m} f\psi_{n_1}\psi_{n_2}d\mu,$$

$$S_{n_1,n_2}f(x^1,x^2) := \sum_{k=0}^{n_1-1}\sum_{l=0}^{n_2-1}\hat{f}^{\psi}(k,l)\psi_k(x^1)\psi_l(x^2),$$

$$D_{n_1,n_2}(x^1,x^2) := D_{n_1}(x^1)D_{n_2}(x^2),$$

$$\mathscr{M}_n f := \frac{1}{n}\sum_{k=0}^{n} S_{k,k}f, \;\; K_n := \frac{1}{n}\sum_{k=0}^{n} D_{k,k}$$

where $n \in \mathbf{P}$.

The two-dimensional logarithmic means and kernels of quadratical partial sums are defined by

$$t_n(f) := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_{k,k}f}{n-k}, \;\; F_n := \frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_{k,k}}{n-k}.$$

Let $\mathscr{A}_{n,n}$ denote the σ-algebra generated by the sets $I_n(x)\times I_n(y)$ $(x,y\in G_m)$ and $E_{n,n}$ the conditional expectation operator with respect to $\mathscr{A}_{n,n}$ $(n\in\mathbf{N})$.

For two-dimensional variable $(x,y)\in G_m\times G_m$ we use the notations

$$\psi_n^1(x,y) = \psi_n(x), \;\; D_n^1(x,y) = D_n(x), \;\; K_n^1(x,y) = K_n(x),$$

$$\psi_n^2(x,y) = \psi_n(y), \;\; D_n^2(x,y) = D_n(y), \;\; K_n^2(x,y) = K_n(y),$$

for any $n\in\mathbf{N}$. From now, let the sequence m be bounded.

2 The a.e. Convergence of a Subsequence of One-Variable Logarithmic Means

Theorem 1 *Let $\{n^k : k\geq 1\}$ be a sequence of positive integers wich satisfies*

$$\sum_{k=1}^{\infty}\frac{\log^2(n^k - n^k_{|n^k|}M_{|n^k|}+1)}{\log n^k} < \infty,$$

where $n^k = \sum_{j=0}^{\infty} n_j^k M_j$. Then the operator $t^*(f) := \sup_{k\geq 1}|t_{n^k}(f)|$ is of weak type $(1,1)$.

Analogue of this result on Walsh-Fourier logarithmic means was given by Goginava [7].

Corollary 1 *Let $\{n^k : k \geq 1\}$ be a sequence of positive integers which satisfies the condition of Theorem 1 and let $f \in L^1(G_m)$, then*

$$t_{n^k}(f,x) \to f(x) \ \text{a.e. as } k \to \infty.$$

Corollary 2 *Let $f \in L^1(G_m)$, then*

$$t_{M_n}(f,x) \to f(x) \ \text{a.e. as } n \to \infty.$$

The basis of the proof of Theorem 1 are the following lemmas.

Lemma 1 *Let $M_A \leq n < M_{A+1}$, then*

$$\begin{aligned} l_n F_n(x) &= l_n D_{n_A M_A}(x) \\ &- \psi_{n_A M_A - 1}(x) \sum_{j=1}^{M_A - 2} \left(\frac{1}{n - n_A M_A + j} - \frac{1}{n - n_A M_A + j + 1} \right) j \overline{K}_j(x) \\ &- \psi_{n_A M_A - 1}(x) \frac{n_A M_A - 1}{n-1} \overline{K}_{n_A M_A - 1}(x) \\ &+ \psi_{n_A M_A}(x) l_{n - n_A M_A} F_{n - n_A M_A}(x). \end{aligned}$$

Proof 1 *During the proof of Lemma 1 we will use the following equation in [11]:*

$$D_{n_A M_A + j} = D_{n_A M_A} + \psi_{n_A M_A} D_j \tag{2}$$

and the equation in [13]

$$D_{n_A M_A - j} = D_{n_A M_A} - \psi_{n_A M_A - 1} \overline{D}_j, \tag{3}$$

for $0 \leq j < n_A M_A$ and $0 \leq n_A < m_A$.

Let $|n| = A$, then

$$l_n F_n(x) = \sum_{j=1}^{n_A M_A} \frac{D_j(x)}{n-j} + \sum_{j=n_A M_A + 1}^{n-1} \frac{D_j(x)}{n-j} =: I + II.$$

First, we discuss II by the help of (2).

$$\begin{aligned} II = &\sum_{j=1}^{n - n_A M_A - 1} \frac{D_{n_A M_A + j}(x)}{n - n_A M_A - j} \\ &= l_{n - n_A M_A} D_{n_A M_A}(x) + \psi_{n_A M_A}(x) \sum_{j=1}^{n - n_A M_A - 1} \frac{D_j(x)}{n - n_A M_A - j} \\ &= l_{n - n_A M_A} D_{n_A M_A}(x) + \psi_{n_A M_A}(x) l_{n - n_A M_A} F_{n - n_A M_A}(x). \end{aligned}$$

By the help of (3) and Abel's transformation we investigate I.

$$\begin{aligned}
I = &\sum_{j=0}^{n_AM_A-1} \frac{D_{n_AM_A-j}(x)}{n-n_AM_A+j}\\
=&\frac{D_{n_AM_A}(x)}{n-n_AM_A} + \sum_{j=1}^{n_AM_A-1} \frac{D_{n_AM_A-j}(x)}{n-n_AM_A+j}\\
=&(l_n - l_{n-n_AM_A})D_{n_AM_A}(x)\\
&-\psi_{n_AM_A-1}(x)\sum_{j=1}^{n_AM_A-2}\left(\frac{1}{n-n_AM_A+j} - \frac{1}{n-n_AM_A+j+1}\right) j\overline{K}_j(x)\\
&-\psi_{n_AM_A-1}(x)\frac{n_AM_A-1}{n-1}\overline{K}_{n_AM_A-1}(x).
\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2 *Let* $\overline{\lim\limits_{k\to\infty}} \frac{\log^2(n^k - n^k_{|n^k|}M_{|n^k|}+1)}{\log n^k} < \infty$, *then*

$$\|F_{n^k}\|_1 \le c < \infty, \quad k = 1,2,\ldots$$

Proof 2 *In [14] we have*

$$\|K_n\|_1 = O(1) \text{ as } n \to \infty. \tag{4}$$

Moreover,

$$\|D_n\|_1 = O(\log n) \text{ as } n \to \infty.$$

(see [10]). These give that

$$\|F_n\|_1 \le \frac{1}{l_n}\sum_{j=1}^{n-1}\frac{\|D_j\|_1}{n-j} \le \frac{c}{l_n}\sum_{j=1}^{n-1}\frac{\ln j}{n-j} = O(l_n).$$

Using Lemma 1, we immediately have

$$\begin{aligned}
\|F_{n^k}\|_1 &\le c + \frac{1}{l_{n^k}}\sum_{j=1}^{n^k_{|n^k|}M_{|n^k|}-2}\frac{\|K_j\|_1}{j} + \frac{1}{l_{n^k}}\|K_{n^k_{|n^k|}M_{|n^k|}-1}\|_1\\
&+ \frac{l_{n^k - n^k_{|n^k|}M_{|n^k|}}}{l_{n^k}}\|F_{n^k - n^k_{|n^k|}M_{|n^k|}}\|_1\\
&= O\left(\frac{\log^2(n^k - n^k_{|n^k|}M_{|n^k|}+1)}{\log n^k}\right) = O(1).
\end{aligned}$$

This completes the proof of Lemma 2.

Proof of Theorem 1: The maximal function $f^* := \sup_{n\in\mathbf{N}} |f * D_{M_n}|$ is of weak type $(1,1)$ [15]. Define the operator T by

$$T f := \sup_{\substack{n,A\in\mathbf{N}\\ |n|\le A}} |f * \overline{K}_n \psi_{n_A M_A - 1}|.$$

In the paper [13] Gát and Goginava proved that

$$\int_{\overline{I}_k} \sup_{n\ge M_k} |K_n| \le c. \tag{5}$$

(4),(5) and the definition

$$T f \le \sup_{n\in\mathbf{N}} |f| * |K_n| =: G f$$

give by standard argument that the operators T, G are of weak type $(1,1)$. At last, let $f \in L^1(G_m)$, supp $f \subset I_l$ and $\int_{I_l} f = 0$. Set $n(l) := \min\{j : |n^j| \ge l\}$.

If $k < n(l)$ then

$$t_{n^k}(f,x) = \int_{G_m} f(y) F_{n^k}(x-y) d\mu(y) = F_{n^k}(x) \int_{G_m} f(y) d\mu(y) = 0.$$

Consequently, set $k \ge n(l)$.

Define the operator N by

$$N f := \sup_{n\ge 1} |f * \psi_{n^k_{|n^k|} M_{|n^k|}} \frac{l_{n^k - n^k_{|n^k|} M_{|n^k|}}}{l_{n^k}} F_{n^k - n^k_{|n^k|} M_{|n^k|}}|.$$

We have that

$$\begin{aligned}
&\int_{\overline{I}_l} \sup_{k\ge n(l)} \frac{l_{n^k - n^k_{|n^k|} M_{|n^k|}}}{l_{n^k}} |F_{n^k - n^k_{|n^k|} M_{|n^k|}}| \\
&\le \sum_{k=1}^{\infty} \frac{\log(n^k - n^k_{|n^k|} M_{|n^k|} + 1)}{\log n^k} \|F_{n^k - n^k_{|n^k|} M_{|n^k|}}\|_1 \\
&\le \sum_{k=1}^{\infty} \frac{\log^2(n^k - n^k_{|n^k|} M_{|n^k|} + 1)}{\log n^k} \le c.
\end{aligned}$$

This implies

$$\begin{aligned}
\int_{\overline{I}_l} N f(x) d\mu(x) &\le \int_{I_l} |f(y)| \left(\int_{\overline{I}_l} \sup_{k\ge n(l)} |F_{n^k - n^k_{|n^k|} M_{|n^k|}}(x-y)| d\mu(x) \right) d\mu(y) \\
&\le c\|f\|_1.
\end{aligned}$$

From Lemma 2 the operator N and t^* is of type (∞,∞). The operator N is sublinear and quasi-local, this gives by standard argument [16] that the operator N is of weak type $(1,1)$.

Lemma 1 and

$$\begin{aligned} t^*(f) &\leq cf^* + \sup_{k\geq 1} \frac{1}{l_{n^k}} \sum_{j=1}^{n^k_{|n^k|} M_{|n^k|}-2} \frac{Tf}{j} + cTf + Nf \\ &\leq cf^* + cTf + cNf \end{aligned}$$

complete the proof of Theorem 1. □

Corollary 3 *The operator t^* is of type (p,p) for all $1 < p \leq \infty$.*

3 The a.e. Convergence of a Subsequence of Logarithmic Means of Quadratical Partial Sums

Define the two-dimensional maximal operator $t_\natural$ by

$$t_\natural f(x^1,x^2) := \sup_{n\in\mathbf{P}} |t_{M_n}(f,x^1,x^2)|.$$

During the proof of Theorem 2 we will use that the two-dimensional maximal function $f^* := \sup_{n\in\mathbf{P}} |f * (D^1_{M_n} D^2_{M_n})|$ is of weak type $(1,1)$ and of type (p,p) for all $1 < p \leq \infty$ [15].

Theorem 2 *The operator $t_\natural$ is of weak type $(1,1)$ and of type (p,p) for all $1 < p \leq \infty$.*

By standard argument we have

Corollary 4 *Let $f \in L^1(G_m \times G_m)$, then*

$$t_{M_n}(f,x^1,x^2) \to f(x^1,x^2) \textit{ a.e. as } n \to \infty.$$

The analogue of this result with respect to Walsh-Fourier logarithmic means was given by Gát and Goginava [8]. To proove Theorem 2 we need the following Calderon-Zygmund decomposition lemma [17].

Lemma 3 *(Calderon-Zygmund decomposition [17]) Let* $f \in L_1(G_m \times G_m)$, $\lambda > \|f\|_1$. *Then there exists* $(u^{(i,1)}, u^{(i,2)}) \in G_m \times G_m$, $k_i \in \mathbf{N}\,(i = 1,2,\ldots,)$ *and a decomposition*

$$f = f_0 + \sum_{i=1}^{\infty} f_i,$$

where

1) $\|f_0\|_\infty \le c\lambda$, $\|f_0\|_1 \le c\|f\|_1$;
2) *supp* $f_i \subset I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)$, $\int\limits_{G_m \times G_m} f_i = 0$, $i = 1,2,\ldots$;
3) $\mu\left(\bigcup\limits_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right)\right) \le c\|f\|_1/\lambda$.

Proof of Theorem 2: First, we decompose the M_n-th logarithmic kernels. by the help of (3)

$$\begin{aligned}
l_{M_n}F_{M_n}(x^1,x^2) = &\sum_{j=1}^{M_n-1} \frac{D_{M_n-j}(x^1)D_{M_n-j}(x^2)}{j} \\
&= l_{M_n}D_{M_n}(x^1)D_{M_n}(x^2) - D_{M_n}(x^1)\psi_{M_n-1}(x^2)\sum_{j=1}^{M_n-1}\frac{\overline{D}_j(x^2)}{j} \\
&- D_{M_n}(x^2)\psi_{M_n-1}(x^1)\sum_{j=1}^{M_n-1}\frac{\overline{D}_j(x^1)}{j} \\
&+ \psi_{M_n-1}(x^1)\psi_{M_n-1}(x^2)\sum_{j=1}^{M_n-1}\frac{\overline{D}_j(x^1)\overline{D}_j(x^2)}{j} \\
&=: l_{M_n}(F^1_{M_n}(x^1,x^2) - F^2_{M_n}(x^1,x^2) - F^3_{M_n}(x^1,x^2) + F^4_{M_n}(x^1,x^2)).
\end{aligned}$$

Since $F^1_{M_n}(x^1,x^2) = D_{M_n}(x^1)D_{M_n}(x^2)$ we have

$$t^1_\natural f := \sup_{n\in\mathbf{P}} |f * F^1_{M_n}| = f^*.$$

To discuss $F^2_{M_n}$ we will use Abel's transformation ($F^3_{M_n}$ goes in the same way)

$$\sum_{j=1}^{M_n-1}\frac{\overline{D}_j}{j} = \sum_{j=1}^{M_n-2}\left(\frac{1}{j} - \frac{1}{j+1}\right)j\overline{K}_j + \overline{K}_{M_n-1} = \sum_{j=1}^{M_n-2}\frac{\overline{K}_j}{j+1} + \overline{K}_{M_n-1}.$$

These implies that

$$F^2_{M_n} = D^1_{M_n}\psi^2_{M_n-1}\sum_{j=1}^{M_n-2}\frac{\overline{K}^2_j}{j+1} + D^1_{M_n}\psi^2_{M_n-1}\overline{K}^2_{M_n-1} =: F^{2,1}_{M_n} + F^{2,2}_{M_n}.$$

Define the operators $t_{\natural}^{2,i}$ and $t_n^{2,i}$ for $i = 1,2$ by

$$t_{\natural}^{2,i} f := \sup_{n \in \mathbf{P}} |f * F_{M_n}^{2,i}|, \ \ t_n^{2,i} f := |f * F_{M_n}^{2,i}| \ (n \in \mathbf{P}).$$

From (1) and (4) we have that the operators $t_{\natural}^{2,i}$ $(i = 1,2)$ are of type (∞,∞). Now, we will discuss the operator $t_{\natural}^{2,1}$ ($t_{\natural}^{2,2}$ goes in the same way). We follow the method of Gát and Goginava [8].

Denote (use the notation of Lemma 3)

$$g\left(t^1,t^2\right) := \sum_{i=1}^{\infty} \frac{\left|f_i\left(t^1,t^2\right)\right|}{l_{M_{k_i}}},$$
$$L(t) := \sum_{i=1}^{\infty} \frac{|K_i(t)|}{i+1}.$$

Let

$$\left(y^1,y^2\right) \in \overline{\bigcup_{i=1}^{\infty} \left(I_{k_i}\left(u^{i,1}\right) \times I_{k_i}\left(u^{i,2}\right)\right)}.$$

Since $\int_{G_m} f_i = 0$ we have

$$t_n^{(2,1)} f_i\left(y^1,y^2\right) = 0 \tag{6}$$

for $n \leq k_i$.

Let $y^1 \in \overline{I_{k_i}\left(u^{i,1}\right)}$. Then from 1 we can write that $t_n^{(2,1)} f_i\left(y^1,y^2\right) = 0$ for $n > k_i$. Hence $t_n^{(2,1)} f_i\left(y^1,y^2\right) \neq 0$ implies that $y^1 \in I_{k_i}\left(u^{i,1}\right)$. Consequently, we can suppose that

$$y^2 \in \bigcap_{i=1}^{\infty} \overline{I_{k_i}\left(u^{i,2}\right)}.$$

Then we write

$$\begin{aligned} D &:= \mu\left\{\left(y^1,y^2\right) \in G_m \times \left(\bigcap_{i=1}^{\infty} \overline{I_{k_i}\left(u^{i,2}\right)}\right) : t_{\natural}^{(2,1)} f\left(y^1,y^2\right) > c\lambda\right\} \\ &\leq \int\limits_{\bigcap_{i=1}^{\infty} \overline{I_{k_i}\left(u^{i,2}\right)}} \mu\left\{y^1 \in G_m : t_{\natural}^{(2,1)}\left(\sum_{i=1}^{\infty} f_i\right)\left(y^1,y^2\right) > c\lambda\right\} d\mu\left(y^2\right). \end{aligned}$$

From (6), we have

$$
\begin{aligned}
&\left| t_n^{(2,1)} \left(\sum_{i=1}^{\infty} f_i \right) \left(y^1, y^2 \right) \right| \\
&\le \sum_{i=1}^{\infty} \int\limits_{I_{k_i}(u^{i,1}) \times I_{k_i}(u^{i,2})} \left| f_i \left(x^1, x^2 \right) \right| \frac{D_{M_n}\left(x^1 - y^1 \right) \left| \psi_{M_n - 1} \left(x^2 - y^2 \right) \right|}{l_{M_n}} \\
&\times \sum_{j=1}^{M_n - 2} \frac{\left| K_j \left(x^2 - y^2 \right) \right|}{j+1} d\mu \left(x^1, x^2 \right) \\
&\le \int\limits_{G_m} \left(\int\limits_{G_m} \sum_{i=1}^{\infty} \frac{\left| f_i \left(x^1, x^2 \right) \right|}{l_{M_{k_i}}} \sum_{j=1}^{M_n - 2} \frac{\left| K_j \left(x^2 - y^2 \right) \right|}{j+1} d\mu \left(x^2 \right) \right) D_{M_n} \left(x^1 - y^1 \right) d\mu \left(x^1 \right) \\
&= \int\limits_{G_m} \left(\int\limits_{G_m} g \left(x^1, x^2 \right) L \left(x^2 - y^2 \right) d\mu \left(x^2 \right) \right) D_{M_n} \left(x^1 - y^1 \right) d\mu \left(x^1 \right).
\end{aligned}
$$

Define the one-dimensional function h_{y^2} for every fixed $y^2 \in G_m$ by $h_{y^2}(x^1) := \int_{G_m} g(x^1, x^2)\, L(x^2 - y^2) d\mu(x^2)$. The one-dimensional operator $\sup_{n \in \mathbf{N}} |S_{M_n} f|$ is of weak type $(1,1)$. We apply this fact for the function $h_{y^2}^*$ Consequently, by the above we can write

$$
\begin{aligned}
D &\le \int\limits_{\overline{\bigcap_{i=1}^{\infty} I_{k_i}(u^{i,2})}} \mu \left\{ y^1 \in G_m : \sup_n \int\limits_{G_m} h_{y^2}(x^1) D_{M_n}\left(x^1 - y^1 \right) d\mu \left(x^1 \right) > c\lambda \right\} d\mu(y^2) \\
&\le \int\limits_{\overline{\bigcap_{i=1}^{\infty} I_{k_i}(u^{i,2})}} \mu \left\{ y^1 \in G_m : h_{y^2}^* > c\lambda \right\} d\mu(y^2) \\
&\le \frac{c}{\lambda} \int\limits_{\overline{\bigcap_{i=1}^{\infty} I_{k_i}(u^{i,2})}} \| h_{y^2} \|_1 d\mu \left(y^2 \right)
\end{aligned}
$$

Now, we investigate $\|h_{y^2}\|_1$ for a fixed $y^2 \in G_m$. By the theorem of Fubini

$$\begin{aligned}
\|h_{y^2}\|_1 &= \int_{G_m} \left(\int_{G_m} g(x^1,x^2) L(x^2-y^2) d\mu(x^2) \right) d\mu(x^1) \\
&= \int_{G_m} \left(\int_{G_m} g(x^1,x^2) d\mu(x^1) \right) L(x^2-y^2) d\mu(x^2) \\
&= \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \int_{I_{k_i}(u^{i,2})} \left(\int_{I_{k_i}(u^{i,1})} |f_i(x^1,x^2)| d\mu(x^1) \right) L(x^2-y^2) d\mu(x^2)
\end{aligned}$$

and

$$\begin{aligned}
D \leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \int_{\overline{I_{k_i}(u^{i,2})}} \left[\int_{I_{k_i}(u^{i,2})} \left(\int_{I_{k_i}(u^{i,1})} |f_i(x^1,x^2)| d\mu(x^1) \right) \right. \\
& \left. \times L(x^2-y^2) d\mu(x^2) \right] d\mu(y^2) \\
\leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \int_{\overline{I_{k_i}(u^{i,2})}} \left[\int_{I_{k_i}(u^{i,2})} \left(\int_{I_{k_i}(u^{i,1})} |f_i(x^1,x^2)| d\mu(x^1) \right) \right. \\
& \left. \times \sum_{j=1}^{M_{k_i}-1} \frac{|K_j(x^2-y^2)|}{j+1} d\mu(x^2) \right] d\mu(y^2) \\
& + \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \int_{\overline{I_{k_i}(u^{i,2})}} \left[\int_{I_{k_i}(u^{i,2})} \left(\int_{I_{k_i}(u^{i,1})} |f_i(x^1,x^2)| d\mu(x^1) \right) \right. \\
& \left. \times \sum_{j=M_{k_i}}^{\infty} \frac{|K_j(x^2-y^2)|}{j+1} d\mu(x^2) \right] d\mu(y^2) \\
& =: D^1 + D^2.
\end{aligned}$$

Since (4) we have

$$
\begin{aligned}
D^1 \leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \int\limits_{I_{k_i}(u^{i,2})} \left[\int\limits_{I_{k_i}(u^{i,1})} \left| f_i\left(x^1, x^2\right) \right| d\mu\left(x^1\right) \right. \\
& \times \left. \int\limits_{I_{k_i}(u^{i,2})} \sum_{j=1}^{M_{k_i}-1} \frac{\left| K_j\left(x^2 - y^2\right) \right|}{j+1} d\mu\left(y^2\right) \right] d\mu\left(x^2\right) \\
\leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \| f_i \|_1 \leq \frac{c}{\lambda} \| f \|_1 .
\end{aligned}
$$

In [13] it was proved for $A, k \in \mathbf{N}, A \geq k$ that

$$
\int_{\overline{I_k}} \sup_{n \geq M_A} |K_n(x)| d\mu(x) \leq c \frac{M_k(A-k+1)}{M_A}. \tag{7}
$$

Using (7) for D^2 we have

$$
\begin{aligned}
D^2 \leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \int\limits_{I_{k_i}(u^{i,2})} \left[\int\limits_{I_{k_i}(u^{i,1})} \left| f_i\left(x^1, x^2\right) \right| d\mu\left(x^1\right) \right. \\
& \times \left. \int\limits_{\overline{I_{k_i}(u^{i,2})}} \sum_{j=M_{k_i}}^{\infty} \frac{\left| K_j\left(x^2 - y^2\right) \right|}{j+1} d\mu\left(y^2\right) \right] d\mu\left(x^2\right) \\
\leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \int\limits_{I_{k_i}(u^{i,2})} \left[\int\limits_{I_{k_i}(u^{i,1})} \left| f_i\left(x^1, x^2\right) \right| d\mu\left(x^1\right) \sum_{r=k_i}^{\infty} \sum_{j=M_r}^{M_{r+1}-1} \frac{1}{j} \right. \\
& \times \left. \int\limits_{\overline{I_{k_i}(u^{i,2})}} \left| K_j\left(x^2 - y^2\right) \right| d\mu\left(y^2\right) \right] d\mu\left(x^2\right) \\
\leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \frac{1}{l_{M_{k_i}}} \left(\sum_{r=k_i}^{\infty} \frac{M_{k_i}(r-k_i+1)}{M_r} \right) \int\limits_{I_{k_i}(u^{i,2})} \int\limits_{I_{k_i}(u^{i,1})} \left| f_i\left(x^1, x^2\right) \right| d\mu\left(x^1, x^2\right) \\
\leq & \frac{c}{\lambda} \sum_{i=1}^{\infty} \| f_i \|_1 \leq \frac{c}{\lambda} \| f \|_1 .
\end{aligned}
$$

These imply

$$\mu\left\{\left(y^1,y^2\right)\in\overline{\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right)\times I_{k_i}\left(u^{i,2}\right)\right)}:t_\natural^{(2,1)}f\left(y^1,y^2\right)>c\lambda\right\}\le\frac{c}{\lambda}\|f\|_1.$$

From Lemma 3, we get

$$\begin{aligned}&\mu\left\{\left(y^1,y^2\right)\in\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right)\times I_{k_i}\left(u^{i,2}\right)\right):t_\natural^{(2,1)}f\left(y^1,y^2\right)>c\lambda\right\}\\ \le\;&\mu\left(\bigcup_{i=1}^{\infty}\left(I_{k_i}\left(u^{i,1}\right)\times I_{k_i}\left(u^{i,2}\right)\right)\right)\le\frac{c}{\lambda}\|f\|_1,\end{aligned}$$

and consequently the operator $t_\natural^{2,1}$ is of weak type $(1,1)$.

Now, we discuss $F^4_{M_n}$. Abel's transformation gives that

$$\sum_{j=1}^{M_n-1}\frac{\overline{D}_j(x^1)\overline{D}_j(x^2)}{j}=\sum_{j=1}^{M_n-2}\frac{\overline{K}_j(x^1,x^2)}{j+1}+\overline{K}_{M_n-1}(x^1,x^2)$$

and

$$F^4_{M_n}=\frac{1}{l_{M_n}}\psi^1_{M_n-1}\psi^2_{M_n-1}\left(\sum_{j=1}^{M_n-2}\frac{\overline{K}_j}{j+1}+\overline{K}_{M_n-1}\right).$$

Now, we will define the operator G,M by the following way

$$Gf:=\sup_{\substack{n,A\in\mathbf{P}\\|n|\le A}}|f*\psi^1_{M_n-1}\psi^2_{M_n-1}\overline{K}_n|\le\sup_{n\in\mathbf{P}}|f|*|K_n|=:Mf.$$

It was proved in [18] that

$$\int_{\overline{I_k\times I_k}}\sup_{n\ge M_k}|K_n|\le c \tag{8}$$

and in [19] that

$$\|K_n\|_1\le c \tag{9}$$

for all $n\in\mathbf{N}$. (9) imply that the operator M (and G) is of type (∞,∞). (8) gives immediately the quasi-locality of the operator M. These and that the operator M is sublinear give by standard argument [16] the operator M (and G) is of weak type $(1,1)$ and of type (p,p) for all $1<p\le\infty$.

$$t_\natural^4 f:=\sup_{n\in\mathbf{P}}|f*F^4_{M_n}|\le\sup_{n\in\mathbf{P}}\frac{1}{l_{M_n}}\sum_{j=1}^{M_n-2}\frac{1}{j+1}Mf+cMf\le cMf$$

imply the same properties of the operator $t_\natural^4$.

This completes the proof of Theorem 2. $\square$

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