ALMOST EXPONENTIAL DECAY OF PERIODIC VISCOUS SURFACE WAVES WITHOUT SURFACE TENSION

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Abstract. We consider a viscous fluid of finite depth below the air, occupying a three-dimensional domain bounded below by a fixed solid boundary and above by a free moving boundary. The fluid dynamics are governed by the gravity-driven incompressible Navier-Stokes equations, and the effect of surface tension is neglected on the free surface. The long time behavior of solutions near equilibrium has been an intriguing question since the work of Beale [3]. This paper is the third in a series of three [13, 14] that answers this question. Here we consider the case in which the free interface is horizontally periodic; we prove that the problem is globally well-posed and that solutions decay to equilibrium at an almost exponential rate. In particular, the free interface decays to a flat surface.

Our framework contains several novel techniques, which include: (1) optimal a priori estimates that utilize a "geometric" reformulation of the equations; (2) a two-tier energy method that couples the boundedness of high-order energy to the decay of low-order energy, the latter of which is necessary to balance out the growth of the highest derivatives of the free interface; (3) a localization procedure that is compatible with the energy method and allows for curved lower surface geometry. Our decay estimates lead to the construction of global-in-time solutions to the surface wave problem.

1. Introduction

1.1. Formulation of the equations in Eulerian coordinates.

We consider a viscous, incompressible fluid evolving in a moving domain

$$\Omega(t) = \{y \in \Sigma \times \mathbb{R} \mid -b(y_1, y_2) < y_3 < \eta(y_1, y_2, t)\}. \tag{1.1}$$

Here we assume that $\Omega(t)$ is horizontally periodic by setting $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the usual $1-$torus and $L_1, L_2 > 0$ the periodicity lengths. The lower boundary $0 < b \in C^\infty(\Sigma)$ is assumed to be fixed and given, but the upper boundary is a free surface that is the graph of the unknown function $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$. For each t, the fluid is described by its velocity and pressure functions $(u, p) : \Omega(t) \to \mathbb{R}^3 \times \mathbb{R}$. We require that (u, p, η) satisfy the gravity-driven incompressible Navier-Stokes equations in $\Omega(t)$ for $t > 0$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \\ \partial_t \eta = u_3 - u_1 \partial_{y_1} \eta - u_2 \partial_{y_2} \eta & \text{on } \{y_3 = \eta(y_1, y_2, t)\} \\ (pI - \mu \mathbb{D}(u))\nu = g\eta\nu & \text{on } \{y_3 = \eta(y_1, y_2, t)\} \\ u = 0 & \text{on } \{y_3 = -b(y_1, y_2)\} \end{cases} \tag{1.2}$$

for ν the outward-pointing unit normal on $\{y_3 = \eta\}$, I the 3×3 identity matrix, $(\mathbb{D}u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u, $g > 0$ the strength of gravity, and $\mu > 0$ the viscosity. The tensor $(pI - \mu\mathbb{D}(u))$ is known as the viscous stress tensor. The third equation in (1.2) implies that the free surface is advected with the fluid. Note that in (1.2) we have shifted the gravitational forcing to the boundary and eliminated the constant atmospheric pressure, p_{atm}, in the usual way by adjusting the actual pressure $\bar{p}$ according to $p = \bar{p} + gy_3 - p_{atm}$.

The problem is augmented with initial data (u_0, η_0) satisfying certain compatibility conditions, which for brevity we will not write now. We will assume that $\eta_0 > -b$ on Σ.

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Without loss of generality, we may assume that $\mu = g = 1$. Indeed, a standard scaling argument allows us to scale so that $\mu = g = 1$, at the price of multiplying b and the periodicity lengths L_1, L_2 by positive constants and rescaling b. This means that, up to renaming b, L_1, and L_2, we arrive at the above problem with $\mu = g = 1$.

We assume that the initial surface function satisfies the "zero average" condition

$$\frac{1}{L_1 L_2}\int_\Sigma \eta_0 = 0. \tag{1.3}$$

If it happens that η_0 does not satisfy (1.3) but does satisfy the extra condition that $\inf_\Sigma b + (\eta_0) > 0$, where we have written (η_0) for the left side of (1.3), then it is possible to shift the problem to obtain a solution to (1.2) with η_0 satisfying (1.3). Indeed, we may change

$$y_3 \mapsto y_3 - (\eta_0), \eta \mapsto \eta - (\eta_0), b \mapsto b + (\eta_0), \text{ and } p \mapsto p - (\eta_0) \tag{1.4}$$

to find a new solution with the initial surface function satisfying (1.3). The data u_0 and $\eta_0 - (\eta_0)$ will still satisfy the compatibility conditions, and $b + (\eta_0) \ge \inf_\Sigma b + (\eta_0) > 0$, so after renaming we arrive at the above problem with η_0 satisfying (1.3). Note that for sufficiently regular solutions to the periodic problem, the condition (1.3) persists in time since $\partial_t \eta = u \cdot \nu \sqrt{1 + (\partial_{y_1}\eta)^2 + (\partial_{y_2}\eta)^2}$:

$$\frac{d}{dt}\int_\Sigma \eta = \int_\Sigma \partial_t \eta = \int_{\{y_3 = \eta(y_1,y_2,t)\}} u \cdot \nu = \int_{\Omega(t)} \operatorname{div} u = 0. \tag{1.5}$$

The zero average of $\eta(t)$ for $t \ge 0$ is analytically useful in that it allows us to apply the Poincaré inequality on Σ for all $t \ge 0$. Moreover, we are interested in the decay $\eta(t) \to 0$ as $t \to \infty$, in say $L^2(\Sigma)$ or $L^\infty(\Sigma)$; due to the conservation of (η_0), we cannot expect this decay unless $(\eta_0) = 0$.

The problem (1.2) possesses a natural physical energy. For sufficiently regular solutions, we have an energy evolution equation that expresses how the change in physical energy is related to the dissipation:

$$\frac{1}{2}\int_{\Omega(t)} |u(t)|^2 + \frac{1}{2}\int_\Sigma |\eta(t)|^2 + \frac{1}{2}\int_0^t \int_{\Omega(s)} |\mathbb{D}u(s)|^2 \, ds = \frac{1}{2}\int_{\Omega(0)} |u_0|^2 + \frac{1}{2}\int_\Sigma |\eta_0|^2 . \tag{1.6}$$

The first two integrals constitute the kinetic and potential energies, while the third constitutes the dissipation. The structure of this energy evolution equation is the basis of the energy method we will use to analyze (1.2).

1.2. **Geometric form of the equations.** In order to work in a fixed domain, we want to flatten the free surface via a coordinate transformation. We will not use a Lagrangian coordinate transformation, but rather a flattening transformation introduced by Beale in [4]. To this end, we consider the fixed equilibrium domain

$$\Omega := \{x \in \Sigma \times \mathbb{R} \mid -b(x_1, x_2) < x_3 < 0\} \tag{1.7}$$

for which we will write the coordinates as $x \in \Omega$. We will think of Σ as the upper boundary of Ω, and we will write $\Sigma_b := \{x_3 = -b(x_1, x_2)\}$ for the lower boundary. We continue to view η as a function on $\Sigma \times \mathbb{R}^+$. We then define

$$\bar{\eta} := \mathcal{P}\eta = \text{harmonic extension of } \eta \text{ into the lower half space,} \tag{1.8}$$

where $\mathcal{P}\eta$ is defined by (A.7). The harmonic extension $\bar{\eta}$ allows us to flatten the coordinate domain via the mapping

$$\Omega \ni x \mapsto (x_1, x_2, x_3 + \bar{\eta}(x,t)(1 + x_3/b(x_1,x_2))) = \Phi(x,t) = (y_1, y_2, y_3) \in \Omega(t). \tag{1.9}$$

Note that $\Phi(\Sigma, t) = \{y_3 = \eta(y_1, y_2, t)\}$ and $\Phi(\cdot, t)|_{\Sigma_b} = Id_{\Sigma_b}$, i.e. Φ maps Σ to the free surface and keeps the lower surface fixed. We have

$$\nabla\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix} \text{ and } \mathcal{A} := (\nabla\Phi^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix} \tag{1.10}$$

for

$$(1.11)\quad \begin{aligned} &A = \partial_1 \bar{\eta} \tilde{b} - (x_3 \bar{\eta} \partial_1 b)/b^2, \quad B = \partial_2 \bar{\eta} \tilde{b} - (x_3 \bar{\eta} \partial_2 b)/b^2, \\ &J = 1 + \bar{\eta}/b + \partial_3 \bar{\eta} \tilde{b}, \quad K = J^{-1}, \\ &\tilde{b} = (1 + x_3/b). \end{aligned}$$

Here $J = \det \nabla \Phi$ is the Jacobian of the coordinate transformation.

If η is sufficiently small (in an appropriate Sobolev space), then the mapping Φ is a diffeomorphism. This allows us to transform the problem to one on the fixed spatial domain Ω for $t \geq 0$. In the new coordinates, the PDE (1.2) becomes

$$(1.12)\quad \begin{cases} \partial_t u - \partial_t \bar{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathcal{A}} u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ S_{\mathcal{A}}(p, u)\mathcal{N} = \eta \mathcal{N} & \text{on } \Sigma \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \\ u(x, 0) = u_0(x), \eta(x', 0) = \eta_0(x'). \end{cases}$$

Here we have written the differential operators $\nabla_{\mathcal{A}}$, $\operatorname{div}_{\mathcal{A}}$, and $\Delta_{\mathcal{A}}$ with their actions given by $(\nabla_{\mathcal{A}} f)_i := \mathcal{A}_{ij} \partial_j f$, $\operatorname{div}_{\mathcal{A}} X := \mathcal{A}_{ij} \partial_j X_i$, and $\Delta_{\mathcal{A}} f = \operatorname{div}_{\mathcal{A}} \nabla_{\mathcal{A}} f$ for appropriate f and X; for $u \cdot \nabla_{\mathcal{A}} u$ we mean $(u \cdot \nabla_{\mathcal{A}} u)_i := u_j \mathcal{A}_{jk} \partial_k u_i$. We have also written $\mathcal{N} := -\partial_1 \eta e_1 - \partial_2 \eta_2 e_2 + e_3$ for the non-unit normal to Σ, and we write $S_{\mathcal{A}}(p, u) = (pI - \mathbb{D}_{\mathcal{A}} u)$ for the stress tensor, where I the 3×3 identity matrix and $(\mathbb{D}_{\mathcal{A}} u)_{ij} = \mathcal{A}_{ik} \partial_k u_j + \mathcal{A}_{jk} \partial_k u_i$ is the symmetric $\mathcal{A}-$gradient. Note that if we extend $\operatorname{div}_{\mathcal{A}}$ to act on symmetric tensors in the natural way, then $\operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) = \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u$ for vector fields satisfying $\operatorname{div}_{\mathcal{A}} u = 0$.

Recall that $\mathcal{A}$ is determined by η through the relation (1.10). This means that all of the differential operators in (1.12) are connected to η, and hence to the geometry of the free surface. This geometric structure is essential to our analysis, as it allows us to control high-order derivatives that would otherwise be out of reach.

1.3. **Previous results and Beale's non-decay theorem.** Many authors have considered problems similar to (1.2), both with and without viscosity and surface tension: [1, 2, 3, 4, 5, 6, 7, 8, 11, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30]. We refer the reader to the introduction of our paper [13] for a discussion of how these results relate to ours. We will only mention the details of those papers most relevant to the present problem.

In [3], Beale developed a local existence theory for the problem (1.2) in Lagrangian coordinates, where the unknowns are replaced with $v = u \circ \zeta$, $q = p \circ \zeta$ for ζ the Lagrangian flow map, which satisfies $\partial_t \zeta = v$. The result showed that (roughly speaking), given $v_0 \in H^{r-1}$ for $r \in (3, 7/2)$, there exists a unique solution on a time interval $(0, T)$, with T depending on v_0, so that $v \in L^2 H^r \cap H^{r/2} L^2$. A second local existence theorem was then proved for small data near equilibrium. It showed that for any fixed $0 < T < \infty$, there exists a collection of sufficiently small data so that a unique solution exists on $(0, T)$ and so that the solutions depend analytically on the data.

The second result suggests that solutions should exist globally in time for small data. If global solutions do exist, it is natural to expect the free surface to decay to 0 as $t \to \infty$. However, Beale's third result in [3] was a non-decay theorem that showed that a "reasonable" extension to small-data global well-posedness with decay of the free surface fails. More precisely, Theorem 6.4 of [3] establishes that it is possible to choose $\Theta \in H^1(\Omega)$ with $\Theta = 0$ on Σ_b so that there cannot exist a curve of solutions in Lagrangian coordinates, written $(v(\varepsilon), q(\varepsilon))$ for ε near 0, so that (among other things)

$$(1.13)\quad v(\varepsilon) \in L^2([0, \infty); H^r(\Omega)) \cap H^{r/2}([0, \infty); L^2(\Omega)) \cap L^1([0, \infty); H^r(\Omega)) \text{ for } r \in (3, 7/2),$$

$$(1.14)\quad \zeta_0(\varepsilon) = Id + \varepsilon \Theta, \ v_0(\varepsilon) = 0,$$

$$(1.15)\quad \lim_{t \to \infty} \zeta_3(\varepsilon)|_{\Sigma} = 0, \text{ and } v(\varepsilon) = \varepsilon v^1 + \varepsilon^2 v^2 + O(\varepsilon^3).$$

The proof, which is a reductio ad absurdum, hinges on the inclusion $v(\varepsilon) \in L^1 H^r$ and Θ satisfying the properties

$$\operatorname{div} \Theta = 0 \text{ and } \int_\Sigma \partial_3 \Theta_3 \cdot \Theta_3 \neq 0. \tag{1.16}$$

The condition (1.14) says that the domain is initially close to equilibrium, and the first condition in (1.15) says that the free surface returns to equilibrium as $t \to \infty$. In the discussion of this result, Beale pointed out that it does not imply the non-existence of global-in-time solutions, but rather that establishing global-in-time results requires stronger or different hypotheses than those imposed in the non-decay theorem. Note that, even though the non-decay theorem is proved in the context of horizontally infinite domains, its proof carries over to horizontally periodic domains.

The non-decay theorem raises two intriguing questions. First, is viscosity alone capable of producing global well-posedness? Second, if global solutions exist, do they decay as $t \to \infty$? Our main result answers both questions in the affirmative. In order to avoid the applicability of the non-decay theorem, we must show why its hypotheses are not satisfied. We would like to highlight two crucial ways in which we do this. The first and most obvious is that we work in a different coordinate system and within a different functional framework. In particular this requires higher regularity of the initial data and imposes more compatibility conditions than are satisfied by the data in the non-decay theorem.

The second difference is found in our assumption that η_0 has zero average in (1.3). We claim that this condition makes (1.16) impossible, i.e. the zero average condition prevents the choice of Θ satisfying (1.16), which then breaks the reductio ad absurdum used to prove the non-decay theorem. The argument in the theorem goes as follows. The expansion of $v(\varepsilon)$ in (1.15), and the $L^1 H^r$ condition in (1.13) imply an expansion $\zeta(\varepsilon) = \varepsilon \zeta^1 + \varepsilon^2 \zeta^2 + O(\varepsilon^3)$. The term v^1 is assumed to be known, and a contradiction is derived in solving for v^2 using the ζ expansion, if Θ is chosen to satisfy (1.16).

To show that the zero average condition prevents the choice of Θ satisfying (1.16), we must first compare the flow map, ζ, to the free surface function, η. Since ζ and η yield the same surface, we must have that as graphs,

$$\{(\zeta_1(x_1,x_2,0,t), \zeta_2(x_1,x_2,0,t), \zeta_3(x_1,x_2,0,t))\} = \{(x_1,x_2,\eta(x_1,x_2,t))\}. \tag{1.17}$$

Let $\psi_i(x_1,x_2,t) = \zeta_i(x_1,x_2,0,t)$ for $i = 1, 2$. If ζ is a diffeomorphism, then it is possible to solve $\psi(y_1,y_2,t) = (x_1,x_2) = x'$ for $y' = (y_1,y_2)$, i.e. $y' = \psi^{-1}(x',t)$. Hence

$$\eta(x_1,x_2,t) = \zeta_3(\psi^{-1}(x_1,x_2,t),0,t) \text{ for all } x' \in \Sigma, t \geq 0. \tag{1.18}$$

At time $t = 0$ we have

$$\psi_0(\varepsilon) = (x_1 + \varepsilon\Theta_1)e_1 + (x_2 + \varepsilon\Theta_2)e_2, \text{ and } e_3 \cdot \zeta_0(\varepsilon)(y',0) = \varepsilon\Theta_3(y',0), \tag{1.19}$$

so that $\eta_0(x') = \varepsilon\Theta_3(\psi_0(\varepsilon)^{-1}(x'),0)$. Using the zero average condition and a change of variables shows that

$$0 = \int_\Sigma \eta_0(x')dx' = \varepsilon \int_\Sigma \Theta_3(\psi_0(\varepsilon)^{-1}(x'),0)dx' = \varepsilon \int_\Sigma \Theta_3(y',0) \left|\det D_y\psi_0(\varepsilon)\right| dy', \tag{1.20}$$

but it is easily verified that for ε near 0,

$$\left|\det D_y\psi_0(\varepsilon)\right| = \det D_y\psi_0(\varepsilon) = 1 + \varepsilon(\partial_1\Theta_1 + \partial_2\Theta_2) + O(\varepsilon^2), \tag{1.21}$$

so that

$$0 = \varepsilon \int_\Sigma \Theta_3 \left(1 + \varepsilon(\partial_1\Theta_1 + \partial_2\Theta_2) + O(\varepsilon^2)\right) dy'. \tag{1.22}$$

Sending $\varepsilon \to 0$, we find that Θ must satisfy

$$0 = \int_\Sigma \Theta_3 dy' \text{ and } 0 = \int_\Sigma \Theta_3(\partial_1\Theta_1 + \partial_2\Theta_2)dy'. \tag{1.23}$$

However, $\operatorname{div}\Theta = 0$ implies that $\partial_3\Theta_3 = -(\partial_1\Theta_1 + \partial_2\Theta_2)$, so that the latter condition becomes

$$0 = \int_\Sigma \Theta_3 \partial_3 \Theta_3, \tag{1.24}$$

in violation of assumption (1.16).

This analysis shows that imposing condition (1.16) on the initial data for the flow map is essentially equivalent to choosing an initial coordinate system in which the average disturbance of the free surface does not vanish. If the system returns to equilibrium, then the map describing the equilibrium surface should be a non-zero constant (whatever the initial average was), and hence we should not expect L^2 or L^∞ decay of this map. Choosing the initial data with zero average circumvents this problem and allows for L^2 and L^∞ decay.

1.4. **Local well-posedness.** The a priori estimates we develop in this paper are done in different coordinates and in a different functional framework from those used by Beale in [3]. As such, we need a local well-posedness theory for (1.12) in our framework. We proved this in Theorem 1.1 of our companion paper [13]. Since we will need the result here, we record it now.

In order to state our result, we must explain our notation for Sobolev spaces and norms. We take $H^k(\Omega)$ and $H^k(\Sigma)$ for $k \ge 0$ to be the usual Sobolev spaces. When we write norms we will suppress the H and Ω or Σ. When we write $\left\|\partial_t^j u\right\|_k$ and $\left\|\partial_t^j p\right\|_k$ we always mean that the space is $H^k(\Omega)$, and when we write $\left\|\partial_t^j \eta\right\|_k$ we always mean that the space is $H^k(\Sigma)$.

In the following we write ${}_0H^1(\Omega) := \{u \in H^1(\Omega) \mid u|_{\Sigma_b} = 0\}$. The compatibility conditions for the initial data are the natural ones that would be satisfied for solutions in our functional framework. They are cumbersome to write, so we shall not record them here. We refer the reader to [13] for their precise definition.

Theorem 1.1. *Let $N \ge 3$ be an integer. Assume that u_0 and η_0 satisfy the bounds $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N+1/2}^2 < \infty$ as well as the appropriate compatibility conditions. There exist $0 < \delta_0, T_0 < 1$ so that if*

$$0 < T \le T_0 \min\left\{1, \frac{1}{\|\eta_0\|_{4N+1/2}^2}\right\}, \tag{1.25}$$

and $\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 \le \delta_0$, then there exists a unique solution (u, p, η) to (1.12) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimates

$$\begin{aligned} \sum_{j=0}^{2N} \sup_{0\le t\le T} \left\|\partial_t^j u\right\|_{4N-2j}^2 + \sum_{j=0}^{2N} \sup_{0\le t\le T} \left\|\partial_t^j \eta\right\|_{4N-2j}^2 + \sum_{j=0}^{2N-1} \sup_{0\le t\le T} \left\|\partial_t^j p\right\|_{4N-2j-1}^2 \\ + \int_0^T \left(\sum_{j=0}^{2N} \left\|\partial_t^j u\right\|_{4N-2j+1}^2 + \left\|\partial_t^{2N+1} u\right\|_{({}_0H^1(\Omega))^*}^2 + \sum_{j=0}^{2N} \left\|\partial_t^j p\right\|_{4N-2j}^2 \right) \\ + \int_0^T \left(\|\eta\|_{4N+1/2}^2 + \|\partial_t \eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \left\|\partial_t^j \eta\right\|_{4N-2j+5/2}^2 \right) \\ \le C\left(\|u_0\|_{4N}^2 + \|\eta_0\|_{4N}^2 + T\|\eta_0\|_{4N+1/2}^2 \right) \end{aligned} \tag{1.26}$$

and

$$\sup_{0\le t\le T} \|\eta\|_{4N+1/2}^2 \le C\left(\|u_0\|_{4N}^2 + (1+T)\|\eta_0\|_{4N+1/2}^2 \right) \tag{1.27}$$

for a universal constant $C > 0$. The solution is unique among functions that achieve the initial data and for which the sum of the first three sums in (1.26) is finite. Moroever, η is such that the mapping $\Phi(\cdot, t)$, defined by (1.9), is a C^{4N-2} diffeomorphism for each $t \in [0, T]$.

Remark 1.2. *All of the computations involved in the a priori estimates that we develop in this paper are justified by Theorem 1.1 and the fact that $\eta(t)$ has zero average for $t \ge 0$. In this sense, Theorem 1.1 is a necessary ingredient in the global analysis of* (1.12)*. We do not believe that our a priori estimates could be justified within a high-regularity modification of the functional framework of* [3].

1.5. **Main result.** In [17], Hataya studied the periodic problem with a flat bottom, $b(x') = b \in (0, \infty)$. Using the parabolic theory pioneered by Beale [3] and Solonnikov [21], it was shown that if η_0 has zero average (1.3), then

$$\int_0^\infty (1+t)^2 \|u(t)\|_{r-1}^2 \, dt + \sup_{t\ge 0} (1+t)^2 \|\eta(t)\|_{r-2}^2 < \infty \tag{1.28}$$

for $r \in (5, 11/2)$. Our result on the periodic problem is an improvement of this in two important ways. First, we allow for a more general non-flat bottom geometry. Second, we establish faster decay rates by working in a higher regularity context.

To state our result, we must first define our energies and dissipations. For any integer $N \ge 3$ we write the high-order energy as

$$\mathcal{E}_{2N} = \sum_{j=0}^{2N} \left(\left\| \partial_t^j u \right\|_{4N-2j}^2 + \left\| \partial_t^j \eta \right\|_{4N-2j}^2 \right) + \sum_{j=0}^{2N-1} \left\| \partial_t^j p \right\|_{4N-2j-1}^2 \tag{1.29}$$

and the corresponding dissipation as

$$\begin{aligned} \mathcal{D}_{2N} = \sum_{j=0}^{2N} \left\| \partial_t^j u \right\|_{4N-2j+1}^2 &+ \sum_{j=0}^{2N-1} \left\| \partial_t^j p \right\|_{4N-2j}^2 \\ &+ \|\eta\|_{4N-1/2}^2 + \|\partial_t \eta\|_{4N-1/2}^2 + \sum_{j=2}^{2N+1} \left\| \partial_t^j \eta \right\|_{4N-2j+5/2}^2 . \end{aligned} \tag{1.30}$$

We write the high-order spatial derivatives of η as

$$\mathcal{F}_{2N} := \|\eta\|_{4N+1/2}^2 . \tag{1.31}$$

We define the low-order energy as

$$\mathcal{E}_{N+2} = \sum_{j=0}^{N+2} \left(\left\| \partial_t^j u \right\|_{2(N+2)-2j}^2 + \left\| \partial_t^j \eta \right\|_{2(N+2)-2j}^2 \right) + \sum_{j=0}^{N+1} \left\| \partial_t^j p \right\|_{2(N+2)-2j-1}^2 . \tag{1.32}$$

Finally, we define total energy

$$\mathcal{G}_{2N}(t) = \sup_{0 \le r \le t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N}(r) dr + \sup_{0\le r \le t} (1+r)^{4N-8} \mathcal{E}_{N+2}(r) + \sup_{0 \le r \le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)}. \tag{1.33}$$

Notice that the low-order terms $\mathcal{E}_{N+2}$ are weighted, so bounds on $\mathcal{G}_{2N}$ imply decay estimates $\mathcal{E}_{N+2}(t) \lesssim (1+t)^{-4N+8}$.

Theorem 1.3. *Suppose the initial data (u_0, η_0) satisfy the compatibility conditions of Theorem 1.1 and that η_0 satisfies the zero average condition* (1.3)*. Let $N \ge 3$ be an integer. There exists a $0 < \kappa = \kappa(N)$ so that if $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa$, then there exists a unique solution (u, p, η) on the interval $[0, \infty)$ that achieves the initial data. The solution obeys the estimate*

$$\mathcal{G}_{2N}(\infty) \le C_1 \left(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \right) < C_1 \kappa, \tag{1.34}$$

where $C_1 > 0$ is a universal constant.

Remark 1.4. *The decay of $\mathcal{E}_{N+2}(t)$ implies that*

$$\sup_{t \ge 0} (1+t)^{4N-8} \left[\|u(t)\|_{2N+4}^2 + \|\eta(t)\|_{2N+4}^2 \right] \le C_1 \kappa. \tag{1.35}$$

Since N may be taken to be arbitrarily large, this decay result can be regarded as an "almost exponential" decay rate.

Remark 1.5. *The surface* η *is sufficiently small to guarantee that the mapping* $\Phi(\cdot, t)$*, defined in* (1.9)*, is a diffeomorphism for each* $t \geq 0$*. As such, we may change coordinates to* $y \in \Omega(t)$ *to produce a global-in-time, decaying solution to* (1.2)*.*

The proof of Theorem 1.3 is completed in Section 9. We now present a summary of the principal difficulties we encounter in our analysis as well as a sketch of the key ideas used in our proof.

Principal difficulties

In the study of the unforced incompressible Navier-Stokes equations in a fixed bounded domain with Dirichlet boundary conditions, it is natural to use the energy method to prove that solutions decay in time. Indeed, one may prove an analogue of (1.6) for sufficiently smooth solutions, which relates the natural energy and dissipation:

$$\partial_t \mathcal{E} + \mathcal{D} := \partial_t \int_\Omega \frac{|u(t)|^2}{2} + \frac{1}{2}\int_\Omega |\mathbb{D}u(t)|^2 = 0. \tag{1.36}$$

Korn's inequality allows us to control $C\mathcal{E}(t) \leq \mathcal{D}(t)$ for a constant $C > 0$ independent of time, which shows that the dissipation is stronger than the energy. From this and Gronwall's lemma we may immediately deduce that the energy $\mathcal{E}$ decays exponentially in time and that we have the estimate $\mathcal{E}(t) \leq \mathcal{E}(0)\exp(-Ct)$.

If one seeks to similarly use the energy method to obtain decay estimates for solutions to (1.2), then one encounters a fundamental obstacle that may already be observed in the differential form of (1.6),

$$\partial_t \left(\int_{\Omega(t)} \frac{|u(t)|^2}{2} + \int_\Sigma \frac{|\eta(t)|^2}{2} \right) + \frac{1}{2}\int_{\Omega(t)} |\mathbb{D}u(t)|^2 = 0. \tag{1.37}$$

The difficulty is that the dissipation provides no direct control of the $\eta-$term in the energy. As such, we must resort to using the equations (1.2) to try to control $\|\eta(t)\|_0$ in terms of $\|\mathbb{D}u(t)\|_0$. From (1.2) we see that there are only two available routes: solving for η in the fourth equation; or using the third equation, which is the kinetic transport equation. If we pursue the first route, then we must be able to control

$$\|p(t)\|^2_{H^0(\Sigma)} + \|\mathbb{D}u(t)\nu\cdot\nu\|^2_{H^0(\Sigma)} \lesssim \|\mathbb{D}u(t)\|^2_{H^0(\Omega(t))}, \tag{1.38}$$

which is not possible. If instead we pursue the second route, then we must estimate η as a solution to the kinematic transport equation. Such an estimate (see Lemma A.5) only allows us to estimate $\|\eta(t)\|_0$ in terms of $\int_0^t \|\mathbb{D}u(s)\|_0 \, ds$. That is, transport estimates do not provide control of the $\eta-$part of the energy in terms of the "instantaneous" dissipation, but rather in terms of the "cumulative" integrated dissipation. From this we see that in our problem the dissipation is actually weaker than the energy, so we cannot argue as above to deduce exponential decay.

We might hope that we could avoid this problem by working with a high-regularity energy method, but we will always encounter the same type of problem as above. Regardless of the level of regularity in the energy, the instantaneous dissipation is always weaker than the instantaneous energy, which prevents us from deducing exponential decay of the energy. Instead we pursue a strategy similar to one employed in [22] for another problem where the dissipation is weaker than the energy. We first show that high-order energies are bounded by using an integrated version or (1.37) for derivatives of the solution. Then we consider a low-order energy and show that an equation of the form (1.37) holds, i.e. $\partial_t \mathcal{E}_{\text{low}} + C\mathcal{D}_{\text{low}} \leq 0$. Now, instead of trying to estimate (1.38) for low-order derivatives, we instead interpolate between low-order derivatives and high-order derivatives, which are bounded. Instead of an estimate $C\mathcal{E}_{\text{low}} \leq \mathcal{D}_{\text{low}}$, we must prove one of the form $C\mathcal{E}_{\text{low}}^{1+\theta} \leq \mathcal{D}_{\text{low}}$ for some $\theta > 0$. We can then use this to derive the differential inequality $\partial_t \mathcal{E}_{\text{low}} + C\mathcal{E}_{\text{low}}^{1+\theta} \leq 0$, which can be integrated to see that $\mathcal{E}_{\text{low}}(t) \lesssim \mathcal{E}_{\text{low}}(0)/(1+t)^{1/\theta}$. We would then find that the low-order energy decays algebraically in time rather than exponentially.

To complete this program, we must overcome a pair of intertwined difficulties. First, to close the high-order energy estimates with, say $\|u\|^2_{4N+1}$ for an integer $N \geq 0$ in the dissipation, we have to control η in $H^{4N+1/2}$. The only option for this is to again appeal to estimates for solutions to the transport equation, which say (roughly speaking) that

$$\sup_{0\le t\le T} \|\eta(t)\|^2_{4N+1/2} \le C\exp\left(C\int_0^T \|Du(t)\|_{H^2(\Sigma)}\,dt\right)\left[\|\eta_0\|^2_{4N+1/2} + T\int_0^T \|u(t)\|^2_{4N+1}\,dt\right]. \tag{1.39}$$

Without knowing a priori that u decays, the right side of this estimate has the potential to grow at the rate of $(1+T)e^{\sqrt{T}}$. Even if u decays rapidly, the right side can still grow like $(1+T)$. This growth is potentially disastrous in closing the high-order, global-in-time estimates. To manage the growth, we must identify a special decaying term that always appears in products with the highest derivatives of η. If the special term decays quickly enough, then we can hope to balance the growth and close the high-order estimates. Due to the growth in (1.39), we believe that it is not possible to construct global-in-time solutions without also deriving a decay result.

This leads us to the second difficulty in this program. The decay rate of the special term is dictated by the decay rate of the low-order energy, so we must make the low-order energy decay sufficiently quickly. This amounts to making the constant $\theta > 0$ appearing in the interpolation estimates above sufficiently small. We must then carefully choose the terms that will appear in the low-order and high-order energies in order to keep θ small enough.

The resolution of these intertwined difficulties requires a delicate and involved analysis. We now sketch some of the techniques we will employ.

Localization and horizontal energy evolution estimates

In order to use the natural energy structure of the problem (given in Eulerian coordinates by (1.6)) to study high-order derivatives, we can only apply derivatives that do not break the structure of the boundary condition $u = 0$ on Σ_b. We allow the lower boundary Σ_b to be curved. This means that spatial derivatives in the x_1 and x_2 directions are not compatible with the boundary condition on Σ_b. This prohibits us from applying, say ∂_1^k, to the equations and studying the evolution of $\partial_1^k u$ and $\partial_1^k \eta$. The only operator that does not break the boundary condition is ∂_t.

To get around this problem we introduce a localization procedure. We localize in a horizontal strip near Σ, and in an area around Σ_b. Near Σ the problem behaves like a free boundary problem with a flat bottom, and we are free to apply all horizontal derivatives. In the lower domain, near Σ_b, the problem behaves like a fixed boundary problem with curved lower boundary. The only derivatives we can apply are temporal, but they are sufficient for controlling all derivatives because of the fixed upper boundary.

We then build our a priori estimates out of sums of these appropriate derivatives in the localizations as well as sums of temporal derivatives in all of Ω. The natural energy structure (1.6) leads us to consider "horizontal" energies and dissipations of the form (see Section 2.5 for precise definitions):

$$\begin{aligned} \bar{\mathcal{E}}_n &= \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^- + \bar{\mathcal{E}}_n^0, \\ \bar{\mathcal{D}}_n &= \bar{\mathcal{D}}_n^+ + \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^0, \end{aligned} \tag{1.40}$$

where we allow $n = 2N$ or $n = N + 2$ for an integer $N \ge 3$. Here $2n$ is the number of temporal derivatives, the superscript $\pm$ indicates the upper or lower localization, the superscript 0 indicates the global temporal derivatives, and the bar indicates horizontal derivatives. After estimating the nonlinear terms that appear from differentiating (1.12), we are eventually led to evolution equations for these energies. Roughly speaking, at high-order we have the estimate

$$\bar{\mathcal{E}}_{2N}(t) + \int_0^t \bar{\mathcal{D}}_{2N}(r)dr \lesssim \mathcal{E}_{2N}(0) + \int_0^t (\mathcal{E}_{2N}(r))^\theta \mathcal{D}_{2N}(r)dr + \int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}dr, \tag{1.41}$$

where $\mathcal{K}$ is of the form

$$\mathcal{K} = \|\nabla u\|^2_{C^1} + \|Du\|^2_{H^2(\Sigma)}, \tag{1.42}$$

and $\theta > 0$; and at low-order we have

$$\partial_t \bar{\mathcal{E}}_{N+2} + \bar{\mathcal{D}}_{N+2} \lesssim \mathcal{E}^\theta_{2N}\mathcal{D}_{N+2}. \tag{1.43}$$

Notice that the product $\mathcal{K}\mathcal{F}_{2N}$ in (1.41) multiplies low-order norms of u against the highest-order norm of η.

The actual derivation of bounds like (1.41)–(1.43) is rather delicate and depends crucially on the geometric structure of the equations given in (1.12). Indeed, if we attempted rewrite (1.12) as a perturbation of the usual constant-coefficient Navier-Stokes equations, then we would fail to achieve the estimate (1.41) because we would be unable to control the interaction between $\partial_t^{2N} p$ and $\operatorname{div} \partial_t^{2N} u$, the latter of which does not vanish in the geometric form of the equations.

Comparison estimates

The next step in the analysis is to replace the horizontal energies and dissipations with the full energies and dissipations. We prove that there is a universal $0 < \delta < 1$ so that if $\mathcal{E}_{2N} \le \delta$, then

$$\begin{gathered} \mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_{2N}, \quad \mathcal{D}_{2N} \lesssim \bar{\mathcal{D}}_{2N} + \mathcal{K}\mathcal{F}_{2N}, \\ \mathcal{E}_{N+2} \lesssim \bar{\mathcal{E}}_{N+2}, \quad \mathcal{D}_{N+2} \lesssim \bar{\mathcal{D}}_{N+2} \end{gathered} \tag{1.44}$$

This estimate is extremely delicate and can only be obtained by carefully using the structure of the equations. We make use of every bit of information from the boundary conditions and the vorticity equations to establish it. There are two structural components of the estimates that are of such importance that we mention them now. First, the equation $\operatorname{div}_{\mathcal{A}} u = 0$ allows us to write $\partial_3 u_3 = -(\partial_1 u_1 + \partial_2 u_2) + G^2$ for some quadratic nonlinearity G^2. This allows us to "trade" a vertical derivative of u_3 for horizontal derivatives of u_1 and u_2, an indispensable trick in our analysis. Second, the interaction between the parabolic scaling of u ($\partial_t u \sim \Delta u$) and the transport scaling of η ($\partial_t \eta \sim u_3|_\Sigma$) allows us to gain regularity for the temporal derivatives of η in the dissipation, and it also gives us control of $\partial_t^{2N+1}\eta$, which is one more time derivative than appears in the energy.

Two-tier energy method

Suppose we know that

$$\mathcal{E}_{N+2}(r) \le \frac{\delta}{(1+r)^{4N-8}} \tag{1.45}$$

for some $0 < \delta < 1$ and $N \ge 3$. It is possible to show that $\mathcal{K} \lesssim \mathcal{E}_{N+2}$, so that $\mathcal{K}$ also decays as in (1.45). Since η satisfies a transport equation, we may use Lemma A.5 to derive an estimate of the form

$$\sup_{0 \le r \le t} \mathcal{F}_{2N}(r) \lesssim \exp\left(C \int_0^t \sqrt{\mathcal{K}(r)} dr \right) \left[\mathcal{F}_{2N}(0) + t \int_0^t \mathcal{D}_{2N}(r) dr \right]. \tag{1.46}$$

Although the right side of this equation could potentially blow up exponentially in time, the decay of $\mathcal{K}$ implied by (1.45) implies that

$$\sup_{0 \le r \le t} \mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + t \int_0^t \mathcal{D}_{2N}(r) dr. \tag{1.47}$$

This estimate allows for $\mathcal{F}_{2N}(t)$ to grow linearly in time, but in the product $\mathcal{K}(r)\mathcal{F}_{2N}(r)$ that appears in (1.41), we can use the decay of $\mathcal{K}$ to balance this growth. Then if $\sup_{0 \le r \le t} \mathcal{E}_{2N}(r) \le \delta$ with δ small enough, we can combine (1.41), (1.44), (1.45), and (1.47) to get an estimate

$$\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}(r) dr \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0). \tag{1.48}$$

This highlights the first step of our two-tier energy method: the decay of low-order terms (i.e. $\mathcal{K}$) can balance the growth of $\mathcal{F}_{2N}$, yielding boundedness of the high-order terms. In order to close this argument, we must use a second step: the boundedness of the high-order terms implies the decay of low-order terms, and in particular the decay of $\mathcal{K}$.

To attain this decay, we combine (1.43) and (1.44) to see that

$$\partial_t \bar{\mathcal{E}}_{N+2} + \frac{1}{2}\mathcal{D}_{N+2} \le 0 \tag{1.49}$$

if $\mathcal{E}_{2N} \le \delta$ for δ small enough. If we could show that $\bar{\mathcal{E}}_{N+2} \lesssim \mathcal{D}_{N+2}$, then this estimate would yield exponential decay of $\bar{\mathcal{E}}_{N+2}$ and $\mathcal{E}_{N+2}$. An inspection of $\bar{\mathcal{E}}_{N+2}$ and $\mathcal{D}_{N+2}$ (see Section 2.5) shows that $\mathcal{D}_{N+2}$ can control every term in $\bar{\mathcal{E}}_{N+2}$ except $\|\eta\|^2_{2(N+2)}$ since $\mathcal{D}_{N+2}$ only controls $\|\eta\|^2_{2(N+2)-1/2}$. In a sense, this means that exponential decay fails precisely because the dissipation fails to control the highest spatial derivatives of η appearing in $\bar{\mathcal{E}}_{N+2}$. In lieu of an estimate of the form $\bar{\mathcal{E}}_{N+2} \lesssim \mathcal{D}_{N+2}$, we instead interpolate between $\mathcal{E}_{2N}$ and $\mathcal{D}_{N+2}$:

$$\bar{\mathcal{E}}_{N+2} \lesssim (\mathcal{D}_{N+2})^{(4N-8)/(4N-7)} (\mathcal{E}_{2N})^{1/(4N-7)}. \tag{1.50}$$

Combining (1.49) with (1.50) and the boundedness of $\mathcal{E}_{2N}$ in terms of the data (1.48) then allows us to deduce that

$$\partial_t \bar{\mathcal{E}}_{N+2} + \frac{C}{(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0))^{1/(4N-8)}} (\bar{\mathcal{E}}_{N+2})^{1+1/(4N-8)} \le 0. \tag{1.51}$$

Gronwall's inequality (along with some auxiliary estimates) then leads us to the bound

$$\mathcal{E}_{N+2}(t) \lesssim \bar{\mathcal{E}}_{N+2}(t) \lesssim \frac{\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)}{(1+t)^{4N-8}}. \tag{1.52}$$

We thus use the boundedness of high-order terms to deduce the decay of low-order terms, completing the second step of the two-tier energy estimates.

Poincaré from the zero average condition

Owing to (1.5), we know that the average of $\eta(t)$ over Σ vanishes for all $t \ge 0$. This allows us to utilize the standard Poincaré inequality on Σ to estimate $\|\eta\|_0^2 \lesssim \|D\eta\|_0^2$. This is useful because we will be able to control $\|D\eta\|_0^2$ with the dissipation (through careful use of the boundary conditions), which means we will gain control of η itself. This plays an essential role in the derivation of the decay rate in our two-tier energy method.

Localization for the curved lower surface

The localization procedure that we employ introduces a difficulty in the form of "localization forces" that appear because the cutoff functions we multiply by to localize do not commute with all of the differential operators. These localization forces can only be controlled in terms of the dissipation by employing the Poincaré inequality for η on Σ. Through a careful balance of how and where we localize, we are able to control the localization forces and close our estimates.

1.6. **Comparison to the horizontally infinite problem.** In our companion paper [14], we prove the analogue of Theorem 1.3 for horizontally infinite domains. In order to compare with Theorem 1.3, we record a version of our result here. In the theorem the terms $\mathcal{E}_{10}$, $\mathcal{F}_{10}$, and $\mathcal{G}_{10}$ are similar to what we use here (setting $N = 5$). However, they differ in two crucial ways: they include terms involving the horizontal Riesz potential, which amounts to negative fractional derivatives; and at low-order they require a minimal number of derivatives in the sums of derivatives. We refer to [14] for precise definitions.

Theorem 1.6. *Suppose the initial data (u_0, η_0) satisfy the compatibility conditions of Theorem 1.1. Let $\lambda \in (0,1)$. There exists a $\kappa > 0$ so that if $\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0) < \kappa$, then there exists a unique solution (u, p, η) on the interval $[0,\infty)$ that achieves the initial data. The solution obeys the estimate*

$$\mathcal{G}_{10}(\infty) \le C_1 (\mathcal{E}_{10}(0) + \mathcal{F}_{10}(0)) < C_1 \kappa, \tag{1.53}$$

where $C_1 > 0$ is a universal constant. For any $0 \le \rho < \lambda$, we have that

$$\sup_{t \ge 0} \left[(1+t)^{2+\rho} \|u(t)\|^2_{C^2(\Omega)} \right] \le C(\rho)\kappa, \tag{1.54}$$

for $C(\rho) > 0$ a constant depending on ρ. Also,

$$\sup_{t\ge 0} \left[(1+t)^{1+\lambda} \|u(t)\|_2^2 + (1+t)^{1+\lambda} \|\eta(t)\|^2_{L^\infty} + \sum_{j=0}^{1} (1+t)^{j+\lambda} \left\|D^j \eta(t)\right\|_0^2 \right] \le C\kappa \tag{1.55}$$

for a universal constant $C > 0$.

Remark 1.7. *For the horizontally infinite problem we require the lower boundary to be completely flat, i.e. we take $b \in (0, \infty)$ to be a constant. If we allowed for a curved lower surface and attempted the same localization procedure that we use here, the lack of a Poincaré inequality for η would prohibit us from controlling the localization forces in terms of the dissipation, and our estimates would fail to close.*

Remark 1.8. *A key difference between the periodic result, Theorem 1.3, and the non-periodic result, Theorem 1.6, is that in the periodic case, increasing N also increases the decay rate. No such gain is possible in the non-periodic case.*

Remark 1.9. *The reader interested in a unified presentation of Theorems 1.1, 1.6, and 1.3 may consult* [12].

1.7. **Comparison to the case with surface tension.** If the effect of surface tension is included at the air-fluid free interface, then the formulation of the PDE must be changed. Surface tension is modeled by modifying the fourth equation in (1.2) to be

$$(pI - \mu\mathbb{D}(u))\nu = g\eta\nu - \sigma H\nu, \tag{1.56}$$

where $H = \partial_i(\partial_i\eta/\sqrt{1+|D\eta|^2})$ is the mean curvature of the surface $\{y_3 = \eta(t)\}$ and $\sigma > 0$ is the surface tension.

In [4], Beale proved small-data global well-posedness for the problem with surface tension in horizontally infinite domains. The flattened coordinate system we employ was introduced in [4] and used in place of Lagrangian coordinates. However, Beale employed a change of unknown velocities that is more complicated than just a coordinate change. Well-posedness was demonstrated with $u \in L^2H^r$ and $\eta \in L^2H^{r+1/2}$, given that $u_0 \in H^{r-1/2}$, $\eta_0 \in H^r$ are sufficiently small for $r \in (3, 7/2)$. In this context it is understood that surface tension leads to the decay of certain modes, thereby aiding global existence.

In [5], Beale-Nishida studied the asymptotic properties of the solutions constructed in [4]. They showed that if $\eta_0 \in L^1(\Sigma)$, then

$$\sup_{t\geq 0}(1+t)^2 \left\|u(t)\right\|_2^2 + \sup_{t\geq 0}\sum_{j=1}^{2}(1+t)^{1+j}\left\|D^j\eta(t)\right\|_0^2 < \infty, \tag{1.57}$$

and that this decay rate is optimal. Taking $\lambda \approx 1$ in our Theorem 1.6, the estimates (1.55) yield almost the same decay rates.

In [19], Nishida-Teramoto-Yoshihara showed that in horizontally periodic domains with surface tension and a flat bottom, if η_0 has zero average, then there exists a $\gamma > 0$ so that

$$\sup_{t\geq 0} e^{\gamma t}\left[\left\|u(t)\right\|_2^2 + \left\|\eta(t)\right\|_3^2\right] < \infty. \tag{1.58}$$

In this case, the equation (1.56) gives a third way of estimating η in terms of the dissipation; using this, it is possible to show that the dissipation is stronger than the energy. Thus, if surface tension is added in the periodic case, fully exponential decay is possible, whereas without surface tension we only recover algebraic decay of arbitrary order in Theorem 1.3.

The comparison of these two results with ours establishes a nice contrast between the surface tension and non-surface tension cases. Without surface tension we can recover "almost" the same decay rate as in the case with surface tension. This shows that viscosity is the basic decay mechanism and that the effect of surface tension serves to enhance the decay rate.

1.8. **Definitions and terminology.** We now mention some of the definitions, bits of notation, and conventions that we will use throughout the paper.

Einstein summation and constants

We will employ the Einstein convention of summing over repeated indices for vector and tensor operations. Throughout the paper $C > 0$ will denote a generic constant that can depend on the parameters of the problem, N, and Ω, but does not depend on the data, etc. We refer to such constants as "universal." They are allowed to change from one inequality to the next.

When a constant depends on a quantity z we will write $C = C(z)$ to indicate this. We will employ the notation $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$.

Norms

We write $H^k(\Omega)$ with $k \geq 0$ and and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for the usual Sobolev spaces. We will typically write $H^0 = L^2$; the exception to this is when we use $L^2([0,T]; H^k)$ notation to indicate the space of square-integrable functions with values in H^k.

To avoid notational clutter, we will avoid writing $H^k(\Omega)$ or $H^k(\Sigma)$ in our norms and typically write only $\|\cdot\|_k$. Since we will do this for functions defined on both Ω and Σ, this presents some ambiguity. We avoid this by adopting two conventions. First, we assume that functions have natural spaces on which they "live." For example, the functions u, p, and $\bar{\eta}$ live on Ω, while η itself lives on Σ. As we proceed in our analysis, we will introduce various auxiliary functions; the spaces they live on will always be clear from the context. Second, whenever the norm of a function is computed on a space different from the one in which it lives, we will explicitly write the space. This typically arises when computing norms of traces onto Σ of functions that live on Ω.

Derivatives

We write $\mathbb{N} = \{0, 1, 2, \dots\}$ for the collection of non-negative integers. When using space-time differential multi-indices, we will write $\mathbb{N}^{1+m} = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m)\}$ to emphasize that the $0-$index term is related to temporal derivatives. For just spatial derivatives we write $\mathbb{N}^m$. For $\alpha \in \mathbb{N}^{1+m}$ we write $\partial^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$. We define the parabolic counting of such multi-indices by writing $|\alpha| = 2\alpha_0 + \alpha_1 + \cdots + \alpha_m$. We will write Df for the horizontal gradient of f, i.e. $Df = \partial_1 f e_1 + \partial_2 f e_2$, while ∇f will denote the usual full gradient.

For a given norm $\|\cdot\|$ and integers $k, m \geq 0$, we introduce the following notation for sums of spatial derivatives:

$$\left\| D_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^2 \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 \text{ and } \left\| \nabla_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^3 \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 . \tag{1.59}$$

The convention we adopt in this notation is that D refers to only "horizontal" spatial derivatives, while ∇ refers to full spatial derivatives. For space-time derivatives we add bars to our notation:

$$\left\| \bar{D}_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^{1+2} \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 \text{ and } \left\| \bar{\nabla}_m^k f \right\|^2 := \sum_{\substack{\alpha \in \mathbb{N}^{1+3} \\ m \leq |\alpha| \leq k}} \|\partial^\alpha f\|^2 . \tag{1.60}$$

When $k = m \geq 0$ we will write

$$\left\| D^k f \right\|^2 = \left\| D_k^k f \right\|^2, \left\| \nabla^k f \right\|^2 = \left\| \nabla_k^k f \right\|^2, \left\| \bar{D}^k f \right\|^2 = \left\| \bar{D}_k^k f \right\|^2, \left\| \bar{\nabla}^k f \right\|^2 = \left\| \bar{\nabla}_k^k f \right\|^2 . \tag{1.61}$$

We allow for composition of derivatives in this counting scheme in a natural way; for example, we write

$$\left\| D D_m^k f \right\|^2 = \left\| D_m^k D f \right\|^2 = \sum_{\substack{\alpha \in \mathbb{N}^2 \\ m \leq |\alpha| \leq k}} \|\partial^\alpha D f\|^2 = \sum_{\substack{\alpha \in \mathbb{N}^2 \\ m+1 \leq |\alpha| \leq k+1}} \|\partial^\alpha f\|^2 . \tag{1.62}$$

1.9. **Plan of paper.** Throughout the paper we assume that $N \geq 3$.

In Section 2 we prove some preliminary lemmas and we define the energies and dissipations. We also describe how we localize to handle the curved boundary $b \in C^\infty(\Sigma)$. In Section 3 we present estimates of the some nonlinear forcing terms G^i (as defined in (2.24)–(2.31)) and some other nonlinearities. In Section 4 we use the geometric form of the equations to estimate the evolution of temporal derivatives. Section 5 concerns similar energy evolution estimates for the localized energies. For these, we employ the linear perturbed framework with the G^i forcing terms. In the upper localization we apply horizontal spatial derivatives as well as temporal derivatives, but in the lower localization we only apply temporal derivatives. Section 6 concerns the comparison estimates, where we show how to estimate the full energies and dissipations in terms of their horizontal counterparts. Section 7 combines all of the analysis of Sections 3–6 into our a priori estimates for solutions to (1.12) in the periodic setting. Section 8 concerns a specialized version of the local well-posedness theorem that guarantees that η has zero average for all time. Finally, in Section 9 we record our global well-posedness and decay result, proving Theorem 1.3.

Below, in (2.52), we will define the total energy $\mathcal{G}_{2N}$ that we use in the global well-posedness analysis. For the purposes of deriving our a priori estimates, we will assume throughout Sections 3–7 that solutions are given on the interval $[0,T]$ and that $\mathcal{G}_{2N}(T) \le \delta$ for $0 < \delta < 1$ as small as in Lemma 2.3 so that its conclusions hold. This also means that $\mathcal{E}_{2N}(t) \le 1$ for $t \in [0,T]$. We will also assume throughout that the solutions satisfy the zero average condition

$$\int_\Sigma \eta(t) = 0 \text{ for all } t \in [0,T]. \tag{1.63}$$

We should remark that Theorem 1.1 does not produce solutions that necessarily satisfy the zero average condition. To guarantee that this holds, we must record a specialized version of the local well-posedness result, Theorem 8.1. We could record this result before the a priori estimates, but we have chosen to postpone it until after the a priori estimates. Note that the bounds of Theorem 8.1 control more than just $\mathcal{G}_{2N}(T)$, and the extra control it provides guarantees that all of the calculations used in the a priori estimates are justified.

2. Preliminaries for the a priori estimates

In this section we present some preliminary results that we will use in our a priori estimates. We first present two forms of equations similar to (1.12) and describe the corresponding energy evolution structure. Then we record a useful lemma, describe our localization procedure, and define the energies and dissipations.

2.1. **Geometric form.** We now give a linear formulation of the PDE (1.12) in its geometric form. Suppose that η, u are known and that $\mathcal{A}, \mathcal{N}, J$, etc are given in terms of η as usual ((1.10), etc). We then consider the linear equation for (v, q, ζ) given by

$$\begin{cases} \partial_t v - \partial_t \bar{\eta} \tilde{b} K \partial_3 v + u \cdot \nabla_{\mathcal{A}} v + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(q,v) = F^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} v = F^2 & \text{in } \Omega \\ S_{\mathcal{A}}(q,v)\mathcal{N} = \zeta \mathcal{N} + F^3 & \text{on } \Sigma \\ \partial_t \zeta - \mathcal{N} \cdot v = F^4 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b. \end{cases} \tag{2.1}$$

Now we record the natural energy evolution associated to solutions v, q, ζ of the geometric form equations (2.1).

Lemma 2.1. *Suppose that u and η are given solutions to* (1.12). *Suppose (v, q, ζ) solve* (2.1). *Then*

$$\partial_t \left(\frac{1}{2} \int_\Omega J |v|^2 + \frac{1}{2} \int_\Sigma |\zeta|^2 \right) + \frac{1}{2} \int_\Omega J |\mathbb{D}_{\mathcal{A}} v|^2 = \int_\Omega J(v \cdot F^1 + qF^2) + \int_\Sigma -v \cdot F^3 + \zeta F^4. \tag{2.2}$$

Proof. We multiply the i^{th} component of the first equation of (2.1) by Jv_i, sum over i and integrate over Ω to find that

$$I + II = III \tag{2.3}$$

for

$$I = \int_\Omega \partial_t v_i J v_i - \partial_t \bar{\eta} \tilde{b} \partial_3 v_i v_i + u_j \mathcal{A}_{jk} \partial_k v_i J v_i, \tag{2.4}$$

$$II = \int_\Omega \mathcal{A}_{jk} \partial_k S_{ij}(v,q) J v_i, \text{ and } III = \int_\Omega F^1 \cdot vJ. \tag{2.5}$$

In order to integrate by parts in I, II we will utilize the geometric identity $\partial_k (J\mathcal{A}_{ik}) = 0$ for each i.

Then

$$I = \partial_t \int_\Omega \frac{|v|^2 J}{2} + \int_\Omega -\frac{|v|^2 \partial_t J}{2} - \partial_t \bar{\eta} \tilde{b} \partial_3 \frac{|v|^2}{2} + u_j \partial_k \left(J \mathcal{A}_{jk} \frac{|v|^2}{2} \right) := I_1 + I_2. \tag{2.6}$$

Since $\tilde{b} = 1 + x_3/b$, an integration by parts and an application of the boundary condition $v = 0$ on Σ_b reveals that

$$\begin{aligned} I_2 = \int_\Omega -\frac{|v|^2 \partial_t J}{2} - \partial_t \bar{\eta} \tilde{b} \partial_3 \frac{|v|^2}{2} + u_j \partial_k \left(J \mathcal{A}_{jk} \frac{|v|^2}{2} \right) = \int_\Omega -\frac{|v|^2 \partial_t J}{2} + \frac{|v|^2}{2} \left(\frac{\partial_t \bar{\eta}}{b} + \tilde{b} \partial_t \partial_3 \bar{\eta} \right) \\ - \int_\Omega \partial_k u_j J \mathcal{A}_{jk} \frac{|v|^2}{2} + \frac{1}{2} \int_\Sigma -\partial_t \eta \, |v|^2 + u_j J \mathcal{A}_{jk} e_3 \cdot e_k \, |v|^2 . \end{aligned} \tag{2.7}$$

It is straightforward to verify that $\partial_t J = \partial_t \bar{\eta}/b + \tilde{b} \partial_t \partial_3 \bar{\eta}$ in Ω and that $J \mathcal{A}_{jk} e_3 \cdot e_k = \mathcal{N}_j$ on Σ. Then since u, η satisfy $\partial_k u_j \mathcal{A}_{jk} = 0$ and $\partial_t \eta = u \cdot \mathcal{N}$, we have $I_2 = 0$. Hence

$$I = \partial_t \int_\Omega \frac{|v|^2 J}{2}. \tag{2.8}$$

A similar integration by parts shows that

$$\begin{aligned} II &= \int_\Omega -\mathcal{A}_{jk} S_{ij}(v,q) J \partial_k v_i + \int_\Sigma J \mathcal{A}_{j3} S_{ij}(v,q) v_i \\ &= \int_\Omega -q \mathcal{A}_{ik} \partial_k v_i J + J \frac{|\mathbb{D}_{\mathcal{A}} v|^2}{2} + \int_\Sigma S_{ij}(v,q) \mathcal{N}_j v_i \end{aligned} \tag{2.9}$$

so that

$$II = \int_\Omega -q J F^2 + J \frac{|\mathbb{D}_{\mathcal{A}} v|^2}{2} + \int_\Sigma \zeta \mathcal{N} \cdot v + v \cdot F^3. \tag{2.10}$$

But

$$\int_\Sigma \zeta \mathcal{N} \cdot v = \int_\Sigma \zeta (\partial_t \zeta - F^4) = \partial_t \int_\Sigma \frac{|\zeta|^2}{2} + \int_\Sigma -\zeta F^4, \tag{2.11}$$

which means

$$II = \int_\Omega -q J F^2 + J \frac{|\mathbb{D}_{\mathcal{A}} v|^2}{2} + \partial_t \int_\Sigma \frac{|\zeta|^2}{2} + \int_\Sigma -\zeta F^4. \tag{2.12}$$

Now (2.2) follows from (2.3), (2.8), and (2.12). □

In order to utilize (2.1) we apply the differential operator $\partial^\alpha = \partial_t^{\alpha_0}$ to (1.12). The resulting equations are (2.1) for $v = \partial^\alpha u$, $q = \partial^\alpha p$, and $\zeta = \partial^\alpha \eta$, where

$$F^1 = F^{1,1} + F^{1,2} + F^{1,3} + F^{1,4} + F^{1,5} + F^{1,6} \tag{2.13}$$

for

$$F_i^{1,1} = \sum_{0 < \beta < \alpha} C_{\alpha,\beta} \partial^\beta (\partial_t \bar{\eta} \tilde{b} K) \partial^{\alpha-\beta} \partial_3 u_i + \sum_{0 < \beta \le \alpha} C_{\alpha,\beta} \partial^{\alpha-\beta} \partial_t \bar{\eta} \partial^\beta (\tilde{b} K) \partial_3 u_i \tag{2.14}$$

$$F_i^{1,2} = -\sum_{0 < \beta \le \alpha} C_{\alpha,\beta} \left(\partial^\beta (u_j \mathcal{A}_{jk}) \partial^{\alpha-\beta} \partial_k u_i + \partial^\beta \mathcal{A}_{ik} \partial^{\alpha-\beta} \partial_k p \right) \tag{2.15}$$

$$F_i^{1,3} = \sum_{0 < \beta \le \alpha} C_{\alpha,\beta} \partial^\beta \mathcal{A}_{j\ell} \partial^{\alpha-\beta} \partial_\ell (\mathcal{A}_{im} \partial_m u_j + \mathcal{A}_{jm} \partial_m u_i) \tag{2.16}$$

$$F_i^{1,4} = \sum_{0 < \beta < \alpha} C_{\alpha,\beta} \mathcal{A}_{jk} \partial_k (\partial^\beta \mathcal{A}_{i\ell} \partial^{\alpha-\beta} \partial_\ell u_j + \partial^\beta \mathcal{A}_{j\ell} \partial^{\alpha-\beta} \partial_\ell u_i) \tag{2.17}$$

$$F_i^{1,5} = \partial^\alpha \partial_t \bar{\eta} \tilde{b} K \partial_3 u_i \text{ and } F_i^{1,6} = \mathcal{A}_{jk} \partial_k (\partial^\alpha \mathcal{A}_{i\ell} \partial_\ell u_j + \partial^\alpha \mathcal{A}_{j\ell} \partial_\ell u_i). \tag{2.18}$$

In these equations, the terms $C_{\alpha,\beta}$ are constants that depend on α and β. The term $F^2 = F^{2,1} + F^{2,2}$ for

$$F^{2,1} = -\sum_{0<\beta<\alpha} C_{\alpha,\beta}\partial^\beta \mathcal{A}_{ij}\partial^{\alpha-\beta}\partial_j u_i \text{ and } F^{2,2} = -\partial^\alpha \mathcal{A}_{ij}\partial_j u_i. \tag{2.19}$$

We write $F^3 = F^{3,1} + F^{3,2}$ for

$$F^{3,1} = \sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta D\eta(\partial^{\alpha-\beta}\eta - \partial^{\alpha-\beta}p) \tag{2.20}$$

$$F_i^{3,2} = \sum_{0<\beta\le\alpha} C_{\alpha,\beta}(\partial^\beta(\mathcal{N}_j\mathcal{A}_{im})\partial^{\alpha-\beta}\partial_m u_j + \partial^\beta(\mathcal{N}_j\mathcal{A}_{jm})\partial^{\alpha-\beta}\partial_m u_i). \tag{2.21}$$

Finally,

$$F^4 = \sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta D\eta\cdot\partial^{\alpha-\beta}u. \tag{2.22}$$

2.2. **Perturbed linear form.** Writing the equations in the form (1.12) is more faithful to the geometry of the free boundary problem, but it is inconvenient for many of our a priori estimates. This stems from the fact that if we want to think of the coefficients of the equations for u, p as being frozen for a fixed free boundary given by η, then the underlying linear operator has non-constant coefficients. This makes it unsuitable for applying differential operators.

To get around this problem, in many parts of the paper we will analyze the PDE in a different formulation, which looks like a perturbation of the linearized problem. The utility of this form of the equations lies in the fact that the linear operators have constant coefficients. The equations in this form are

$$\begin{cases} \partial_t u + \nabla p - \Delta u = G^1 & \text{in } \Omega \\ \operatorname{div} u = G^2 & \text{in } \Omega \\ (pI - \mathbb{D}u - \eta I)e_3 = G^3 & \text{on } \Sigma \\ \partial_t \eta - u_3 = G^4 & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b. \end{cases} \tag{2.23}$$

Here we have written $G^1 = G^{1,1} + G^{1,2} + G^{1,3} + G^{1,4} + G^{1,5}$ for

$$G_i^{1,1} = (\delta_{ij} - \mathcal{A}_{ij})\partial_j p \tag{2.24}$$

$$G_i^{1,2} = u_j\mathcal{A}_{jk}\partial_k u_i \tag{2.25}$$

$$G_i^{1,3} = [K^2(1+A^2+B^2) - 1]\partial_{33}u_i - 2AK\partial_{13}u_i - 2BK\partial_{23}u_i \tag{2.26}$$

$$G_i^{1,4} = [-K^3(1+A^2+B^2)\partial_3 J + AK^2(\partial_1 J + \partial_3 A) + BK^2(\partial_2 J + \partial_3 B) - K(\partial_1 A + \partial_2 B)]\partial_3 u_i \tag{2.27}$$

$$G_i^{1,5} = \partial_t\bar{\eta}(1 + x_3/b)K\partial_3 u_i. \tag{2.28}$$

G^2 is the function

$$G^2 = AK\partial_3 u_1 + BK\partial_3 u_2 + (1-K)\partial_3 u_3, \tag{2.29}$$

and G^3 is the vector

$$G^3 := \partial_1\eta\begin{pmatrix} p - \eta - 2(\partial_1 u_1 - AK\partial_3 u_1) \\ -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ -\partial_1 u_3 - K\partial_3 u_1 + AK\partial_3 u_3 \end{pmatrix} + \partial_2\eta\begin{pmatrix} -\partial_2 u_1 - \partial_1 u_2 + BK\partial_3 u_1 + AK\partial_3 u_2 \\ p - \eta - 2(\partial_2 u_2 - BK\partial_3 u_2) \\ -\partial_2 u_3 - K\partial_3 u_2 + BK\partial_3 u_3 \end{pmatrix} + \begin{pmatrix} (K-1)\partial_3 u_1 + AK\partial_3 u_3 \\ (K-1)\partial_3 u_2 + BK\partial_3 u_3 \\ 2(K-1)\partial_3 u_3 \end{pmatrix}. \tag{2.30}$$

Finally,

$$G^4 = -D\eta\cdot u. \tag{2.31}$$

At several points in our analysis, we will need to localize (2.23) by multiplying by a cutoff function. This leads us to consider the energy evolution for a minor modification of (2.23).

Lemma 2.2. *Suppose* (v, q, ζ) *solve*

$$\begin{cases} \partial_t v + \nabla q - \Delta v = \Phi^1 & \text{in } \Omega \\ \operatorname{div} v = \Phi^2 & \text{in } \Omega \\ (qI - \mathbb{D}v)e_3 = a\zeta e_3 + \Phi^3 & \text{on } \Sigma \\ \partial_t \zeta - v_3 = \Phi^4 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{2.32}$$

where either $a = 0$ *or* $a = 1$. *Then*

$$\partial_t \left(\frac{1}{2} \int_\Omega |v|^2 + \frac{1}{2} \int_\Sigma a |\zeta|^2 \right) + \frac{1}{2} \int_\Omega |\mathbb{D}v|^2 = \int_\Omega v \cdot \Phi^1 + q\Phi^2 + \int_\Sigma -v \cdot \Phi^3 + a\zeta\Phi^4. \tag{2.33}$$

Proof. We take the inner-product of the first equation in (2.32) with v and integrate over Ω to find

$$\partial_t \int_\Omega \frac{|v|^2}{2} - \int_\Omega (qI - \mathbb{D}v) : \nabla u + \int_\Sigma (qI - \mathbb{D}v)e_3 \cdot u = \int_\Omega v \cdot \Phi^1. \tag{2.34}$$

We then use the second equation in (2.32) to compute

$$\int_\Omega -(qI - \mathbb{D}v) : \nabla u = \int_\Omega -q \operatorname{div} v + \frac{|\mathbb{D}v|^2}{2} = \int_\Omega -q\Phi^2 + \frac{|\mathbb{D}v|^2}{2}. \tag{2.35}$$

The boundary conditions in (2.32) provide the equality

$$\int_\Sigma (qI - \mathbb{D}v)e_3 \cdot v = \int_\Sigma a\zeta v_3 + v \cdot \Phi^3 = \partial_t \int_\Sigma a\frac{|\zeta|^2}{2} + \int_\Sigma -a\zeta\Phi^4 + v \cdot \Phi^3. \tag{2.36}$$

Combining (2.34)–(2.36) then yields (2.33). □

2.3. **An initial lemma.** The following result is useful for removing the appearance of J factors.

Lemma 2.3. *There exists a universal* $0 < \delta < 1$ *so that if* $\|\eta\|_{5/2}^2 \le \delta$, *then*

$$\|J - 1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \le \frac{1}{2}, \text{ and } \|K\|_{L^\infty}^2 + \|\mathcal{A}\|_{L^\infty}^2 \lesssim 1. \tag{2.37}$$

Proof. According to the definitions of A, B, J given in (1.11) and Lemma A.4, we may bound

$$\|J - 1\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \lesssim \|\bar{\eta}\|_3^2 \lesssim \|\eta\|_{5/2}^2. \tag{2.38}$$

Then if δ is sufficiently small, we find that the first inequality in (2.37) holds. As a consequence $\|K\|_{L^\infty}^2 + \|\mathcal{A}\|_{L^\infty}^2 \lesssim 1$, which is the second inequality in (2.37). □

2.4. **Localization.** Let $0 < b_- := \inf_{x'} b(x')$ and $\sup_{x'} b(x') = b_+ < \infty$. Let $\chi_i \in C_c^\infty(\mathbb{R})$ for $i = 1, 2, 3$ with the property that

$$\begin{cases} \chi_1 = 1 \text{ on } [-3b_-/4, 1] & \text{and } \chi_1 = 0 \text{ on } (-\infty, -7b_-/8) \\ \chi_2 = 1 \text{ on } [-(b_+ + 1), -b_-/2] & \text{and } \chi_2 = 0 \text{ on } (-3b_-/8, \infty) \\ \chi_3 = 1 \text{ on } [-b_-/2, 1] & \text{and } \chi_3 = 0 \text{ on } (-\infty, -5b_-/8). \end{cases} \tag{2.39}$$

We then define the subsets $\Omega_i \subset \Omega$ by

$$\begin{aligned} \Omega_1 &= \{-3b_-/4 \le x_3 \le 0\} \cap \Omega, \\ \Omega_2 &= \{-b_+ \le x_3 \le -b_-/2\} \cap \Omega, \\ \Omega_3 &= \{-b_-/2 \le x_3 \le 0\} \cap \Omega. \end{aligned} \tag{2.40}$$

We will view the functions $\chi_i(x) = \chi_i(x_3)$ as cutoff functions in the vertical direction. They are constructed so that $\chi_1 = 1$ on Ω_i and so that $\Omega = \Omega_1 \cup \Omega_2 = \Omega_3 \cup \Omega_2$, $\Omega_3 \subset \Omega_1$, and $\operatorname{supp}(\nabla\chi_2) \subset \Omega_3$.

When we multiply the equations in (2.23) by χ_i, $i = 1, 2$, we find that $(\chi_i u, \chi_i p, \eta)$ solve

$$\begin{cases} \partial_t(\chi_i u) + \nabla(\chi_i p) - \Delta(\chi_i u) = \chi_i G^1 + H^{1,i} & \text{in } \Omega \\ \operatorname{div}(\chi_i u) = \chi_i G^2 + H^{2,i} & \text{in } \Omega \\ ((\chi_i p)I - \mathbb{D}(\chi_i u))e_3 = \delta_{i,1}\left(\eta e_3 + G^3\right) & \text{on } \Sigma \\ \partial_t \eta - (\chi_1 u_3) = G^4 & \text{on } \Sigma \\ \chi_i u = 0 & \text{on } \Sigma_b, \end{cases} \tag{2.41}$$

where $\delta_{i,1}$ is the Kronecker delta and

$$H^{1,i} = \partial_3 \chi_i (p e_3 - 2\partial_3 u) - \partial_3^2 \chi_i u \text{ and } H^{2,i} = \partial_3 \chi_i u_3. \tag{2.42}$$

The H functions have this form since χ_i is only a function of x_3.

2.5. **Energies and dissipations.** We will consider energies and dissipates at both the $N+2$ and $2N$ levels. To define both at once we consider a generic integer $n \geq 3$. Recall that we use the derivative conventions described in Section 1.8. We define the energy as

$$\mathcal{E}_n = \sum_{j=0}^{n} \left(\left\| \partial_t^j u \right\|_{2n-2j}^2 + \left\| \partial_t^j \eta \right\|_{2n-2j}^2 \right) + \sum_{j=0}^{n-1} \left\| \partial_t^j p \right\|_{2n-2j-1}^2. \tag{2.43}$$

The corresponding dissipation is

$$\begin{aligned} \mathcal{D}_n = \sum_{j=0}^{n} \left\| \partial_t^j u \right\|_{2n-2j+1}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j p \right\|_{2n-2j}^2 \\ + \|\eta\|_{2n-1/2}^2 + \|\partial_t \eta\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \left\| \partial_t^j \eta \right\|_{2n-2j+5/2}^2. \end{aligned} \tag{2.44}$$

For our "horizontal" energies and dissipations, we must use different types of derivatives depending on the localization. In the whole domain we only consider temporal derivatives, writing

$$\bar{\mathcal{E}}_n^0 = \sum_{j=0}^{n} \left\| \sqrt{J} \partial_t^j u \right\|_0^2 + \sum_{j=0}^{n} \left\| \partial_t^j \eta \right\|_0^2 \text{ and } \bar{\mathcal{D}}_n^0 = \sum_{j=0}^{n} \left\| \mathbb{D} \partial_t^j u \right\|_0^2. \tag{2.45}$$

Remark 2.4. *According to Lemma 2.3, if* $\|\eta\|_{5/2}^2 \leq \delta$, *then*

$$\frac{1}{2} \sum_{j=0}^{n} \left\| \partial_t^j u \right\|_0^2 \leq \sum_{j=0}^{n} \left\| \sqrt{J} \partial_t^j u \right\|_0^2 \leq \frac{3}{2} \sum_{j=0}^{n} \left\| \partial_t^j u \right\|_0^2. \tag{2.46}$$

In the upper localization we allow both horizontal spatial derivatives and temporal derivatives, but we do not allow the highest order temporal derivatives:

$$\bar{\mathcal{E}}_n^+ = \left\| \bar{D}_0^{2n-1}(\chi_1 u) \right\|_0^2 + \left\| D\bar{D}^{2n-1}(\chi_1 u) \right\|_0^2 + \left\| \bar{D}_0^{2n-1} \eta \right\|_0^2 + \left\| D\bar{D}^{2n-1} \eta \right\|_0^2, \tag{2.47}$$

$$\bar{\mathcal{D}}_n^+ = \left\| \bar{D}_0^{2n-1} \mathbb{D}(\chi_1 u) \right\|_0^2 + \left\| D\bar{D}^{2n-1} \mathbb{D}(\chi_1 u) \right\|_0^2. \tag{2.48}$$

In the lower localization we only take temporal derivatives, but not all the way to the highest order:

$$\bar{\mathcal{E}}_n^- = \sum_{j=0}^{n-1} \left\| \partial_t^j (\chi_2 u) \right\|_0^2 \text{ and } \bar{\mathcal{D}}_n^- = \sum_{j=0}^{n-1} \left\| \mathbb{D} \partial_t^j (\chi_2 u) \right\|_0^2. \tag{2.49}$$

Our specialized energy terms are

$$\mathcal{F}_{2N} = \|\eta\|_{4N+1/2}^2 \tag{2.50}$$

and

$$\mathcal{K} := \|\nabla u\|_{L^\infty}^2 + \left\|\nabla^2 u\right\|_{L^\infty}^2 + \sum_{i=1}^{2} \|Du_i\|_{H^2(\Sigma)}^2 . \tag{2.51}$$

The total energy we will use in our global well-posedness result is

$$\mathcal{G}_{2N}(t) = \sup_{0\le r\le t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N}(r)dr + \sup_{0\le r\le t} (1+r)^{4N-8}\mathcal{E}_{N+2}(r) + \sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)}. \tag{2.52}$$

3. Nonlinear estimates

3.1. **Estimates of G^i at the $N+2$ level.** We now estimate the G^i terms defined in (2.24)–(2.31) at the $N+2$ level.

Theorem 3.1. *Then there exists a $\theta > 0$ so that*

$$\left\|\bar{\nabla}_0^{2(N+2)-2} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{2(N+2)-2} G^2\right\|_1^2 + \left\|\bar{D}_0^{2(N+2)-2} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{2(N+2)-2} G^4\right\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{E}_{N+2} \tag{3.1}$$

and

$$\left\|\bar{\nabla}_0^{2(N+2)-1} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{2(N+2)-1} G^2\right\|_1^2 + \left\|\bar{D}_0^{2(N+2)-1} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{2(N+2)-1} G^4\right\|_{1/2}^2 + \left\|\bar{D}^{2(N+2)-2} \partial_t G^4\right\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2}. \tag{3.2}$$

Proof. The estimates of these nonlinearities are fairly routine to derive, so for the sake of brevity we present only a sketch. First we note that all terms are quadratic or of higher order. Then we apply the differential operator and expand using the Leibniz rule; each term in the resulting sum is also at least quadratic. We then estimate one term in H^k ($k = 0, 1/2$, or 1 depending on G^i) and the other term in L^∞ or H^m for m depending on k, using Sobolev embeddings, trace theory, and Lemmas A.1, A.3, and A.4. The derivative count in the differential operators is chosen in order to allow estimation by $\mathcal{E}_{N+2}$ in (3.1) and by $\mathcal{D}_{N+2}$ in (3.2). □

3.2. **Estimates of G^i at the $2N$ level.** Now we estimate G^i at the $2N$ level.

Theorem 3.2. *Then there exists a $\theta > 0$ so that*

$$\left\|\bar{\nabla}_0^{4N-2} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{4N-2} G^2\right\|_1^2 + \left\|\bar{D}_0^{4N-2} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{4N-2} G^4\right\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^{1+\theta}, \tag{3.3}$$

$$\begin{aligned} &\left\|\bar{\nabla}_0^{4N-2} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{4N-2} G^2\right\|_1^2 + \left\|\bar{D}_0^{4N-2} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{4N-2} G^4\right\|_{1/2}^2 \\ &+ \left\|\bar{\nabla}^{4N-3} \partial_t G^1\right\|_0^2 + \left\|\bar{\nabla}^{4N-3} \partial_t G^2\right\|_1^2 + \left\|\bar{D}^{4N-3} \partial_t G^3\right\|_{1/2}^2 + \left\|\bar{D}^{4N-2} \partial_t G^4\right\|_{1/2}^2 \\ &\lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{2N}, \end{aligned} \tag{3.4}$$

and

$$\left\|\nabla^{4N-1} G^1\right\|_0^2 + \left\|\nabla^{4N-1} G^2\right\|_1^2 + \left\|D^{4N-1} G^3\right\|_{1/2}^2 + \left\|D^{4N-1} G^4\right\|_{1/2}^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}. \tag{3.5}$$

Proof. The proof of (3.3) and (3.4) proceeds as in Theorem 3.1, using Sobolev embeddings, trace theory, and Lemmas A.1, A.3, and A.4 to estimate $\partial^\alpha G^i$.

We now turn to the derivation of (3.5). Consider $\partial^\alpha G^i$ with $|\alpha| = 4N-1$ and $\alpha_0 = 0$, i.e. purely spatial derivatives, and expand $\partial^\alpha G^i$ using the Leibniz rule. With two exceptions, we may argue as in the derivation of (3.4) to estimate the desired norms of all of the resulting terms by $\mathcal{E}_{2N}^\theta \mathcal{D}_{2N}$ for $\theta > 0$. The exceptional terms are ones involving either $\nabla^{4N+1}\bar{\eta}$ in Ω or $D^{4N}\eta$ on Σ. We will now show how to estimate the exceptional terms with $\mathcal{K}\mathcal{F}_{2N}$, as defined by (2.51) and (2.50).

In $\nabla^{4N-1}G^1$, there are terms of the form $\partial^\beta \bar{\eta} Q \partial^\gamma u$, with

$$Q = Q(A, B, J, K, \nabla A, \nabla B, \nabla J) \tag{3.6}$$

a polynomial and $\beta, \gamma \in \mathbb{N}^3$ with $|\beta| = 4N + 1$ and $|\gamma| = 1$. To estimate such a term, we use Lemma A.3 to bound

$$\left\| \nabla^{4N+1} \bar{\eta} \right\|_0^2 \lesssim \left\| D^{4N+1/2} \eta \right\|_0^2 \lesssim \mathcal{F}_{2N}. \tag{3.7}$$

Sobolev embeddings imply that $\|Q\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N}^\theta \lesssim 1$ for some $\theta > 0$, so

$$\left\| \partial^\beta \bar{\eta} Q \partial^\gamma u \right\|_0^2 \lesssim \left\| \nabla^{4N+1} \bar{\eta} \right\|_0^2 \|\nabla u\|_{L^\infty}^2 \|Q\|_{L^\infty}^2 \lesssim \left\| D^{4N+1/2} \eta \right\|_0^2 \|\nabla u\|_{L^\infty}^2 \lesssim \mathcal{F}_{2N} \mathcal{K}. \tag{3.8}$$

This estimate then yields the G^1 estimate in (3.5).

In $\nabla^{4N-1}G^2$ there are terms of the form $\partial^\beta \bar{\eta} Q \partial^\gamma u$ with $Q = Q(A, B, K)$ a polynomial and $\beta, \gamma \in \mathbb{N}^3$ with $|\beta| = 4N$, $|\gamma| = 1$. Again, Sobolev embeddings imply that $\|Q\|_{C^1(\Omega)}^2 \lesssim \mathcal{E}_{2N}^\theta \lesssim 1$, so

$$\begin{aligned} \left\| \partial^\beta \bar{\eta} Q \partial^\gamma u \right\|_1^2 &\lesssim \|Q\|_{C^1(\Omega)}^2 \left\| \partial^\beta \bar{\eta} \partial^\gamma u \right\|_1^2 \lesssim \left\| \partial^\beta \bar{\eta} \partial^\gamma u \right\|_0^2 + \left\| \partial^\beta \bar{\eta} \nabla \partial^\gamma u \right\|_0^2 + \left\| \nabla \partial^\beta \bar{\eta} \partial^\gamma u \right\|_0^2 \\ &\lesssim \left\| \nabla^{4N} \bar{\eta} \right\|_0^2 \|\nabla u\|_{C^1(\Omega)}^2 + \left\| \nabla^{4N+1} \bar{\eta} \right\|_0^2 \|\nabla u\|_{L^\infty}^2 \lesssim \|\eta\|_{4N-1/2}^2 \|\nabla u\|_3^2 + \mathcal{K} \mathcal{F}_{2N} \\ &\lesssim \mathcal{E}_{2N} \mathcal{D}_{2N} + \mathcal{K} \mathcal{F}_{2N}, \end{aligned} \tag{3.9}$$

where again we have used Lemmas A.3, A.4, and Sobolev embeddings. This estimate yields the G^2 estimate in (3.5).

In $D^{4N-1}G^3$ there are terms of the form $\partial^\beta \eta Q \partial^\gamma u$, where $\beta \in \mathbb{N}^2$ with $|\beta| = 4N$, $\gamma \in \mathbb{N}^3$ with $|\gamma| = 1$, and Q is a term for which we can estimate $\|Q\|_{C^1(\Sigma)}^2 \lesssim \mathcal{E}_{2N}^\theta \lesssim 1$. Then Lemma A.2 implies that

$$\left\| \partial^\beta \eta Q \partial^\gamma u \right\|_{1/2}^2 \lesssim \left\| \partial^\beta \eta \right\|_{1/2}^2 \|Q \partial^\gamma u\|_{C^1}^2 \lesssim \|\eta\|_{4N+1/2}^2 \|Q\|_{C^1}^2 \|\nabla u\|_{C^1(\Sigma)}^2 \lesssim \mathcal{F}_{2N} \mathcal{K}, \tag{3.10}$$

where in the last inequality we have used $\|\nabla u\|_{C^1(\Sigma)}^2 \lesssim \mathcal{K}$, which follows since ∇u and $\nabla^2 u$ are continuous on the closure of Ω. This estimate yields the G^3 estimate in (3.5).

In $D^{4N-1}G^4$ the exceptional terms are of the form $\partial^\beta u_i$, where $\beta \in \mathbb{N}^2$ with $|\beta| = 4N$ and $i = 1, 2$. Then Lemma A.1 implies that

$$\left\| \partial^\beta \eta u_1 \right\|_{1/2}^2 \lesssim \left\| \partial^\beta \eta \right\|_{1/2}^2 \|u_i\|_{H^2(\Sigma)}^2 \lesssim \mathcal{F}_{2N} \mathcal{K}. \tag{3.11}$$

This estimate yields the G^4 estimate in (3.5). □

3.3. **Other nonlinearities.** Now we provide an estimate of for $\partial_t^j \mathcal{A}$ when $j = 2N + 1$ and when $j = N + 3$.

Lemma 3.3. *We have that*

$$\left\| \partial_t^{2N+1} \mathcal{A} \right\|_0^2 \lesssim \mathcal{D}_{2N}, \text{ and } \left\| \partial_t^{N+3} \mathcal{A} \right\|_0^2 \lesssim \mathcal{D}_{N+2}. \tag{3.12}$$

Proof. We will only prove the first estimate in (3.12); the second follows from similar analysis. Since $\left\| \partial_t^{2N+1} \eta \right\|_{1/2}^2 \le \mathcal{D}_{2N}$ and temporal derivatives commute with the Poisson integral, we may employ Lemma A.3 to bound

$$\left\| \partial_t^{2N+1} \bar{\eta} \right\|_1^2 = \left\| \partial_t^{2N+1} \bar{\eta} \right\|_0^2 + \left\| \nabla \partial_t^{2N+1} \bar{\eta} \right\|_0^2 \lesssim \left\| \partial_t^{2N+1} \eta \right\|_{1/2}^2 \le \mathcal{D}_{2N}. \tag{3.13}$$

From this we easily deduce that

$$\left\| \partial_t^{2N+1} J \right\|_0^2 + \left\| \partial_t^{2N+1} K \right\|_0^2 \lesssim \mathcal{D}_{2N}. \tag{3.14}$$

This, the previous bound, and the Sobolev embeddings then imply the first estimate in (3.12) since the components of $\mathcal{A}$ are either unity, K, AK, or BK. □

4. Global energy evolution in the geometric form

4.1. **Estimates of the perturbations when $\partial^\alpha = \partial_t^{\alpha_0}$ is applied to** (1.12)**.** We now present estimates for the perturbations (2.13)–(2.22) when $\partial^\alpha = \partial_t^{\alpha_0}$ for $\alpha_0 \le 2N$.

Theorem 4.1. *Let $\partial^\alpha = \partial_t^{\alpha_0}$ with $\alpha_0 \le 2N$ and let F^1, F^2, F^3, F^4 be defined by* (2.13)*–*(2.22)*. Then*

$$\left\|F^1\right\|_0^2 + \left\|F^2\right\|_0^2 + \left\|\partial_t(JF^2)\right\|_0^2 + \left\|F^3\right\|_0^2 + \left\|F^4\right\|_0 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}. \tag{4.1}$$

Also,

$$\left\|F^2\right\|_0^2 \lesssim \mathcal{E}_{2N}^2. \tag{4.2}$$

Proof. We first consider the F^1 estimate in (4.1). Each term in the sums that define F^1 is at least quadratic. It is straightforward to see that each such term can be written in the form XY, where we X involves fewer temporal derivatives than Y, and we may use the usual Sobolev embeddings and Lemmas A.1, A.3, and A.4 along with the definitions of $\mathcal{E}_{2N}$ and $\mathcal{D}_{2N}$ to estimate

$$\|X\|_{L^\infty}^2 \lesssim \mathcal{E}_{2N} \text{ and } \|Y\|_0^2 \lesssim \mathcal{D}_{2N}. \tag{4.3}$$

Then $\|XY\|_0^2 \le \|X\|_{L^\infty}^2 \|Y\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}$, and the F^1 estimate in (4.1) follows by summing. The first F^2 estimate and the F^2 estimate in (4.2) follow similarly. A similar argument, also employing trace estimates, yields the F^3 and F^4 estimates in (4.1).

The same analysis also works for $\partial_t(JF^{2,1})$ and shows that $\left\|\partial_t(JF^{2,1})\right\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}$. To handle $\partial_t(JF^{2,2})$ when $\alpha_0 = 2N$ we must also be able to estimate $\left\|\partial_t^{2N+1}\mathcal{A}\right\|_0^2 \lesssim \mathcal{D}_{2N}$, but this is possible due to Lemma 3.3. Then a similar splitting into L^∞ and H^0 estimates shows that $\left\|\partial_t(JF^{2,2})\right\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{D}_{2N}$, and then the $\partial_t(JF^2)$ estimate in (4.1) follows since $F^2 = F^{2,1} + F^{2,2}$. □

We now present estimates for these perturbations when $\partial^\alpha = \partial_t^{\alpha_0}$ with $\alpha_0 \le N+2$. The proof may be carried out as in Theorem 4.1, and is thus omitted.

Theorem 4.2. *Let $\partial^\alpha = \partial_t^{\alpha_0}$ with $\alpha_0 \le N+2$ and let F^1, F^2, F^3, F^4 be defined by* (2.13)*–*(2.22)*. Then*

$$\left\|F^1\right\|_0^2 + \left\|F^2\right\|_0^2 + \left\|\partial_t(JF^2)\right\|_0^2 + \left\|F^3\right\|_0^2 + \left\|F^4\right\|_0 \lesssim \mathcal{E}_{2N}\mathcal{D}_{N+2}. \tag{4.4}$$

Also,

$$\left\|F^2\right\|_0^2 \lesssim \mathcal{E}_{2N}\mathcal{E}_{N+2}. \tag{4.5}$$

4.2. **Global energy evolution with only temporal derivatives.** Now we present the applications of Theorems 4.1 and 4.2.

Proposition 4.3. *There exists a $\theta > 0$ so that*

$$\bar{\mathcal{E}}_{2N}^0(t) + \int_0^t \bar{\mathcal{D}}_{2N}^0 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N}. \tag{4.6}$$

Proof. We apply $\partial^\alpha = \partial_t^{\alpha_0}$ with $0 \le \alpha_0 \le 2N$ to (1.12). Then $v = \partial_t^{\alpha_0} u$, $q = \partial_t^{\alpha_0} p$, and $\zeta = \partial_t^{\alpha_0}\eta$ solve (2.1) with F^i, $i = 1,2,3,4$ given by (2.13)–(2.22). Applying Lemma 2.1 to these functions and then integrating in time from 0 to t gives

$$\begin{aligned}\frac{1}{2}\int_\Omega J\left|\partial_t^{\alpha_0}u(t)\right|^2 + \frac{1}{2}\int_\Sigma \left|\partial_t^{\alpha_0}\eta(t)\right|^2 + \frac{1}{2}\int_0^t\int_\Omega J\left|\mathbb{D}_{\mathcal{A}}\partial_t^{\alpha_0}u\right|^2 = \frac{1}{2}\int_\Omega J\left|\partial_t^{\alpha_0}u(0)\right|^2 \\ + \frac{1}{2}\int_\Sigma \left|\partial_t^{\alpha_0}\eta(0)\right|^2 + \int_0^t\int_\Omega J(\partial_t^{\alpha_0}u\cdot F^1 + \partial_t^{\alpha_0}pF^2) + \int_0^t\int_\Sigma -\partial_t^{\alpha_0}u\cdot F^3 + \partial_t^{\alpha_0}\eta F^4.\end{aligned} \tag{4.7}$$

We claim that for $0 \le \alpha_0 \le 2N$ we have the estimate

$$\left\|\sqrt{J}\partial_t^{\alpha_0}u(t)\right\|_0^2 + \left\|\partial_t^{\alpha_0}\eta(t)\right\|_0^2 + \int_0^t \left\|\mathbb{D}\partial_t^{\alpha_0}u\right\|_0^2 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N}. \tag{4.8}$$

Once the claim is established, we may sum over α_0 to deduce (4.6).

We will estimate all of the terms involving F^i on the right side of (4.7), beginning with the F^1 term. According to Theorem 4.1 and Lemma 2.3, we may bound

$$\int_0^t \int_\Omega J\partial_t^{\alpha_0} u\cdot F^1 \le \int_0^t \|\partial_t^{\alpha_0} u\|_0 \|J\|_{L^\infty} \left\|F^1\right\|_0 \lesssim \int_0^t \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} = \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{4.9}$$

Similarly, we use Theorem 4.1 and trace theory to handle the F^3 and F^4 terms:

$$\begin{aligned} \int_0^t \int_\Sigma -\partial_t^{\alpha_0} u\cdot F^3 + \partial_t^{\alpha_0}\eta F^4 &\le \int_0^t \|\partial_t^{\alpha_0} u\|_{H^0(\Sigma)} \left\|F^3\right\|_0 + \|\partial_t^{\alpha_0}\eta\|_0 \left\|F^4\right\|_0 \\ &\lesssim \int_0^t \left(\|\partial_t^{\alpha_0} u\|_1 + \|\partial_t^{\alpha_0}\eta\|_0\right)\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} \lesssim \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \end{aligned} \tag{4.10}$$

For the term $\partial_t^{\alpha_0} pF^2$ we must consider the cases $\alpha_0 < 2N$ and $\alpha_0 = 2N$ separately. The case $\alpha_0 = 2N$ is more delicate, so we begin with it. In this case, there is one more time derivative on p than can be controlled by $\mathcal{D}_{2N}$. We are then forced to integrate by parts in time:

$$\int_0^t \int_\Omega \partial_t^{2N} pJF^2 = -\int_0^t \int_\Omega \partial_t^{2N-1} p\partial_t(JF^2) + \int_\Omega (\partial_t^{2N-1} pJF^2)(t) - \int_\Omega (\partial_t^{2N-1} pJF^2)(0). \tag{4.11}$$

Then according to Theorem 4.1 we may estimate

$$-\int_0^t \int_\Omega \partial_t^{2N-1} p\partial_t(JF^2) \lesssim \int_0^t \left\|\partial_t^{2N-1} p\right\|_0 \left\|\partial_t(JF^2)\right\|_0 \lesssim \int_0^t \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} = \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{4.12}$$

On the other hand, it is easy to verify using (4.2) that

$$\int_\Omega (\partial_t^{2N-1} pJF^2)(t) - \int_\Omega (\partial_t^{2N-1} pJF^2)(0) \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2}. \tag{4.13}$$

Hence

$$\int_0^t \int_\Omega \partial_t^{2N} pJF^2 \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{4.14}$$

On the other hand, if $0 \le \alpha_0 < 2N$, then we may control $\partial_t^{\alpha_0} p$ directly using the F^2 estimate in Theorem 4.1:

$$\int_0^t \int_\Omega \partial_t^{\alpha_0} pJF^2 \lesssim \int_0^t \|\partial_t^{\alpha_0} p\|_0 \left\|F^2\right\|_0 \lesssim \int_0^t \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} = \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N}. \tag{4.15}$$

Now we combine (4.9), (4.10), and (4.14)–(4.15) to deduce that

$$\begin{aligned} \frac{1}{2}\int_\Omega J\left|\partial_t^{\alpha_0} u(t)\right|^2 + \frac{1}{2}\int_\Sigma \left|\partial_t^{\alpha_0}\eta(t)\right|^2 + \frac{1}{2}\int_0^t \int_\Omega J\left|\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u\right|^2 \\ \lesssim \mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N} \end{aligned} \tag{4.16}$$

for all $0 \le \alpha_0 \le 2N$.

We now seek to replace $J\left|\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u\right|^2$ with $\left|\mathbb{D}\partial_t^{\alpha_0} u\right|^2$ in (4.16). To this end we write

$$J\left|\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u\right|^2 = \left|\mathbb{D}\partial_t^{\alpha_0} u\right|^2 + (J-1)\left|\mathbb{D}\partial_t^{\alpha_0} u\right|^2 + J\left(\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u + \mathbb{D}\partial_t^{\alpha_0} u\right) : \left(\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u - \mathbb{D}\partial_t^{\alpha_0} u\right) \tag{4.17}$$

and estimate the last three terms on the right side. For the last term we note that

$$\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u \pm \mathbb{D}\partial_t^{\alpha_0} u = (\mathcal{A}_{ik} \pm \delta_{ik})\partial_k\partial_t^{\alpha_0} u_j + (\mathcal{A}_{jk} \pm \delta_{jk})\partial_k\partial_t^{\alpha_0} u_i \tag{4.18}$$

so that Sobolev embeddings and Lemmas A.3 and A.4 provide the bounds

$$\left|\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u - \mathbb{D}\partial_t^{\alpha_0} u\right| \lesssim \sqrt{\mathcal{E}_{2N}}\left|\nabla\partial_t^{\alpha_0} u\right| \text{ and } \left|\mathbb{D}_\mathcal{A}\partial_t^{\alpha_0} u + \mathbb{D}\partial_t^{\alpha_0} u\right| \lesssim (1+\sqrt{\mathcal{E}_{2N}})\left|\nabla\partial_t^{\alpha_0} u\right|. \tag{4.19}$$

We then get

$$(4.20)\quad \int_0^t \int_\Omega \left| J \left(\mathbb{D}_{\mathcal{A}} \partial_t^{\alpha_0} u + \mathbb{D} \partial_t^{\alpha_0} u \right) : \left(\mathbb{D}_{\mathcal{A}} \partial_t^{\alpha_0} u - \mathbb{D} \partial_t^{\alpha_0} u \right) \right| \lesssim \int_0^t (\sqrt{\mathcal{E}_{2N}} + \mathcal{E}_{2N}) \int_\Omega \left| \nabla \partial_t^{\alpha_0} u \right|^2 \lesssim \int_0^t \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}.$$

Similarly,

$$(4.21)\quad \int_0^t \int_\Omega |J - 1| \left| \mathbb{D} \partial_t^{\alpha_0} u \right|^2 \lesssim \int_0^t \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}.$$

We may then use (4.17) and (4.20)–(4.21) to replace in (4.16) and derive the bound (4.8). This completes the proof of the claim and of the proposition. □

Now we present the corresponding estimate at the $N + 2$ level.

Proposition 4.4. *Let F^2 be given by (2.19) with $\partial^\alpha = \partial_t^{N+2}$. Then*

$$(4.22)\quad \partial_t \left(\bar{\mathcal{E}}_{N+2}^0 - 2 \int_\Omega J \partial_t^{N+1} p F^2 \right) + \bar{\mathcal{D}}_{N+2}^0 \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}.$$

Proof. We apply $\partial^\alpha = \partial_t^{\alpha_0}$ to (1.12) for $0 \le \alpha_0 \le N + 2$. Then $v = \partial_t^{\alpha_0} u$, $q = \partial_t^{\alpha_0} p$, and $\zeta = \partial_t^{\alpha_0} \eta$ solve (2.1) with F^i, $i = 1, 2, 3, 4$ given by (2.13)–(2.22). Applying Lemma 2.1 to these functions gives

$$(4.23)\quad \partial_t \left(\frac{1}{2} \int_\Omega J \left| \partial_t^{\alpha_0} u \right|^2 + \frac{1}{2} \int_\Sigma \left| \partial_t^{\alpha_0} \eta \right|^2 \right) + \frac{1}{2} \int_\Omega J \left| \mathbb{D}_{\mathcal{A}} \partial_t^{\alpha_0} u \right|^2 = \int_\Omega J (\partial_t^{\alpha_0} u \cdot F^1 + \partial_t^{\alpha_0} p F^2) + \int_\Sigma -\partial_t^{\alpha_0} u \cdot F^3 + \partial_t^{\alpha_0} \eta F^4.$$

We claim that for $0 \le \alpha_0 \le N + 2$ we have the estimate

$$(4.24)\quad \partial_t \left(\left\| \sqrt{J} \partial_t^{\alpha_0} u(t) \right\|_0^2 + \left\| \partial_t^{\alpha_0} \eta(t) \right\|_0^2 - \delta_{\alpha_0, N+2} 2 \int_\Omega J \partial_t^{N+1} p F^2 \right) + \left\| \mathbb{D} \partial_t^{\alpha_0} u \right\|_0^2 \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2},$$

where $\delta_{\alpha_0, N+2} = 1$ if $\alpha_0 = N + 2$ and 0 otherwise. Once the claim is established, we may sum over α_0 to deduce (4.22).

We will estimate all of the terms involving F^i on the right side of (4.23) as in Proposition 4.3. We begin with the F^1 term. According to Theorem 4.2 and Lemma 2.3, we may bound

$$(4.25)\quad \int_\Omega J \partial_t^{\alpha_0} u \cdot F^1 \le \left\| \partial_t^{\alpha_0} u \right\|_0 \|J\|_{L^\infty} \left\| F^1 \right\|_0 \lesssim \sqrt{\mathcal{D}_{N+2}} \sqrt{\mathcal{E}_{2N} \mathcal{D}_{N+2}} = \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}.$$

Similarly, we use Theorem 4.2 and trace theory to handle the F^3 and F^4 terms:

$$(4.26)\quad \int_\Sigma -\partial_t^{\alpha_0} u \cdot F^3 + \partial_t^{\alpha_0} \eta F^4 \le \left\| \partial_t^{\alpha_0} u \right\|_{H^0(\Sigma)} \left\| F^3 \right\|_0 + \left\| \partial_t^{\alpha_0} \eta \right\|_0 \left\| F^4 \right\|_0 \lesssim \left(\left\| \partial_t^{\alpha_0} u \right\|_1 + \left\| \partial_t^{\alpha_0} \eta \right\|_0 \right) \sqrt{\mathcal{E}_{2N} \mathcal{D}_{N+2}} \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}.$$

For the term $\partial_t^{\alpha_0} p F^2$ we must consider the cases $\alpha_0 = N + 2$ and $0 \le \alpha_0 < N + 2$ separately. When $\alpha_0 = N + 2$ there is one more time derivative on p than can be controlled by $\mathcal{D}_{N+2}$. We are then forced to pull out a time derivative:

$$(4.27)\quad \int_\Omega \partial_t^{N+2} p J F^2 = \partial_t \int_\Omega \partial_t^{N+1} p J F^2 - \int_\Omega \partial_t^{N+1} p \partial_t (J F^2).$$

Then according to Theorem 4.2 we may estimate

$$(4.28)\quad -\int_\Omega \partial_t^{N+1} p \partial_t (J F^2) \lesssim \left\| \partial_t^{N+1} p \right\|_0 \left\| \partial_t (J F^2) \right\|_0 \lesssim \sqrt{\mathcal{D}_{N+2}} \sqrt{\mathcal{E}_{2N} \mathcal{D}_{N+2}} = \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}.$$

Hence

$$(4.29)\quad \int_0^t \int_\Omega \partial_t^{2N} p J F^2 \lesssim \partial_t \int_\Omega \partial_t^{N+1} p J F^2 + \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}.$$

On the other hand, when $0 \le \alpha_0 < N+2$ we may control $\partial_t^{\alpha_0} p$ directly:

$$\int_\Omega J \partial_t^{\alpha_0} p F^2 \lesssim \left\| \partial_t^{\alpha_0} p \right\|_0 \left\| F^2 \right\|_0 \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}. \tag{4.30}$$

Now we combine (4.23)–(4.26) and (4.29)–(4.30) to deduce that

$$\partial_t \left(\frac{1}{2} \int_\Omega J \left| \partial_t^{\alpha_0} u \right|^2 + \frac{1}{2} \int_\Sigma \left| \partial_t^{\alpha_0} \eta \right|^2 - \delta_{\alpha_0, N+2} \int_\Omega \partial_t^{N+1} p J F^2 \right) + \frac{1}{2} \int_\Omega J \left| \mathbb{D}_{\mathcal{A}} \partial_t^{\alpha_0} u \right|^2 \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}. \tag{4.31}$$

We may argue as in (4.17)–(4.21) of Theorem 4.3 to show that

$$\frac{1}{2} \int_\Omega \left| \mathbb{D} \partial_t^{\alpha_0} u \right|^2 \lesssim \frac{1}{2} \int_\Omega J \left| \mathbb{D}_{\mathcal{A}} \partial_t^{\alpha_0} u \right|^2 + \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}. \tag{4.32}$$

Then (4.24) follows from (4.31) and (4.32), which completes the proof of the claim and the proposition. □

5. Localized energy evolution using the perturbed linear form

5.1. Upper localization. We now estimate how the upper-localization energies evolve. In order to analyze the upper localization, we will use the equation (2.41) with $i = 1$.

First we need a technical lemma that estimates the term $\partial^\alpha \eta \partial^\alpha G^4$ when α is the highest spatial derivatives.

Lemma 5.1. *Let $\alpha \in \mathbb{N}^2$ be such that $|\alpha| = 4N$, i.e. let ∂^α be $4N$ spatial derivatives in the x_1, x_2 directions. Then*

$$\left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right| \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N} \mathcal{K} \mathcal{F}_{2N}}. \tag{5.1}$$

Proof. Throughout the proof β will always denote an element of $\mathbb{N}^2$, and we will write $Df \cdot \partial^\beta u = \partial_1 f \partial^\beta u_1 + \partial_2 f \partial^\beta u_2$ for a function f defined on Σ. Then by the Leibniz rule, we have that

$$\partial^\alpha G^4 = \partial^\alpha (D\eta \cdot u) = D\partial^\alpha \eta \cdot u + \sum_{\substack{0 < \beta \le \alpha \\ |\beta| = 1}} C_{\alpha,\beta} D \partial^{\alpha - \beta} \eta \cdot \partial^\beta u + \sum_{\substack{0 < \beta \le \alpha \\ |\beta| \ge 2}} C_{\alpha,\beta} D \partial^{\alpha - \beta} \eta \cdot \partial^\beta u \tag{5.2}$$

for constants $C_{\alpha,\beta}$ depending on α and β. We will analyze each of the three terms on the right separately.

For the first term, we integrate by parts to see that.

$$\int_\Sigma \partial^\alpha \eta D \partial^\alpha \eta \cdot u = \frac{1}{2} \int_\Sigma D \left| \partial^\alpha \eta \right|^2 \cdot u = -\frac{1}{2} \int_\Sigma \partial^\alpha \eta \partial^\alpha \eta (\partial_1 u_1 + \partial_2 u_2). \tag{5.3}$$

This then allows us to use (A.3) of Lemma A.1 to bound

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta D \partial^\alpha \eta \cdot u \right| &\lesssim \left\| \partial^\alpha \eta \right\|_{1/2} \left\| \partial^\alpha \eta (\partial_1 u_1 + \partial_2 u_2) \right\|_{H^{-1/2}(\Sigma)} \\ &\le \left\| \eta \right\|_{4N+1/2} \left\| \partial^\alpha \eta \right\|_{-1/2} \left\| \partial_1 u_1 + \partial_2 u_2 \right\|_{H^2(\Sigma)} \\ &\le \left\| \eta \right\|_{4N+1/2} \left\| D\eta \right\|_{4N-3/2} \left\| \partial_1 u_1 + \partial_2 u_2 \right\|_{H^2(\Sigma)} \le \sqrt{\mathcal{F}_{2N} \mathcal{D}_{2N} \mathcal{K}}. \end{aligned} \tag{5.4}$$

Similarly, for the second term we estimate

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta \sum_{\substack{0 < \beta \le \alpha \\ |\beta| = 1}} C_{\alpha,\beta} D \partial^{\alpha - \beta} \eta \cdot \partial^\beta u \right| &\lesssim \left\| D^{4N} \eta \right\|_{1/2} \left\| D^{4N} \eta \right\|_{-1/2} \sum_{i=1}^2 \left\| D u_i \right\|_{H^2(\Sigma)} \\ &\le \left\| \eta \right\|_{4N+1/2} \left\| D\eta \right\|_{4N-3/2} \sum_{i=1}^2 \left\| D u_i \right\|_{H^2(\Sigma)} \le \sqrt{\mathcal{F}_{2N} \mathcal{D}_{2N} \mathcal{K}}. \end{aligned} \tag{5.5}$$

For the third term we first note that $\|\partial^\alpha \eta\|_{-1/2} \le \|D\eta\|_{4N-3/2} \le \sqrt{\mathcal{D}_{2N}}$, which allows us to bound

$$\left| \int_\Sigma \partial^\alpha \eta D \partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right| \le \|\partial^\alpha \eta\|_{-1/2} \left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)} \le \sqrt{\mathcal{D}_{2N}} \left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)}. \tag{5.6}$$

We estimate the last term on the right using Lemma A.1, but in different ways depending on $|\beta|$:

$$\left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)} \lesssim \begin{cases} \left\| D\partial^{\alpha-\beta}\eta \right\|_{1/2} \left\| \partial^\beta u \right\|_{H^2(\Sigma)} & \text{for } 2 \le |\beta| \le 2N \\ \left\| D\partial^{\alpha-\beta}\eta \right\|_{2} \left\| \partial^\beta u \right\|_{H^{1/2}(\Sigma)} & \text{for } 2N+1 \le |\beta| \le 4N \end{cases}$$
$$\lesssim \begin{cases} \|D\eta\|_{4N-3/2} \|u\|_{2N+3} & \text{for } 2 \le |\beta| \le 2N \\ \|D\eta\|_{2N+1} \|u\|_{4N+1} & \text{for } 2N+1 \le |\beta| \le 4N \end{cases}, \tag{5.7}$$

so that $\left\| D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right\|_{H^{1/2}(\Sigma)} \lesssim \sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}}$ for all $0 < \beta \le \alpha$ with $|\beta| \ge 2$. Hence

$$\left| \int_\Sigma \partial^\alpha \eta \sum_{\substack{0<\beta\le\alpha \\ |\beta|\ge 2}} C_{\alpha,\beta} D\partial^{\alpha-\beta}\eta \cdot \partial^\beta u \right| \lesssim \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}} = \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{2N}. \tag{5.8}$$

The estimate (5.1) then follows from (5.4), (5.5), and (5.8). □

Now we estimate the upper-localization energy at the $2N$ level.

Proposition 5.2. *Let $\alpha \in \mathbb{N}^{1+2}$ so that $\alpha_0 \le 2N-1$ and $|\alpha| \le 4N$. Then for any $\varepsilon \in (0,1)$ it holds that*

$$\|\partial^\alpha(\chi_1 u)\|_0^2 + \|\partial^\alpha \eta\|_0^2 + \int_0^t \|\mathbb{D}\partial^\alpha(\chi_1 u)\|_0^2 \lesssim \bar{\mathcal{E}}_{2n}^+(0) + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2n}\mathcal{K}\mathcal{F}_{2N}} + \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}_{2N}^0. \tag{5.9}$$

In particular,

$$\bar{\mathcal{E}}_{2N}^+(t) + \int_0^t \bar{\mathcal{D}}_{2N}^+ \lesssim \bar{\mathcal{E}}_{2N}^+(0) + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2n}\mathcal{K}\mathcal{F}_{2N}} + \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}_{2N}^0. \tag{5.10}$$

Proof. We divide the proof into several steps.

Step 1 – Evolution equation

We apply Lemma 2.2 to $v = \chi_1 \partial^\alpha u$, $q = \chi_1 \partial^\alpha p$, $\zeta = \partial^\alpha \eta$ with $a = 1$, $\Phi^1 = \chi_1 \partial^\alpha G^1 + \partial^\alpha H^{1,1}$, $\Phi^2 = \chi_1 \partial^\alpha G^2 + \partial^\alpha H^{2,1}$, $\Phi^3 = \partial^\alpha G^3$, and $\Phi^4 = \partial^\alpha G^4$ to find

$$\partial_t \left(\frac{1}{2}\int_\Omega |\partial^\alpha(\chi_1 u)|^2 + \frac{1}{2}\int_\Sigma |\partial^\alpha \eta|^2 \right) + \frac{1}{2}\int_\Omega |\mathbb{D}\partial^\alpha(\chi_1 u)|^2 = \int_\Omega \chi_1 \partial^\alpha u \cdot (\chi_1 \partial^\alpha G^1 + \partial^\alpha H^{1,1}) + \int_\Omega \chi_1 \partial^\alpha p (\chi_1 \partial^\alpha G^2 + \partial^\alpha H^{2,1}) + \int_\Sigma -\partial^\alpha u \cdot \partial^\alpha G^3 + \partial^\alpha \eta \partial^\alpha G^4. \tag{5.11}$$

Here $H^{1,1}$ and $H^{2,1}$ are given by (2.42).

Step 2 – Estimates of terms involving $H^{1,1}$ and $H^{2,1}$

We will estimate the terms on the right side of (5.11), beginning with the terms involving $H^{1,1}$ and $H^{2,1}$. Since χ_1 is only a function of x_3, we have that

$$\partial^\alpha H^{1,1} = \partial_3\chi_1(\partial^\alpha p e_3 - 2\partial^\alpha \partial_3 u) - \partial_3^2 \chi_1 \partial^\alpha u \text{ and } \partial^\alpha H^{2,1} = \partial_3 \chi_1 \partial^\alpha u_3. \tag{5.12}$$

This and the constraints on α allow us to estimate

$$(5.13)\quad \int_\Omega \chi_1 \partial^\alpha u \cdot \partial^\alpha H^{1,1} + \chi_1 \partial^\alpha p \partial^\alpha H^{2,1} \lesssim \|\partial^\alpha u\|_0 \left(\|\partial^\alpha p\|_0 + \|\partial^\alpha u\|_1\right) + \|\partial^\alpha p\|_0 \|\partial^\alpha u\|_0$$
$$\lesssim \|\partial^\alpha u\|_0 \left(\|\partial^\alpha p\|_0 + \|\partial^\alpha u\|_1\right) \lesssim \left\| D_0^{4N-2\alpha_0} \partial_t^{\alpha_0} u \right\|_0 \sqrt{\mathcal{D}_{2N}} \lesssim \sqrt{\mathcal{D}_{2N}} \left\|\partial_t^{\alpha_0} u\right\|_{4N-2\alpha_0}$$

We estimate the $4N - 2\alpha_0$ norm with standard Sobolev interpolation:

$$(5.14)\quad \|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0} \lesssim \|\partial_t^{\alpha_0} u\|_0^\theta \|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0+1}^{1-\theta} \le (\bar{\mathcal{D}}_{2N}^0)^{\theta/2} (\mathcal{D}_{2N})^{(1-\theta)/2},$$

where $\theta = (4N - 2\alpha_0 + 1)^{-1} \in (0, 1)$. Then Young's inequality allows us to further bound

$$(5.15)\quad \sqrt{\mathcal{D}_{2N}} \|\partial_t^{\alpha_0} u\|_{4N-2\alpha_0} \lesssim \sqrt{\mathcal{D}_{2N}} (\bar{\mathcal{D}}_{2N}^0)^{\theta/2} (\mathcal{D}_{2N})^{(1-\theta)/2} = (\bar{\mathcal{D}}_{2N}^0)^{\theta/2} (\mathcal{D}_{2N})^{1-\theta/2}$$
$$\le \varepsilon \left(1 - \frac{\theta}{2}\right) \mathcal{D}_{2N} + \frac{\theta}{2} \varepsilon^{(\theta-2)/\theta} \bar{\mathcal{D}}_{2N}^0 \le \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1} \bar{\mathcal{D}}_{2N}^0,$$

where in the last inequality we have used the fact that $(2 - \theta)/\theta = 8N - 4\alpha_0 + 1$ to find the largest power of $1/\varepsilon$ when $0 \le \alpha_0 \le 2N$. Chaining together (5.13) and (5.15) then yields the bound

$$(5.16)\quad \int_\Omega \chi_1 \partial^\alpha u \cdot \partial^\alpha H^{1,1} + \chi_1 \partial^\alpha p \partial^\alpha H^{2,1} \lesssim \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1} \bar{\mathcal{D}}_{2N}^0.$$

Step 3 – Terms involving G^i, $1 \le i \le 4$

We now turn to estimates of the terms involving G^i, $1 \le i \le 4$. We claim that

$$(5.17)\quad \int_\Omega \chi_1^2 \left(\partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2\right) \lesssim (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N} \mathcal{K} \mathcal{F}_{2N}}$$

and

$$(5.18)\quad \int_\Sigma -\partial^\alpha u \cdot \partial^\alpha G^3 + \partial^\alpha \eta \partial^\alpha G^4 \lesssim (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N} \mathcal{K} \mathcal{F}_{2N}}$$

for some $\theta > 0$.

To prove the claim, we assume initially that $1 \le |\alpha| \le 4N - 1$. Then according to the estimates (3.4)–(3.5) of Theorem 3.2 and the definition of $\mathcal{D}_{2N}$, we have

$$(5.19)\quad \left| \int_\Omega \chi_1^2 \left(\partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2\right) \right| \lesssim \|\partial^\alpha u\|_0 \left\|\partial^\alpha G^1\right\|_0 + \|\partial^\alpha p\|_0 \left\|\partial^\alpha G^2\right\|_0$$
$$\lesssim \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K} \mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N} \mathcal{K} \mathcal{F}_{2N}},$$

where in the last equality we have written $\kappa = \theta/2$ for $\theta > 0$ the number provided by Theorem 3.2. Similarly, we may use Theorem 3.2 along with the trace estimate $\|\partial^\alpha u\|_{H^0(\Sigma)} \lesssim \|\partial^\alpha u\|_1 \le \sqrt{\mathcal{D}_{2N}}$ to find that

$$(5.20)\quad \left| \int_\Sigma -\partial^\alpha u \cdot \partial^\alpha G^3 + \partial^\alpha \eta \partial^\alpha G^4 \right| \le \|\partial^\alpha u\|_{H^0(\Sigma)} \left\|\partial^\alpha G^3\right\|_0 + \|\partial^\alpha \eta\|_0 \left\|\partial^\alpha G^4\right\|_0$$
$$\lesssim \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K} \mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N} \mathcal{K} \mathcal{F}_{2N}}.$$

Now assume that $|\alpha| = 4N$. Since $\alpha_0 \le 2N - 1$, we may write $\alpha = \beta + (\alpha - \beta)$ for some $\beta \in \mathbb{N}^2$ with $|\beta| = 1$, i.e. ∂^α involves at least one spatial derivative. Since $|\alpha - \beta| = 4N - 1$, we can then integrate by parts and use (3.5) of Theorem 3.2 to see that

$$(5.21)\quad \left| \int_\Omega \chi_1^2 \partial^\alpha u \cdot \partial^\alpha G^1 \right| = \left| \int_\Omega \chi_1^2 \partial^{\alpha+\beta} u \cdot \partial^{\alpha-\beta} G^1 \right| \lesssim \left\|\partial^{\alpha+\beta} u\right\|_0 \left\|\partial^{\alpha-\beta} G^1\right\|_0$$
$$\le \|\partial^\alpha u\|_1 \left\|\bar{\nabla}^{4N-1} G^1\right\|_0 \lesssim \sqrt{\mathcal{D}_{2N}} \sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K} \mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N} \mathcal{K} \mathcal{F}_{2N}}.$$

For the pressure term we do not need to integrate by parts:

$$\left|\int_\Omega \chi_1^2 \partial^\alpha p \partial^\alpha G^2\right| \lesssim \|\partial^\alpha p\|_0 \left\|\partial^{\alpha-\beta}\partial^\beta G^1\right\|_0 \le \|\partial^\alpha p\|_0 \left\|\bar\nabla^{4N-1} G^1\right\|_1 \\ \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \tag{5.22}$$

We integrate by parts and use the trace estimate $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$ to see that

$$\left|\int_\Sigma \partial^\alpha u \cdot \partial^\alpha G^3\right| = \left|\int_\Sigma \partial^{\alpha+\beta} u \cdot \partial^{\alpha-\beta} G^3\right| \le \left\|\partial^{\alpha+\beta} u\right\|_{H^{-1/2}(\Sigma)} \left\|\partial^{\alpha-\beta} G^3\right\|_{1/2} \\ \le \|\partial^\alpha u\|_{H^{1/2}(\Sigma)} \left\|\bar D^{4N-1} G^3\right\|_{1/2} \le \|\partial^\alpha u\|_1 \left\|\bar D^{4N-1} G^3\right\|_{1/2} \\ \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \tag{5.23}$$

For the term $\partial^\alpha \eta \partial^\alpha G^4$ we must split to two cases: $\alpha_0 \ge 1$ and $\alpha_0 = 0$. In the former case, there is at least one temporal derivative in ∂^α, so $\|\partial^\alpha \eta\|_{1/2} \le \sqrt{\mathcal{D}_{2N}}$, and hence

$$\left|\int_\Sigma \partial^\alpha \eta \partial^\alpha G^4\right| = \left|\int_\Sigma \partial^{\alpha+\beta} \eta \partial^{\alpha-\beta} G^4\right| \le \left\|\partial^{\alpha+\beta}\eta\right\|_{-1/2} \left\|\partial^{\alpha-\beta} G^4\right\|_{1/2} \\ \le \|\partial^\alpha \eta\|_{1/2} \left\|\bar D^{4N-1} G^3\right\|_{1/2} \lesssim \sqrt{\mathcal{D}_{2N}}\sqrt{\mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}} \lesssim \mathcal{E}_{2N}^\kappa \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \tag{5.24}$$

In the latter case, $\alpha_0 = 0$, so that ∂^α involves only spatial derivatives; in this case we use Lemma 5.1 to bound

$$\left|\int_\Sigma \partial^\alpha \eta \partial^\alpha G^4\right| \lesssim \sqrt{\mathcal{E}_{2N}}\mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}. \tag{5.25}$$

Now, owing to the estimates in (5.19)–(5.25) we know that (5.17) and (5.18) hold. This completes the proof of the claim.

Step 4 – Conclusion

Now, in light of (5.11) and (5.16)–(5.18), we have

$$\partial_t \left(\int_\Omega |\partial^\alpha(\chi_1 u)|^2 + \int_\Sigma |\partial^\alpha \eta|^2 \right) + \int_\Omega |\mathbb{D}\partial^\alpha(\chi_1 u)|^2 \\ \lesssim (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}} + \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}_{2N}^0 \tag{5.26}$$

for all $|\alpha| \le 4N$ with $\alpha_0 \le 2N-1$. The estimate (5.9) then follows from (5.26) by integrating in time from 0 to t, and then (5.10) follows from (5.9) by summing over α. □

Now we prove a similar estimate at the $N+2$ level.

Proposition 5.3. *Let $\alpha \in \mathbb{N}^{1+2}$ so that $\alpha_0 \le N+1$ and $|\alpha| \le 2(N+2)$. Then for any $\varepsilon \in (0,1)$ it holds that*

$$\partial_t \left(\|\partial^\alpha(\chi_1 u)\|_0^2 + \|\partial^\alpha \eta\|_0^2 \right) + \|\mathbb{D}\partial^\alpha(\chi_1 u)\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2} + \varepsilon \mathcal{D}_{N+2} + \varepsilon^{-4N-9}\bar{\mathcal{D}}_{N+2}^0. \tag{5.27}$$

In particular,

$$\partial_t \bar{\mathcal{E}}_{N+2}^+ + \bar{\mathcal{D}}_{N+2}^+ \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2} + \varepsilon \mathcal{D}_{N+2} + \varepsilon^{-4N-9}\bar{\mathcal{D}}_{N+2}^0. \tag{5.28}$$

Proof. We divide the proof into steps as in Proposition 5.2.

Step 1 – Energy evolution

We argue as in Step 1 of Proposition 5.2 to see that (5.11) holds for the present range of α.

Step 2 – Estimates of terms involving $H^{1,1}$ and $H^{2,1}$

We have the estimate

$$\int_\Omega \chi_1 \partial^\alpha u \cdot \partial^\alpha H^{1,1} + \chi_1 \partial^\alpha p \partial^\alpha H^{2,1} \lesssim \varepsilon \mathcal{D}_{N+2} + \varepsilon^{-4N-9}\bar{\mathcal{D}}_{N+2}^0. \tag{5.29}$$

To derive this, we may argue as in Step 2 of Proposition 5.2; the only difference is that now when we interpolate we have

$$\|\partial_t^{\alpha_0} u\|_{2N+4-2\alpha_0} \lesssim \|\partial_t^{\alpha_0} u\|_0^\theta \|\partial_t^{\alpha_0} u\|_{2N+5-2\alpha_0}^{1-\theta} \tag{5.30}$$

for $\theta = (2N+5-2\alpha_0)^{-1} \in (0,1)$ so that $(2-\theta)/\theta = 4N+9-2\alpha_0 \le 4N+9$, which gives the power of $1/\varepsilon$ in the estimate.

Step 3 – Terms involving G^i, $1 \le i \le 4$

We claim that

$$\int_\Omega \chi_1^2 \left(\partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2 \right) \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2} \tag{5.31}$$

and

$$\int_\Sigma -\partial^\alpha u \cdot \partial^\alpha G^3 + \partial^\alpha \eta \partial^\alpha G^4 \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}. \tag{5.32}$$

To prove the claim, we argue as in Step 3 of Proposition 5.2, using the estimates of Theorem 3.1 in place of those of Theorem 3.2. This is sufficient to estimate all of the terms except

$$\int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \text{ when } |\alpha| = 2(N+2). \tag{5.33}$$

For this, in place of Lemma 5.1, we argue as follows.

First, we write $\partial^\alpha G^4 = I + II + III$ where I, II, III are the first, second, and third terms on the right of (5.2), respectively. We may argue as in the proof of Lemma 5.1 to see that

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta (I + II) \right| &\lesssim \|\eta\|_{2(N+2)+1/2} \|\eta\|_{2(N+2)-1/2} \|u\|_4 \\ &\lesssim \sqrt{\mathcal{E}_{2N}} \sqrt{\mathcal{D}_{N+2}} \sqrt{\mathcal{D}_{N+2}} = \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2} \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} \left| \int_\Sigma \partial^\alpha \eta (III) \right| &\lesssim \|\eta\|_{2(N+2)-1/2} \|u\|_{2(N+2)+1} \|\eta\|_{2(N+2)+1} \\ &\lesssim \sqrt{\mathcal{D}_{N+2}} \sqrt{\mathcal{D}_{N+2}} \sqrt{\mathcal{E}_{2N}} = \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}. \end{aligned} \tag{5.35}$$

Hence

$$\left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^4 \right| \lesssim \sqrt{\mathcal{E}_{2N}} \mathcal{D}_{N+2}, \tag{5.36}$$

which serves as the replacement for Lemma 5.1 in the present case. Using this, we can complete the proof of the claim.

Step 4 – Conclusion

We now combine the above as in Proposition 5.2 to deduce the estimate (5.27). Then (5.28) follows from (5.27) by summing over α. □

5.2. **Lower localization.** We now consider the evolution of the lower-localization energies at the $2N$ level.

Proposition 5.4. *Let j be an integer satisfying $0 \le j \le 2N-1$. Then for any $\varepsilon \in (0,1)$ it holds that*

$$\left\| \partial_t^j (\chi_2 u) \right\|_0^2 + \int_0^t \left\| \mathbb{D} \partial_t^j (\chi_1 u) \right\|_0^2 \lesssim \bar{\mathcal{E}}_{2N}^-(0) + \int_0^t \mathcal{E}_{2N}^\theta \mathcal{D}_{2N} + \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1} \bar{\mathcal{D}}_{2N}^0. \tag{5.37}$$

In particular,

$$\bar{\mathcal{E}}_{2N}^-(t) + \int_0^t \bar{\mathcal{D}}_{2N}^- \lesssim \bar{\mathcal{E}}_{2N}^-(0) + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1} \bar{\mathcal{D}}_{2N}^0. \tag{5.38}$$

Proof. We apply Lemma 2.2 to $v = \chi_2 \partial_t^j u$, $q = \chi_2 \partial_t^j p$, $\zeta = \partial_t^j \eta$ with $a = 0$, $\Phi^1 = \chi_2 \partial_t^j G^1 + \partial_t^j H^{1,2}$, $\Phi^2 = \chi_2 \partial_t^j G^2 + \partial_t^j H^{2,2}$, $\Phi^3 = 0$, and $\Phi^4 = 0$ to find

$$\partial_t \left(\frac{1}{2} \int_\Omega \left| \partial_t^j (\chi_2 u) \right|^2 \right) + \frac{1}{2} \int_\Omega \left| \mathbb{D} \partial_t^j (\chi_2 u) \right|^2 = \int_\Omega \chi_2 \partial_t^j u \cdot (\chi_2 \partial_t^j G^1 + \partial_t^j H^{1,2}) + \int_\Omega \chi_2 \partial_t^j p (\chi_2 \partial_t^j G^2 + \partial_t^j H^{2,2}). \tag{5.39}$$

Here $H^{1,2}$ and $H^{2,2}$ are given by (2.42). The right hand side may then be estimated as in Proposition 5.2, using only the temporal derivative estimates of Theorem 3.2. In particular, we have the estimates

$$\int_\Omega \chi_2 \partial_t^j u \cdot \partial_t^j H^{1,2} + \chi_2 \partial_t^j p \partial_t^j H^{2,2} \lesssim \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1} \bar{\mathcal{D}}_{2N}^0 \tag{5.40}$$

and

$$\int_\Omega \chi_2^2 (\partial_t^j u \cdot \partial_t^j G^1 + \partial_t^j p \partial_t^j G^2) \lesssim (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N}, \tag{5.41}$$

which yield (5.37) when combined with (5.39) and integrated in time from 0 to t. Then (5.38) follows from (5.37) by summing over $0 \le j \le 2N-1$. □

Now we prove the corresponding result at the $N+2$ level.

Proposition 5.5. *Let j be an integer satisfying $0 \le j \le N+1$. Then for any $\varepsilon \in (0,1)$ it holds that*

$$\partial_t \left(\left\| \partial_t^j (\chi_2 u) \right\|_0^2 \right) + \left\| \mathbb{D} \partial_t^j (\chi_1 u) \right\|_0^2 \lesssim \mathcal{E}_{2N}^\theta \mathcal{D}_{N+2} + \varepsilon \mathcal{D}_{N+2} + \varepsilon^{-4N-9} \bar{\mathcal{D}}_{N+2}^0. \tag{5.42}$$

In particular,

$$\partial_t \bar{\mathcal{E}}_{N+2}^- + \bar{\mathcal{D}}_{N+2}^- \lesssim (\mathcal{E}_{2N})^\theta \mathcal{D}_{N+2} + \varepsilon \mathcal{D}_{N+2} + \varepsilon^{-4N-9} \bar{\mathcal{D}}_{N+2}^0. \tag{5.43}$$

Proof. The proof proceeds as in Proposition 5.4, following Proposition 5.3 rather than Proposition 5.2, and using the $\partial_t^j G^i$ estimates of Theorem 3.1 rather than of Theorem 3.2. □

6. Comparison results

We now show that, up to some error terms, the instantaneous energy $\mathcal{E}_{2N}$ is comparable to the sum $\bar{\mathcal{E}}_{2N}^0 + \bar{\mathcal{E}}_{2N}^+$ and that the dissipation rate $\mathcal{D}_{2N}$ is comparable to the sum $\bar{\mathcal{D}}_{2N}^0 + \bar{\mathcal{D}}_{2N}^- + \bar{\mathcal{D}}_{2N}^+$. We also prove similar results with $2N$ replaced by $N+2$.

6.1. Instantaneous energy.

We begin with the result for the instantaneous energy.

Theorem 6.1. *There exists a $\theta > 0$ so that*

$$\mathcal{E}_{2N} \lesssim \bar{\mathcal{E}}_{2N}^+ + \bar{\mathcal{E}}_{2N}^0 + (\mathcal{E}_{2N})^{1+\theta} \tag{6.1}$$

and

$$\mathcal{E}_{N+2} \lesssim \bar{\mathcal{E}}_{N+2}^+ + \bar{\mathcal{E}}_{N+2}^0 + (\mathcal{E}_{2N})^\theta \mathcal{E}_{N+2}. \tag{6.2}$$

Proof. In order to prove the result at both the $2N$ and $N+2$ levels at the same time, we will generically write n to refer to either quantity. In the proof we will write

$$\mathcal{W}_n = \sum_{j=0}^{n-1} \left\| \partial_t^j G^1 \right\|_{2n-2j-2}^2 + \left\| \partial_t^j G^2 \right\|_{2n-2j-1}^2 + \left\| \partial_t^j G^3 \right\|_{2n-2j-3/2}^2. \tag{6.3}$$

Note that the definitions of $\bar{\mathcal{E}}_n^+$ and $\bar{\mathcal{E}}_n^0$ guarantee that

$$\sum_{j=0}^{n} \left\| \partial_t^j \eta \right\|_{2n-2j}^2 \lesssim \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0. \tag{6.4}$$

The key to proving the result is the following elliptic estimate. Let $j = 0, \dots, n-1$. Then we may apply ∂_t^j to the equations of (2.23) and use Lemma A.8 to see that

$$\left\|\partial_t^j u\right\|_{2n-2j}^2 + \left\|\partial_t^j p\right\|_{2n-2j-1}^2 \lesssim \left\|\partial_t^{j+1} u\right\|_{2n-2(j+1)}^2 + \left\|\partial_t^j G^1\right\|_{2n-2j-2}^2 + \left\|\partial_t^j G^2\right\|_{2n-2j-1}^2 + \left\|\partial_t^j \eta\right\|_{2n-2j-3/2}^2 + \left\|\partial_t^j G^3\right\|_{2n-2j-3/2}^2 \lesssim \left\|\partial_t^{j+1} u\right\|_{2n-2(j+1)}^2 + \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n. \tag{6.5}$$

In the last inequality of (6.5) we have used (6.4) and the definition of $\mathcal{W}_n$.

We claim that

$$\mathcal{E}_n \lesssim \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n. \tag{6.6}$$

To prove this claim, we will use estimate (6.5) and a finite induction. For $j = n-1$ we employ the definition of $\bar{\mathcal{E}}_n^0$ and Remark 2.4 in (6.5) to get

$$\left\|\partial_t^{n-1} u\right\|_2^2 + \left\|\partial_t^{n-1} p\right\|_1^2 \lesssim \left\|\partial_t^n u\right\|_0^2 + \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n \lesssim \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n. \tag{6.7}$$

Now suppose that the inequality

$$\left\|\partial_t^{n-\ell} u\right\|_{2\ell}^2 + \left\|\partial_t^{n-\ell} p\right\|_{2\ell-1}^2 \lesssim \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n \tag{6.8}$$

holds for $1 \le \ell < n$. We apply (6.5) with $j = n - \ell - 1$ and use the induction hypothesis (6.8) to find

$$\left\|\partial_t^{n-\ell-1} u\right\|_{2(\ell+1)}^2 + \left\|\partial_t^{n-\ell-1} p\right\|_{2(\ell+1)-1}^2 \lesssim \left\|\partial_t^{n-\ell} u\right\|_{2\ell}^2 + \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n \lesssim \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n. \tag{6.9}$$

Hence (6.8) holds with ℓ replaced by $\ell + 1$, and by finite induction,

$$\sum_{j=0}^{n-1} \left\|\partial_t^j u\right\|_{2n-2j}^2 + \left\|\partial_t^j p\right\|_{2n-2j-1}^2 \lesssim \bar{\mathcal{E}}_n^+ + \bar{\mathcal{E}}_n^0 + \mathcal{W}_n. \tag{6.10}$$

We then sum (6.4), (6.10), and the trivial inequality $\left\|\partial_t^n u\right\|_0^2 \le \bar{\mathcal{E}}_n^0$ to deduce that (6.6) holds.

To conclude, we must estimate $\mathcal{W}_n$ for $n = 2N$ and $n = N+2$. When $n = N+2$, we use (3.1) of Theorem 3.1 to bound $\mathcal{W}_{N+2} \lesssim (\mathcal{E}_{2N})^\theta \mathcal{E}_{N+2}$, and when $n = 2N$ we use (3.3) of Theorem 3.2 to bound $\mathcal{W}_{2N} \lesssim (\mathcal{E}_{2N})^{1+\theta}$. These two estimates and (6.6) then imply (6.1) and (6.2). □

6.2. **Dissipation.** Now we consider the dissipation rate.

Theorem 6.2. *For $n = N+2$ or $n = 2N$, write*

$$\mathcal{Y}_n = \left\|\bar{\nabla}_0^{2n-1} G^1\right\|_0^2 + \left\|\bar{\nabla}_0^{2n-1} G^2\right\|_1^2 + \left\|\bar{D}_0^{2n-1} G^3\right\|_{1/2}^2 + \left\|\bar{D}_0^{2n-1} G^4\right\|_{1/2}^2 + \left\|\bar{D}_0^{2n-2} \partial_t G^4\right\|_{1/2}^2. \tag{6.11}$$

Then

$$\mathcal{D}_n \lesssim \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n. \tag{6.12}$$

In particular, there is a $\theta > 0$ so that

$$\mathcal{D}_{2N} \lesssim \bar{\mathcal{D}}_{2N}^0 + \bar{\mathcal{D}}_{2N}^- + \bar{\mathcal{D}}_{2N}^+ + (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N} \tag{6.13}$$

and

$$\mathcal{D}_{N+2} \lesssim \bar{\mathcal{D}}_{N+2}^0 + \bar{\mathcal{D}}_{N+2}^- + \bar{\mathcal{D}}_{N+2}^+ + (\mathcal{E}_{2N})^\theta \mathcal{D}_{N+2} \tag{6.14}$$

Proof. In this proof we use a separate counting for spatial and temporal derivatives, so unlike elsewhere, we now use $\alpha \in \mathbb{N}^2$ to refer only to spatial derivatives. In order to compactly write our estimates, throughout the proof we will write

$$\mathcal{Z} := \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n. \tag{6.15}$$

The proof is divided into several steps.

Step 1 – Application of Korn's inequality

First note that according to Lemma A.7 we have

$$\left\|\bar{D}_0^{2n-1}u\right\|_{H^1(\Omega_1)}^2 + \left\|D\bar{D}^{2n-1}u\right\|_{H^1(\Omega_1)}^2 \lesssim \left\|\bar{D}_0^{2n-1}(\chi_1 u)\right\|_1^2 + \left\|D\bar{D}^{2n-1}(\chi_1 u)\right\|_1^2 \lesssim \bar{\mathcal{D}}_n^+ \tag{6.16}$$

and

$$\sum_{j=0}^{n} \left\|\partial_t^j u\right\|_1^2 \lesssim \bar{\mathcal{D}}_n^0. \tag{6.17}$$

Here, we recall that $\Omega_1 \subset \Omega$ is defined in (2.40). Summing these yields the bound

$$\left\|\bar{D}_0^{2n}u\right\|_{H^1(\Omega_1)}^2 \lesssim \bar{\mathcal{D}}_n^+ + \bar{\mathcal{D}}_n^0. \tag{6.18}$$

Step 2 – Initial estimates of the pressure and improvement of u estimates

Recall that χ_3 is given by (2.39), $\Omega_3 \subset \Omega_1$ is given by (2.40), and $\chi_3 = 1$ on Ω_3. We claim that we have the estimate

$$\left\|\bar{D}_0^{2n-1}u\right\|_{H^2(\Omega_3)}^2 + \left\|\bar{D}_0^{2n-1}\nabla p\right\|_{H^0(\Omega_3)}^2 \lesssim \mathcal{Z}. \tag{6.19}$$

To prove this, we will first use the structure of the equations (2.23) to derive various estimates of terms involving ∂_3. Then we use elliptic estimates for $\omega = \operatorname{curl} u$ to recover other terms with ∂_3.

Let $0 \le j \le n-1$ and $\alpha \in \mathbb{N}^2$ be such that

$$0 \le 2j + |\alpha| \le 2n-1. \tag{6.20}$$

Note that if $2j + |\alpha| = 2n-1$, then the condition $j \le n-1$ implies that $|\alpha| \ge 1$. This means that we are free to use (6.18) to bound

$$\left\|\partial^\alpha \partial_t^{j+1} u\right\|_{H^0(\Omega_1)} \le \left\|\bar{D}_0^{2n}u\right\|_{H^1(\Omega_1)} \lesssim \mathcal{Z}. \tag{6.21}$$

In order to extract further information, we apply the operator $\partial_t^j \partial^\alpha$ to the first two equations in (2.23) to find that

$$\partial^\alpha \partial_t^{j+1} u - \Delta \partial^\alpha \partial_t^j u + \nabla \partial^\alpha \partial_t^j p = \partial^\alpha \partial_t^j G^1 \tag{6.22}$$

$$\operatorname{div} \partial^\alpha \partial_t^j u = \partial^\alpha \partial_t^j G^2. \tag{6.23}$$

Because of the constraints on j, α given by (6.20) we may control

$$\left\|\partial^\alpha \partial_t^j G^1\right\|_0^2 + \left\|\partial^\alpha \partial_t^j G^2\right\|_1^2 \le \left\|\bar{D}_0^{2n-1}G^1\right\|_0^2 + \left\|\bar{D}_0^{2n-1}G^2\right\|_1^2 \le \mathcal{Z}. \tag{6.24}$$

We will utilize the structure of (6.22)–(6.23) in conjunction with (6.21) and (6.24) in order to improve our estimates.

We begin by utilizing (6.23) to control one of the terms in the third component of (6.22). We have

$$\partial^\alpha \partial_t^j (\partial_3 u_3) = \partial^\alpha \partial_t^j (-\partial_1 u_1 - \partial_2 u_2 + G^2) \tag{6.25}$$

so that (6.18) and (6.24) imply

$$\left\|\partial_3^2 \partial^\alpha \partial_t^j u_3\right\|_{H^0(\Omega_1)} \lesssim \left\|\bar{D}_0^{2n}u\right\|_{H^1(\Omega_1)} + \left\|\bar{D}_0^{2n-1}G^2\right\|_1^2 \lesssim \mathcal{Z}. \tag{6.26}$$

A further application of (6.18) to control $(\partial_1^2 + \partial_2^2)\partial^\alpha \partial_t^j u_3$ then provides the estimate

$$\left\|\Delta \partial^\alpha \partial_t^j u_3\right\|_{H^0(\Omega_1)} \lesssim \mathcal{Z}. \tag{6.27}$$

Applying the bounds (6.21), (6.24), and (6.27) to the third component of (6.22), we arrive at a partial bound for the pressure:

$$\left\|\partial_3 \partial^\alpha \partial_t^j p\right\|_{H^0(\Omega_1)} \lesssim \mathcal{Z}. \tag{6.28}$$

It remains to control the terms $\partial_i \partial^\alpha \partial_t^j p$ and $\partial_3^2 \partial^\alpha \partial_t^j u_i$ for $i = 1, 2$. To accomplish this, we employ an elliptic estimate of $\operatorname{curl} u := \omega$. Taking the curl of (6.22) eliminates the pressure gradient and yields

$$\partial^\alpha \partial_t^{j+1} \omega = \Delta \partial^\alpha \partial_t^j \omega + \operatorname{curl}(\partial^\alpha \partial_t^j G^1). \tag{6.29}$$

We only need the first two components $\omega_1 = \partial_2 u_3 - \partial_3 u_2$, $\omega_2 = \partial_3 u_1 - \partial_1 u_3$, for which we use the Σ boundary condition (2.23)

$$\partial_i u_3 + \partial_3 u_i = \mathbb{D} u e_3 \cdot e_i = -G^3 \cdot e_i \text{ for } i = 1, 2 \tag{6.30}$$

to derive the boundary conditions

$$\begin{cases} \omega_1 = 2\partial_2 u_3 + G^3 \cdot e_2 & \text{on } \Sigma \\ \omega_2 = -2\partial_1 u_3 - G^3 \cdot e_1 & \text{on } \Sigma. \end{cases} \tag{6.31}$$

The functions $\chi_3 \omega_i$, $i = 1, 2$, satisfy

$$\Delta \partial^\alpha \partial_t^j (\chi_3 \omega_i) = \chi_3 (\partial^\alpha \partial_t^{j+1} \omega_i) + 2(\partial_3 \chi_3)(\partial_3 \partial^\alpha \partial_t^j \omega_i) + (\partial_3^2 \chi_3)(\partial^\alpha \partial_t^j \omega_i) - \chi_3 \operatorname{curl}(\partial^\alpha \partial_t^j G^1) \tag{6.32}$$

in Ω as well as the boundary conditions

$$\begin{cases} \partial^\alpha \partial_t^j (\chi_3 \omega_1) = 2\partial_2 \partial^\alpha \partial_t^j u_3 + \partial^\alpha \partial_t^j G^3 \cdot e_2 & \text{on } \Sigma \\ \partial^\alpha \partial_t^j (\chi_3 \omega_2) = -2\partial_1 \partial^\alpha \partial_t^j u_3 - \partial^\alpha \partial_t^j G^3 \cdot e_1 & \text{on } \Sigma \\ \partial^\alpha \partial_t^j (\chi_3 \omega_1) = \partial^\alpha \partial_t^j (\chi_3 \omega_2) = 0 & \text{on } \Sigma_b. \end{cases} \tag{6.33}$$

In order to employ an elliptic estimate of $\partial^\alpha \partial_t^j (\chi_3 \omega_i)$ we must first prove two auxiliary estimates.

First we derive an estimate of the $H^{-1}(\Omega) = (H_0^1(\Omega))^*$ norm of each term on the right side of equation (6.32). Let $\varphi \in H_0^1(\Omega)$. When $\alpha \neq 0$ we may write $\alpha = \beta + (\alpha - \beta)$ with $|\beta| = 1$ and integrate by parts to bound

$$\left| \int_\Omega \varphi \chi_3 \partial^\alpha \partial_t^{j+1} \omega_i \right| = \left| \int_\Omega \partial^\beta \varphi \chi_3 \partial^{\alpha-\beta} \partial_t^{j+1} \omega_i \right| \le \|\varphi\|_1 \left\| \chi_3 \bar{D}_0^{2n} \omega_i \right\|_0 \tag{6.34}$$

since $2(j+1) + |\alpha - \beta| = 2j + |\alpha| + 1 \in [1, 2n]$. We may use (6.18) for

$$\left\| \chi_3 \bar{D}_0^{2n} \omega_i \right\|_0^2 \lesssim \left\| \bar{D}_0^{2n} u \right\|_{H^1(\Omega_1)} \lesssim \mathcal{Z}. \tag{6.35}$$

Chaining these inequalities together when $\alpha \neq 0$ and taking the supremum over all φ such that $\|\varphi\|_1 \le 1$, we get

$$\left\| \partial^\alpha \partial_t^{j+1} \omega_i \right\|_{H^{-1}}^2 \lesssim \mathcal{Z}. \tag{6.36}$$

A similar argument without an integration by parts shows that (6.36) is also true when $\alpha = 0$ since in this case the condition $j \le n-1$ implies that $2 \le 2(j+1) \le 2n$. Similarly integrating by parts with ∂_3 in the dual-pairing, we may estimate the second term on the right side of (6.32):

$$\left\| 2(\partial_3 \chi_3)(\partial_3 \partial^\alpha \partial_t^j \omega_i) \right\|_{H^{-1}}^2 \lesssim (\|\partial_3 \chi_3\|_{L^\infty}^2 + \left\| \partial_3^2 \chi_3 \right\|_{L^\infty}^2) \left\| \bar{D}_0^{2n} \omega_i \right\|_{H^0(\Omega_3)} \lesssim \left\| \bar{D}_0^{2n} u \right\|_{H^1(\Omega_1)} \lesssim \mathcal{Z}. \tag{6.37}$$

The third term may be estimated without integration by parts in the dual-pairing:

$$\left\| (\partial_3^2 \chi_3)(\partial^\alpha \partial_t^j \omega_i) \right\|_{H^{-1}}^2 \lesssim \left\| \partial_3^2 \chi_3 \right\|_{L^\infty}^2 \left\| \bar{D}_0^{2n} \omega_i \right\|_{H^0(\Omega_3)} \lesssim \left\| \bar{D}_0^{2n} u \right\|_{H^1(\Omega_1)} \lesssim \mathcal{Z}. \tag{6.38}$$

The fourth term is estimated by integrating by parts with the curl operator and using (6.24):

$$\left\| \chi_3 \operatorname{curl}(\partial^\alpha \partial_t^j G^1) \right\|_{H^{-1}}^2 \lesssim (\|\chi_3\|_{L^\infty}^2 + \|\partial_3 \chi_3\|_{L^\infty}^2) \left\| \bar{D}_0^{2n-1} G^1 \right\|_0^2 \lesssim \mathcal{Z}. \tag{6.39}$$

Combining these four estimates of the right hand side of (6.32) yields

$$\left\| \Delta \partial^\alpha \partial_t^j (\chi_3 \omega_i) \right\|_{H^{-1}}^2 \lesssim \mathcal{Z} \text{ for } i = 1, 2. \tag{6.40}$$

Next, to complete the elliptic estimate of $\partial^\alpha \partial_t^j (\chi_3 \omega_i)$, we also need $H^{1/2}(\Sigma)$ estimates for the boundary terms on the right side of the first two equations in (6.33). We may estimate the $\partial_i u_3$, $i = 1, 2$, terms with the embedding $H^1(\Omega) \hookrightarrow H^{1/2}(\Sigma)$:

$$(6.41) \qquad \left\| \partial^\alpha \partial_t^j \partial_1 u_3 \right\|_{H^{1/2}(\Sigma)}^2 + \left\| \partial^\alpha \partial_t^j \partial_2 u_3 \right\|_{H^{1/2}(\Sigma)}^2 \lesssim \left\| \bar{D}_0^{2n} u \right\|_{H^1(\Omega_1)} \lesssim \mathcal{Z}.$$

On the other hand, estimates of G^3 are already built into $\mathcal{Z}$:

$$(6.42) \qquad \left\| \partial^\alpha \partial_t^j G^3 \right\|_{1/2}^2 \le \left\| \bar{D}_0^{2n-1} G^3 \right\|_{1/2}^2 \le \mathcal{Y}_{n,m} \le \mathcal{Z}.$$

Since $\chi_3 \omega_i = 0$ on Σ_b for $i = 1, 2$ we then deduce that

$$(6.43) \qquad \left\| \partial^\alpha \partial_t^j (\chi_3 \omega_i) \right\|_{H^{1/2}(\partial\Omega)}^2 \lesssim \mathcal{Z} \text{ for } i = 1, 2.$$

Now according to (6.40), (6.43), standard elliptic estimates, and the fact that $\chi_3 = 1$ on Ω_3 (defined by (2.40)) we have

$$(6.44) \qquad \left\| \partial^\alpha \partial_t^j \omega_i \right\|_{H^1(\Omega_3)}^2 \lesssim \left\| \partial^\alpha \partial_t^j (\chi_3 \omega_i) \right\|_1^2 \lesssim \mathcal{Z} \text{ for } i = 1, 2.$$

We may then rewrite

$$(6.45) \qquad \partial_3^2 \partial^\alpha \partial_t^j u_1 = \partial_3 \partial^\alpha \partial_t^j (\omega_2 + \partial_1 u_3) \text{ and } \partial_3^2 \partial^\alpha \partial_t^j u_2 = \partial_3 \partial^\alpha \partial_t^j (\partial_2 u_3 - \omega_1)$$

and deduce from (6.44) and (6.18) that for $i = 1, 2$ we have

$$(6.46) \qquad \left\| \partial_3^2 \partial^\alpha \partial_t^j u_i \right\|_{H^0(\Omega_3)}^2 \lesssim \left\| \bar{D}_0^{2n} u_3 \right\|_{H^1(\Omega_1)} + \sum_{k=1}^2 \left\| \partial^\alpha \partial_t^j \omega_k \right\|_{H^1(\Omega_3)}^2 \lesssim \mathcal{Z}.$$

We then apply this estimate along with (6.18) and (6.24) to the first two components of equation (6.22) to find that

$$(6.47) \qquad \left\| \partial_i \partial^\alpha \partial_t^j p \right\|_{H^0(\Omega_3)}^2 \lesssim \mathcal{Z} \text{ for } i = 1, 2.$$

Now we sum the estimates (6.18), (6.26), (6.48), (6.58), and (6.47) over all $j \le n-1$ and $\alpha \in \mathbb{N}^2$ with $0 \le 2j + |\alpha| \le 2n - 1$ to deduce that (6.19) holds. This proves the claim.

Step 3 – Bootstrapping, η estimates, and improved pressure estimates

Now we make use of Lemma 6.3 to bootstrap from (6.19) to

$$(6.48) \qquad \sum_{j=0}^{n-1} \left\| \partial_t^j u \right\|_{H^{2n-2j+1}(\Omega_3)}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j \nabla p \right\|_{H^{2n-2j-1}(\Omega_3)}^2 \lesssim \mathcal{Z}.$$

With this estimate in hand, we may derive some estimates for η on Σ by employing the boundary conditions of (2.23):

$$(6.49) \qquad \eta = p - 2\partial_3 u_3 - G_3^3,$$

$$(6.50) \qquad \partial_t \eta = u_3 + G^4.$$

We differentiate (6.49) and employ (6.48) to find that

$$(6.51) \qquad \begin{aligned} \left\| D\eta \right\|_{2n-3/2}^2 &\lesssim \left\| Dp \right\|_{H^{2n-3/2}(\Sigma)}^2 + \left\| D\partial_3 u_3 \right\|_{H^{2n-3/2}(\Sigma)}^2 + \left\| DG^3 \right\|_{2n-3/2}^2 \\ &\lesssim \left\| Dp \right\|_{H^{2n-1}(\Omega_3)}^2 + \left\| D\partial_3 u_3 \right\|_{H^{2n-1}(\Omega_3)}^2 + \left\| G^3 \right\|_{2n-1/2}^2 \lesssim \mathcal{Z}, \end{aligned}$$

so that by the usual Poincaré inequality on Σ (we have that η has zero average) we know

$$(6.52) \qquad \left\| \eta \right\|_{2n-1/2}^2 \lesssim \left\| \eta \right\|_0^2 + \left\| D\eta \right\|_{2n-3/2}^2 \lesssim \left\| D\eta \right\|_{2n-3/2}^2 \lesssim \mathcal{Z}.$$

Similarly, for $j = 2, \ldots, n+1$ we may apply ∂_t^{j-1} to (6.50) and estimate

$$\left\| \partial_t^j \eta \right\|_{2n-2j+5/2}^2 \lesssim \left\| \partial_t^{j-1} u_3 \right\|_{H^{2n-2j+5/2}(\Sigma)}^2 + \left\| \partial_t^{j-1} G^4 \right\|_{2n-2j+5/2}^2 \lesssim \left\| \partial_t^{j-1} u \right\|_{H^{2n-2(j-1)+1}(\Omega_3)}^2 + \left\| \partial_t^{j-1} G^4 \right\|_{2n-2(j-1)+1/2}^2 \lesssim \mathcal{Z}. \tag{6.53}$$

It remains only to control $\partial_t \eta$, which we do again using (6.50):

$$\left\| \partial_t \eta \right\|_{2n-1/2}^2 \lesssim \left\| u_3 \right\|_{H^{2n-1/2}(\Sigma)}^2 + \left\| G^4 \right\|_{2n-1/2}^2 \lesssim \left\| u_3 \right\|_{H^{2n}(\Omega_3)}^2 + \mathcal{Z} \lesssim \mathcal{Z}. \tag{6.54}$$

Summing estimates (6.52)–(6.54) then yields

$$\left\| \eta \right\|_{2n-1/2}^2 + \left\| \partial_t \eta \right\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \left\| \partial_t^j \eta \right\|_{2n-2j+5/2}^2 \lesssim \mathcal{Z}. \tag{6.55}$$

The η estimates (6.55) now allow us to further improve the estimates for the pressure. Indeed, for $j = 0, \ldots, n-1$ we may use Lemma A.6 and (6.49) to bound

$$\left\| \partial_t^j p \right\|_{H^0(\Omega_3)}^2 \lesssim \left\| \partial_t^j \eta \right\|_0^2 + \left\| \partial_3 \partial_t^j u_3 \right\|_{H^0(\Sigma)}^2 + \left\| \partial_t^j G^3 \right\|_0^2 + \left\| \partial_t^j \nabla p \right\|_{H^0(\Omega_3)}^2 \lesssim \left\| \partial_t^j u_3 \right\|_{H^2(\Omega_3)}^2 + \mathcal{Z} \lesssim \mathcal{Z}. \tag{6.56}$$

This, (6.18), and (6.55) allow us to improve (6.48) to

$$\sum_{j=0}^{n} \left\| \partial_t^j u \right\|_{H^{2n-2j+1}(\Omega_3)}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j p \right\|_{H^{2n-2j}(\Omega_3)}^2 + \left\| \eta \right\|_{2n-1/2}^2 + \left\| \partial_t \eta \right\|_{2n-1/2}^2 + \sum_{j=2}^{n+1} \left\| \partial_t^j \eta \right\|_{2n-2j+5/2}^2 \lesssim \mathcal{Z}. \tag{6.57}$$

Step 4 – Estimates in Ω_2

We now extend our estimates to the lower domain, Ω_2, by initially applying Lemma 6.4 for

$$\sum_{j=0}^{n} \left\| \partial_t^j (\chi_2 u) \right\|_{2n-2j+1}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j (\chi_2 p) \right\|_{2n-2j}^2 \lesssim \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^0 + \mathcal{X}_n + \mathcal{Y}_n, \tag{6.58}$$

where $\mathcal{X}_n$ is defined by

$$\mathcal{X}_n = \sum_{j=0}^{n-1} \left\| \partial_t^j H^{1,2} \right\|_{2n-2j-1}^2 + \left\| \partial_t^j H^{2,2} \right\|_{2n-2j}^2 \tag{6.59}$$

for $H^{1,2}$ and $H^{2,2}$ given by (2.42). We must now estimate $\mathcal{X}_n$. For this we note that by construction $\operatorname{supp}(\nabla \chi_2) \subset \Omega_3$, which implies that $\operatorname{supp}(H^{1,2}) \cup \operatorname{supp}(H^{2,2}) \subset \Omega_3$. This allows us to use the estimate (6.57) to bound

$$\mathcal{X}_n \lesssim \sum_{j=0}^{n-1} \left(\left\| \partial_t^j u \right\|_{H^{2n-2j+1}(\Omega_3)}^2 + \left\| \partial_t^j p \right\|_{H^{2n-2j}(\Omega_3)}^2 \right) \lesssim \mathcal{Z}. \tag{6.60}$$

Then estimates (6.58) and (6.60) may be combined to get

$$\sum_{j=0}^{n} \left\| \partial_t^j u \right\|_{H^{2n-2j+1}(\Omega_2)}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j p \right\|_{H^{2n-2j}(\Omega_2)}^2 \lesssim \sum_{j=0}^{n} \left\| \partial_t^j (\chi_2 u) \right\|_{2n-2j+1}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j (\chi_2 p) \right\|_{2n-2j}^2 \lesssim \bar{\mathcal{D}}_n^- + \mathcal{Z}. \tag{6.61}$$

Step 5 –Estimates on all of Ω and conclusion

We recall that $\Omega = \Omega_3 \cup \Omega_2$. This allows us to add the localized estimates (6.57) and (6.61) to deduce (6.12). In order to deduce (6.13) and (6.14) from (6.12), we must only estimate $\mathcal{Y}_n$ for $n = 2N$ and $n = N+2$. In the case $n = 2N$, Theorem 3.2 provides the estimate $\mathcal{Y}_{2N} \lesssim (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N}$, and (6.13) follows. In the case $n = N+2$ we use Theorem 3.1 for $\mathcal{Y}_{N+2} \lesssim (\mathcal{E}_{2N})^\theta \mathcal{D}_{N+2}$, and (6.14) follows. □

The next result is a key bootstrap estimate used in the proof of Theorem 6.2.

Lemma 6.3. *Let $\mathcal{Y}_n$ be as defined in Theorem 6.2. Suppose that*

$$\left\| \bar{D}_0^{2n-2r+1} u \right\|_{H^{2r}(\Omega_3)}^2 + \left\| \bar{D}_0^{2n-2r+1} \nabla p \right\|_{H^{2r-2}(\Omega_3)}^2 \lesssim \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n \tag{6.62}$$

for an integer $r \in \{1 \dots, n-1\}$. Then

$$\begin{aligned} &\left\| \partial_t^{n-r} u \right\|_{H^{2r+1}(\Omega_3)}^2 + \left\| \partial_t^{n-r} \nabla p \right\|_{H^{2r-1}(\Omega_3)}^2 \\ &\quad + \left\| \bar{D}_0^{2n-2(r+1)+1} u \right\|_{H^{2r+2}(\Omega_3)}^2 + \left\| \bar{D}_0^{2n-2(r+1)+1} \nabla p \right\|_{H^{2r}(\Omega_3)}^2 \lesssim \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n. \end{aligned} \tag{6.63}$$

Moreover, if (6.62) *holds with $r = 1$, then*

$$\sum_{j=0}^{n-1} \left\| \partial_t^j u \right\|_{H^{2n-2j+1}(\Omega_3)}^2 + \sum_{j=0}^{n-1} \left\| \partial_t^j \nabla p \right\|_{H^{2n-2j-1}(\Omega_3)}^2 \lesssim \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n. \tag{6.64}$$

Proof. We divide the proof into two steps.

Step 1 – (6.62) implies (6.63)

Throughout the proof we will write $\mathcal{Z} := \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n$. Let $\ell \in \{1, 2\}$ and take $0 \le j \le n-1-r$ and $\alpha \in \mathbb{N}^2$ so that $0 \le 2j + |\alpha| \le 2n - 2r + 1 - \ell$. We apply the differential operator $\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j$ to the first equation in (2.23) and split into separate equations for its third and first two components; after some rearrangement, these read

$$\partial_3^{2r-1+\ell} \partial^\alpha \partial_t^j p = -\partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u_3 + \Delta \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j u_3 + \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G_3^1 \tag{6.65}$$

and

$$\Delta \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j u_i = \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u_i + \partial_i \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j p - \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G_i^1 \tag{6.66}$$

for $i = 1, 2$. Notice that the constraints on $r, j, |\alpha|$ imply that $0 \le |\alpha| + (2r-2+\ell) + 2j \le 2n-1$, so we may estimate

$$\left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G^1 \right\|_0^2 + \left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^j G^2 \right\|_1^2 \le \mathcal{Y}_n \le \mathcal{Z}. \tag{6.67}$$

Since $2r - 2 + \ell \ge 0$, we know that

$$\left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u \right\|_{H^0(\Omega_3)}^2 \le \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_3)}^2. \tag{6.68}$$

If $\ell = 2$ then $|\alpha| + 2(j+1) \le 2n - 2r + 1$ so that

$$\left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_3)}^2 = \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r}(\Omega_3)}^2 \le \left\| \bar{D}_0^{2n-2r+1} u \right\|_{H^{2r}(\Omega_3)}^2 \le \mathcal{Z}. \tag{6.69}$$

On the other hand, if $\ell = 1$, then either $\alpha = 0$, in which case the bound on j implies that $2(j+1) \le 2n - 2r$, and hence

$$\left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_3)}^2 = \left\| \partial_t^{j+1} u \right\|_{H^{2r-1}(\Omega_3)}^2 \le \left\| \bar{D}_0^{2n-2r+1} u \right\|_{H^{2r}(\Omega_3)}^2 \le \mathcal{Z}, \tag{6.70}$$

or else $|\alpha| \ge 1$, and so $\alpha = \beta + (\alpha - \beta)$ for $|\beta| = 1$, which implies that

$$\begin{aligned} \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-2+\ell}(\Omega_3)}^2 &= \left\| \partial^\alpha \partial_t^{j+1} u \right\|_{H^{2r-1}(\Omega_3)}^2 \le \left\| \partial^{\alpha-\beta} \partial_t^{j+1} u \right\|_{H^{2r}(\Omega_3)}^2 \\ &\le \left\| \bar{D}_0^{2n-2r+1} u \right\|_{H^{2r}(\Omega_3)}^2 \le \mathcal{Z}. \end{aligned} \tag{6.71}$$

Then in either case,

$$\left\| \partial_3^{2r-2+\ell} \partial^\alpha \partial_t^{j+1} u \right\|_{H^0(\Omega_3)}^2 \le \mathcal{Z}. \tag{6.72}$$

We have written the equations (6.65)–(6.66) in this form so as to be able to employ the estimates (6.62), (6.67), (6.72) to derive (6.63). We must consider the case of $\ell = 1$ and $\ell = 2$ separately, starting with $\ell = 1$.

Let $\ell = 1$. According to the equation $\operatorname{div} u = G^2$ (the second of (2.23)) and the bounds (6.62) and (6.67) we may estimate

$$\begin{aligned} \left\| \partial_3^{2r+1} \partial^\alpha \partial_t^j u_3 \right\|_{H^0(\Omega_3)}^2 &= \left\| \partial_3^{2r} \partial^\alpha \partial_t^j (G^2 - \partial_1 u_1 - \partial_2 u_2) \right\|_{H^0(\Omega_3)}^2 \\ &\lesssim \left\| \partial_3^{2r-1} \partial^\alpha \partial_t^j G^2 \right\|_1^2 + \left\| \partial^\alpha \partial_t^j (\partial_1 u_1 + \partial_2 u_2) \right\|_{H^{2r}(\Omega_3)}^2 \lesssim \mathcal{Z}, \end{aligned} \tag{6.73}$$

and hence

$$\left\| \Delta(\partial_3^{2r-1} \partial^\alpha \partial_t^j u_3) \right\|_{H^0(\Omega_3)}^2 \lesssim \left\| \partial_3^{2r+1} \partial^\alpha \partial_t^j u_3 \right\|_{H^0(\Omega_3)}^2 + \left\| \partial_3^{2r-1} (\partial_1^2 + \partial_2^2) \partial^\alpha \partial_t^j u_3 \right\|_{H^0(\Omega_3)}^2 \lesssim \mathcal{Z}. \tag{6.74}$$

We may then use (6.67), (6.72), and (6.74) in (6.65) for the pressure estimate

$$\left\| \partial_3^{2r} \partial^\alpha \partial_t^j p \right\|_{H^0(\Omega_3)}^2 \lesssim \mathcal{Z}. \tag{6.75}$$

Turning now to the $i = 1, 2$ components, we note that by (6.62)

$$\begin{aligned} \left\| \partial_i \partial_3^{2r-1} \partial^\alpha \partial_t^j p \right\|_{H^0(\Omega_3)}^2 &+ \left\| (\partial_1^2 + \partial_2^2) \partial_3^{2r-1} \partial^\alpha \partial_t^j u_i \right\|_{H^0(\Omega_3)}^2 \\ &\lesssim \left\| \bar{D}_0^{2n-2r+1} \nabla p \right\|_{H^{2r-2}(\Omega_3)}^2 + \left\| \bar{D}_0^{2n-2r+1} u \right\|_{H^{2r}(\Omega_3)}^2 \lesssim \mathcal{Z} \end{aligned} \tag{6.76}$$

for $i = 1, 2$. Plugging this, (6.67), and (6.72) into (6.66) then shows that

$$\left\| \partial_3^{2r+1} \partial^\alpha \partial_t^j u_i \right\|_{H^0(\Omega_3)}^2 \lesssim \mathcal{Z} \text{ for } i = 1, 2. \tag{6.77}$$

Upon summing (6.73), (6.75), and (6.77) over $0 \le j \le 2n - r - 1$ and α satisfying $0 \le 2j + |\alpha| \le 2n - 2r$, we deduce, in light of (6.62), that

$$\left\| \bar{D}_0^{2n-2r} u \right\|_{H^{2r+1}(\Omega_3)}^2 + \left\| \bar{D}_0^{2n-2r} \nabla p \right\|_{H^{2r-1}(\Omega_3)}^2 \lesssim \mathcal{Z}. \tag{6.78}$$

In the case $\ell = 2$ we may argue as in the case $\ell = 1$, utilizing both (6.62) and (6.78) to derive the bound

$$\left\| \bar{D}_0^{2n-2r-1} u \right\|_{H^{2r+2}(\Omega_3)}^2 + \left\| \bar{D}_0^{2n-2r-1} \nabla p \right\|_{H^{2r}(\Omega_3)}^2 \lesssim \mathcal{Z}. \tag{6.79}$$

Then we may add (6.78) to (6.79) to deduce (6.63).

Step 2 – The proof of (6.64)

Now we turn to the proof of (6.64), assuming that (6.62) holds with $r = 1$. By (6.63) we may iterate with $r = 2, \dots, n - 1$ to deduce that

$$\begin{aligned} \left\| D_0^1 u \right\|_{H^{2n}(\Omega_3)}^2 &+ \sum_{j=1}^n \left\| \partial_t^j u \right\|_{H^{2n-2j+1}(\Omega_3)}^2 \\ &+ \left\| D_0^1 \nabla p \right\|_{H^{2n-2}(\Omega_3)}^2 + \sum_{j=1}^{n-1} \left\| \partial_t^j \nabla p \right\|_{H^{2n-2j-1}(\Omega_3)}^2 \lesssim \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n. \end{aligned} \tag{6.80}$$

Let $1 \le \ell \le 2n - 1$. We apply the operator ∂_3^ℓ to the first equation of (2.23) and split into components to get

$$\partial_3^{\ell+1} p = -\partial_t \partial_3^\ell u_3 + \Delta \partial_3^\ell u_3 + \partial_3^\ell G_3^1, \text{ and} \tag{6.81}$$

$$\partial_3^{\ell+2} u_i = -(\partial_1^2 + \partial_2^2)\partial_3^\ell u_i + \partial_t \partial_3^\ell u_i + \partial_i \partial_3^\ell p - \partial_3^\ell G_i^1 \text{ for } i = 1, 2. \tag{6.82}$$

Then (6.80), together with (6.81)–(6.82) and the equation $\partial_3 u_3 = G^4 - \partial_1 u_1 - \partial_2 u_2$, allows us to derive the estimates

$$\left\|\partial_3^{\ell+2} u\right\|_{H^0(\Omega_3)}^2 + \left\|\partial_3^{\ell+1} p\right\|_{H^0(\Omega_3)}^2 \lesssim \bar{\mathcal{D}}_n^0 + \bar{\mathcal{D}}_n^+ + \mathcal{Y}_n. \tag{6.83}$$

This and (6.80) yield (6.64). □

The following result is based on an argument similar to the one used in Theorem 6.1.

Lemma 6.4. *Let $\mathcal{Y}_n$ be as defined in Theorem 6.2. Let $H^{1,2}$ and $H^{2,2}$ be given by (2.42), and write*

$$\mathcal{X}_n = \sum_{j=0}^{n-1} \left\|\partial_t^j H^{1,2}\right\|_{2n-2j-1}^2 + \left\|\partial_t^j H^{2,2}\right\|_{2n-2j}^2. \tag{6.84}$$

Then

$$\sum_{j=0}^{n} \left\|\partial_t^j (\chi_2 u)\right\|_{2n-2j+1}^2 + \sum_{j=0}^{n-1} \left\|\partial_t^j (\chi_2 p)\right\|_{2n-2j}^2 \lesssim \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^0 + \mathcal{X}_n + \mathcal{Y}_n. \tag{6.85}$$

Proof. First note that by Lemma A.7 we may bound

$$\sum_{j=0}^{n} \left\|\partial_t^j (\chi_2 u)\right\|_1^2 \lesssim \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^0. \tag{6.86}$$

When we localize with χ_2 we find that $\chi_2 u$ and $\chi_2 p$ solve

$$\begin{cases} \partial_t(\chi_2 u) - \Delta(\chi_2 u) + \nabla(\chi_2 p) = \chi_2 G^1 + H^{1,2} & \text{in } \Omega \\ \operatorname{div}(\chi_2 u) = \chi_2 G^2 + H^{2,2} & \text{in } \Omega \\ ((\chi_2 p)I - \mathbb{D}(\chi_2 u))e_3 = 0 & \text{on } \Sigma \\ \chi_2 u = 0 & \text{on } \Sigma_b. \end{cases} \tag{6.87}$$

Then for any $j = 0, \dots, n-1$ we may apply Lemma A.8 to see that

$$\begin{aligned} &\left\|\partial_t^j (\chi_2 u)\right\|_{2n-2j+1}^2 + \left\|\partial_t^j (\chi_2 p)\right\|_{2n-2j}^2 \\ &\lesssim \left\|\partial_t^{j+1} (\chi_2 u)\right\|_{2n-2(j+1)+1}^2 + \left\|\partial_t^j (\chi_2 G^1 + H^{1,2})\right\|_{2n-2j-1}^2 + \left\|\partial_t^j (\chi_2 G^2 + H^{2,2})\right\|_{2n-2j}^2 \\ &\lesssim \left\|\partial_t^{j+1} (\chi_2 u)\right\|_{2n-2(j+1)+1}^2 + \mathcal{Y}_n + \mathcal{X}_n. \end{aligned} \tag{6.88}$$

We will use estimate (6.88) and a finite induction to prove (6.85). For $j = n-1$ we use (6.86) to get

$$\left\|\partial_t^{n-1} (\chi_2 u)\right\|_3^2 + \left\|\partial_t^{n-1} (\chi_2 p)\right\|_2^2 \lesssim \left\|\partial_t^n (\chi_2 u)\right\|_1^2 + \mathcal{Y}_n \lesssim \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^0 + \mathcal{Y}_n + \mathcal{X}_n. \tag{6.89}$$

Now suppose that the inequality

$$\left\|\partial_t^{n-\ell} (\chi_2 u)\right\|_{2\ell+1}^2 + \left\|\partial_t^{n-\ell} (\chi_2 p)\right\|_{2\ell}^2 \lesssim \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^0 + \mathcal{Y}_n + \mathcal{X}_n \tag{6.90}$$

holds for $1 \le \ell < n$. We claim that (6.90) holds with ℓ replaced by $\ell+1$. We apply (6.88) with $j = n - \ell - 1$ to get

$$\begin{aligned} \left\|\partial_t^{n-\ell-1} (\chi_2 u)\right\|_{2(\ell+1)+1}^2 + \left\|\partial_t^{n-\ell-1} (\chi_2 p)\right\|_{2(\ell+1)}^2 &\lesssim \left\|\partial_t^{n-\ell} (\chi_2 u)\right\|_{2\ell+1}^2 + \mathcal{Y}_n + \mathcal{X}_n \\ &\lesssim \bar{\mathcal{D}}_n^- + \bar{\mathcal{D}}_n^0 + \mathcal{Y}_n + \mathcal{X}_n, \end{aligned} \tag{6.91}$$

where in the last inequality we have employed the induction hypothesis (6.90). This proves the claim, so by finite induction the bound (6.90) holds for all $\ell = 1, \dots, n$. Summing this bound over $\ell = 1, \dots, n$ and adding (6.86) then yields (6.85). □

7. A priori estimates

In this section we will combine our energy evolution estimates with the comparison estimates to derive a priori estimates for the full energy, $\mathcal{G}_{2N}$, defined by (2.52).

7.1. Estimates involving $\mathcal{F}_{2N}$ and $\mathcal{K}$.

Our first result is an estimate of $\mathcal{F}_{2N}$.

Lemma 7.1. *It holds that*

$$\sup_{0\le r\le t}\mathcal{F}_{2N}(r)\lesssim\exp\left(C\int_0^t\sqrt{\mathcal{K}(r)}dr\right)\times\left[\mathcal{F}_{2N}(0)+t\int_0^t(1+\mathcal{E}_{2N}(r))\mathcal{D}_{2N}(r)dr+\left(\int_0^t\sqrt{\mathcal{K}(r)\mathcal{F}_{2N}(r)}dr\right)^2\right]. \tag{7.1}$$

Proof. Throughout this proof we will write $u=\tilde{u}+u_3e_3$, i.e. we write $\tilde{u}$ for the part of u parallel to Σ. Then η solves the transport equation $\partial_t\eta+\tilde{u}\cdot D\eta=u_3$ on Σ. We may then use Lemma A.5 with $s=1/2$ to estimate

$$\sup_{0\le r\le t}\|\eta(r)\|_{1/2}\le\exp\left(C\int_0^t\|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)}dr\right)\left[\|\eta_0\|_{1/2}+\int_0^t\|u_3(r)\|_{H^{1/2}(\Sigma)}dr\right]. \tag{7.2}$$

By the definition of $\mathcal{K}$, (2.51), we may bound $\|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)}\le\sqrt{\mathcal{K}(r)}$, but we may also use trace theory to bound $\|u_3(r)\|_{H^{1/2}(\Sigma)}\lesssim\mathcal{D}_{2N}(r)$. This allows us to square both sides of (7.2) and utilize Cauchy-Schwarz to deduce that

$$\sup_{0\le r\le t}\|\eta(r)\|_{1/2}^2\lesssim\exp\left(2C\int_0^t\sqrt{K(r)}dr\right)\left[\|\eta_0\|_{1/2}^2+t\int_0^t\mathcal{D}_{2N}(r)dr\right]. \tag{7.3}$$

To go to higher regularity we let $\alpha\in\mathbb{N}^2$ with $|\alpha|=4N$. Then we apply the operator ∂^α to the equation $\partial_t\eta+\tilde{u}\cdot D\eta=u_3$ to see that $\partial^\alpha\eta$ solves the transport equation

$$\partial_t(\partial^\alpha\eta)+\tilde{u}\cdot D(\partial^\alpha\eta)=\partial^\alpha u_3-\sum_{0<\beta\le\alpha}C_{\alpha,\beta}\partial^\beta\tilde{u}\cdot D\partial^{\alpha-\beta}\eta:=G^\alpha \tag{7.4}$$

with the initial condition $\partial^\alpha\eta_0$. We may then apply Lemma A.5 with $s=1/2$ to find that

$$\sup_{0\le r\le t}\|\partial^\alpha\eta(r)\|_{1/2}\le\exp\left(C\int_0^t\|D\tilde{u}(r)\|_{H^{3/2}(\Sigma)}dr\right)\left[\|\partial^\alpha\eta_0\|_{1/2}+\int_0^t\|G^\alpha(r)\|_{1/2}dr\right]. \tag{7.5}$$

We will now estimate $\|G^\alpha\|_{H^{1/2}}$.

For $\beta\in\mathbb{N}^2$ satisfying $2N+1\le|\beta|\le4N$ we may apply Lemma A.1 with $s_1=r=1/2$ and $s_2=2$ to bound

$$\left\|\partial^\beta\tilde{u}D\partial^{\alpha-\beta}\eta\right\|_{1/2}\lesssim\left\|\partial^\beta\tilde{u}\right\|_{H^{1/2}(\Sigma)}\left\|D\partial^{\alpha-\beta}\eta\right\|_2. \tag{7.6}$$

This and trace theory then imply that

$$\sum_{\substack{0<\beta\le\alpha\\2N+1\le|\beta|\le4N}}\left\|C_{\alpha,\beta}\partial^\beta\tilde{u}\cdot D\partial^{\alpha-\beta}\eta\right\|_{1/2}\lesssim\left\|D_{2N+1}^{4N}u\right\|_1\left\|D_1^{2N}\eta\right\|_2\lesssim\sqrt{\mathcal{D}_{2N}\mathcal{E}_{2N}}. \tag{7.7}$$

On the other hand, if β satisfies $1\le|\beta|\le2N$ then we use Lemma A.1 to bound

$$\left\|\partial^\beta\tilde{u}D\partial^{\alpha-\beta}\eta\right\|_{1/2}\lesssim\left\|\partial^\beta\tilde{u}\right\|_{H^2(\Sigma)}\left\|D\partial^{\alpha-\beta}\eta\right\|_{1/2} \tag{7.8}$$

so that

$$\begin{aligned}\sum_{\substack{0<\beta\le\alpha\\1\le|\beta|\le2N}}\left\|C_{\alpha,\beta}\partial^\beta\tilde{u}\cdot D\partial^{\alpha-\beta}\eta\right\|_{1/2}&\lesssim\left\|D_1^{2N}u\right\|_3\left\|D_{2N+1}^{4N-1}\eta\right\|_2+\|D\tilde{u}\|_{H^2(\Sigma)}\left\|D^{4N}\eta\right\|_{1/2}\\&\lesssim\sqrt{\mathcal{E}_{2N}\mathcal{D}_{2N}}+\sqrt{\mathcal{K}\mathcal{F}_{2N}}.\end{aligned} \tag{7.9}$$

The only remaining term in G^α is $\partial^\alpha u_3$, which we estimate with trace theory:

$$\left\| \partial^\alpha u_3 \right\|_{H^{1/2}(\Sigma)} \lesssim \left\| D^{4N} u_3 \right\|_1 \lesssim \sqrt{\mathcal{D}_{2N}}. \tag{7.10}$$

We may then combine (7.7), (7.9), and (7.10) for

$$\left\| G^\alpha \right\|_{1/2} \lesssim (1 + \sqrt{\mathcal{E}_{2N}})\sqrt{\mathcal{D}_{2N}} + \sqrt{\mathcal{K}\mathcal{F}_{2N}}. \tag{7.11}$$

Returning now to (7.5), we square both sides and employ (7.11) and our previous estimate of the term in the exponential to find that

$$\begin{aligned} \sup_{0\le r\le t} \left\| \partial^\alpha \eta(r) \right\|_{1/2}^2 \le \exp\left(2C \int_0^t \sqrt{\mathcal{K}(r)} dr \right) \\ \times \left[\left\| \partial^\alpha \eta_0 \right\|_{1/2}^2 + t\int_0^t (1+\mathcal{E}_{2N}(r))\mathcal{D}_{2N}(r) dr + \left(\int_0^t \sqrt{\mathcal{K}(r)\mathcal{F}_{2N}(r)} dr \right)^2 \right]. \end{aligned} \tag{7.12}$$

Then the estimate (7.1) follows by summing (7.12) over all $|\alpha| = 4N$, adding the resulting inequality to (7.3), and using the fact that $\left\| \eta \right\|_{4N+1/2}^2 \lesssim \left\| \eta \right\|_{1/2}^2 + \left\| D^{4N}\eta \right\|_{1/2}^2$. □

Now we use this result to derive a stronger result.

Proposition 7.2. *There exists a universal constant* $0 < \delta < 1$ *so that if* $\mathcal{G}_{2N}(T) \le \delta$*, then*

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N} \tag{7.13}$$

for all $0 \le t \le T$.

Proof. Suppose $\mathcal{G}_{2N}(T) \le \delta \le 1$, for δ to be chosen later. Fix $0 \le t \le T$. The Sobolev and trace embeddings allow us to estimate $\mathcal{K} \lesssim \mathcal{E}_{N+2}$. Then

$$\int_0^t \sqrt{\mathcal{K}(r)} dr \lesssim \int_0^t \sqrt{\mathcal{E}_{N+2}(r)} dr \le \sqrt{\delta} \int_0^\infty \frac{1}{(1+r)^{2N-4}} dr \lesssim \sqrt{\delta}. \tag{7.14}$$

Since $\delta \le 1$, this implies that for any constant $C > 0$,

$$\exp\left(C\int_0^t \sqrt{\mathcal{K}(r)} dr \right) \lesssim 1. \tag{7.15}$$

Similarly,

$$\left(\int_0^t \sqrt{\mathcal{K}(r)\mathcal{F}_{2N}(r)} dr \right)^2 \lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \right) \left(\int_0^t \sqrt{\mathcal{K}(r)} dr \right)^2 \lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \right) \delta. \tag{7.16}$$

Then (7.14)–(7.16) and Lemma 7.1 imply that

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \le C\left(\mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}_{2N} \right) + C\delta \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \right), \tag{7.17}$$

for some $C > 0$. Then if δ is small enough so that $C\delta \le 1/2$, we may absorb the right-hand $\mathcal{F}_{2N}$ term onto the left and deduce (7.13). □

This bound on $\mathcal{F}_{2N}$ allows us to estimate the integral of $\mathcal{K}\mathcal{F}_{2N}$ and $\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}$.

Corollary 7.3. *There exists a universal constant* $0 < \delta < 1$ *so that if* $\mathcal{G}_{2N}(T) \le \delta$*, then*

$$\int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r) dr \lesssim \delta\mathcal{F}_{2N}(0) + \delta \int_0^t \mathcal{D}_{2N}(r) dr \tag{7.18}$$

and

$$\int_0^t \sqrt{\mathcal{D}_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)} dr \lesssim \mathcal{F}_{2N}(0) + \sqrt{\delta}\int_0^t \mathcal{D}_{2N}(r) dr \tag{7.19}$$

for $0 \le t \le T$.

Proof. Let $\mathcal{G}_{2N}(T) \le \delta$ with δ as small as in Proposition 7.2 so that estimate (7.13) holds. As in Proposition 7.2, we have that $\mathcal{K}(r) \lesssim \mathcal{E}_{N+2}(r) \le \delta(1+r)^{-4N+8}$. This and (7.13) then easily imply (7.18). These two and Cauchy-Schwarz also imply (7.19). □

7.2. **Boundedness at the $2N$ level.**

Theorem 7.4. *There exists a universal constant $\delta > 0$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\sup_{0\le r\le t} \mathcal{E}_{2N}(r) + \int_0^t \mathcal{D}_{2N} + \sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(r)}{(1+r)} \lesssim \mathcal{E}_{2n}(0) + \mathcal{F}_{2N}(0) \tag{7.20}$$

for all $0 \le t \le T$.

Proof. Fix $0 \le t \le T$. For any $\varepsilon \in (0,1)$ we may sum the bounds of Propositions 5.2 and 5.4 to find

$$\begin{aligned} \bar{\mathcal{E}}^+_{2N}(t) + \bar{\mathcal{E}}^-_{2N}(t) + \int_0^t \bar{\mathcal{D}}^+_{2N} + \bar{\mathcal{D}}^-_{2N} \\ \le C_1 \left(\mathcal{E}_{2N}(0) + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}} + \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}^0_{2N} \right). \end{aligned} \tag{7.21}$$

for a constant $C_1 > 0$ independent of ε. On the other hand, Proposition 4.3 provides the estimate

$$\bar{\mathcal{E}}^0_{2N}(t) + \int_0^t \bar{\mathcal{D}}^0_{2N} \le C_2 \left(\mathcal{E}_{2N}(0) + (\mathcal{E}_{2N}(t))^{3/2} + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} \right) \tag{7.22}$$

for a constant $C_2 > 0$. We multiply (7.22) by $1 + C_*$ for a constant $C_* > 0$ (with precise value to be chosen later) and add the resulting inequality to (7.21) for

$$\begin{aligned} \bar{\mathcal{E}}^+_{2N}(t) + \bar{\mathcal{E}}^-_{2N}(t) + (1+C_*)\bar{\mathcal{E}}^0_{2N}(t) + \int_0^t \bar{\mathcal{D}}^+_{2N} + \bar{\mathcal{D}}^-_{2N} + (1+C_*)\bar{\mathcal{D}}^0_{2N} \\ \le C_2(1+C_*)(\mathcal{E}_{2N}(t))^{3/2} + (C_1 + C_2(1+C_*)) \left(\mathcal{E}_{2N}(0) + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} \right) \\ + C_1 \int_0^t \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}} + \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}^0_{2N}. \end{aligned} \tag{7.23}$$

From Theorem 6.1 we know that

$$\mathcal{E}_{2N}(t) \le C_3 \left(\bar{\mathcal{E}}^+_{2N}(t) + \bar{\mathcal{E}}^0_{2N}(t) + (\mathcal{E}_{2N}(t))^{1+\theta} \right), \tag{7.24}$$

and from Theorem 6.2 we know that

$$\int_0^t \mathcal{D}_{2N} \le C_3 \int_0^t \left(\bar{\mathcal{D}}^0_{2N} + \bar{\mathcal{D}}^-_{2N} + \bar{\mathcal{D}}^+_{2N} + (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \mathcal{K}\mathcal{F}_{2N} \right) \tag{7.25}$$

for a constant $C_3 > 0$. We may then combine (7.23)–(7.25) to see that

$$\begin{aligned} \frac{1}{C_3} \left(\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N} \right) + C_* \left(\bar{\mathcal{E}}^0_{2N}(t) + \int_0^t \bar{\mathcal{D}}^0_{2N} \right) \le C_2(1+C_*)(\mathcal{E}_{2N}(t))^{3/2} \\ + (\mathcal{E}_{2N}(t))^{1+\theta} + (1 + C_1 + C_2(1+C_*)) \left(\mathcal{E}_{2N}(0) + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} \right) \\ + \int_0^t C_1 \sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}} + \mathcal{K}\mathcal{F}_{2N} + C_1 \int_0^t \varepsilon \mathcal{D}_{2N} + \varepsilon^{-8N-1}\bar{\mathcal{D}}^0_{2N}. \end{aligned} \tag{7.26}$$

Now we choose

$$\varepsilon = \min\left\{ \frac{1}{2}, \frac{1}{2C_1C_3} \right\} \Rightarrow \varepsilon \in (0,1) \text{ and } \frac{1}{2C_3} \le \frac{1}{C_3} - C_1\varepsilon, \tag{7.27}$$

and then we choose $C_* = C_1 \varepsilon^{-8N-1}$. With this choice of ε and C_*, (7.26) reduces to

$$\frac{1}{2C_3}\left(\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}\right) \le C_2(1+C_*)(\mathcal{E}_{2N}(t))^{3/2} + (\mathcal{E}_{2N}(t))^{1+\theta} + (1+C_1+C_2(1+C_*))\left(\mathcal{E}_{2N}(0) + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N}\right) + \int_0^t C_1\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}} + \mathcal{K}\mathcal{F}_{2N}. \tag{7.28}$$

Let us assume that $\delta \in (0,1)$ is as small as in Corollary 7.3; this allows us to estimate the integrals involving $\mathcal{K}\mathcal{F}_{2N}$ and $\sqrt{\mathcal{D}_{2N}\mathcal{K}\mathcal{F}_{2N}}$ in (7.28) to bound

$$\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N} \le C_4\left(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) + (\mathcal{E}_{2N}(t))^{1+\psi} + \int_0^t (\mathcal{E}_{2N})^\theta \mathcal{D}_{2N} + \sqrt{\delta}\int_0^t \mathcal{D}_{2N}\right) \tag{7.29}$$

for $C_4 > 0$ and $\psi = \min\{1/2, \theta\} > 0$. Now we further assume that δ is small enough so that

$$C_4\sqrt{\delta} \le \frac{1}{4},\ C_4\delta^\theta \le \frac{1}{4},\ \text{and } C_4\delta^\psi \le \frac{1}{2}. \tag{7.30}$$

Then since $\sup_{0\le r\le t}\mathcal{E}_{N+2}(r) \le \mathcal{G}_{2N}(T) \le \delta$, (7.29) implies that

$$\frac{1}{2}\left(\mathcal{E}_{2N}(t) + \int_0^t \mathcal{D}_{2N}\right) \le C_4\left(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)\right). \tag{7.31}$$

If δ is further restricted to be as small as in Proposition 7.2, then we also have that

$$\sup_{0\le r\le t}\frac{\mathcal{F}_{2N}(r)}{(1+r)} \lesssim \sup_{0\le r\le t}\frac{\mathcal{F}_{2N}(0)}{(1+r)} + \sup_{0\le r\le t}\frac{r}{(1+r)}\int_0^r \mathcal{D}_{2N}(s)ds \lesssim \mathcal{F}_{2N}(0) + \int_0^t \mathcal{D}_{2N}(r)dr \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0). \tag{7.32}$$

Then (7.20) follows by summing (7.31) and (7.32). □

7.3. **Decay at the $N+2$ level.** Before showing the decay estimates, we first need an interpolation result.

Proposition 7.5. *There exists a universal $0<\delta<1$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\mathcal{D}_{N+2}(t) \lesssim \bar{\mathcal{D}}^0_{N+2}(t) + \bar{\mathcal{D}}^+_{N+2}(t) + \bar{\mathcal{D}}^-_{N+2}(t),\ \mathcal{E}_{N+2}(t) \lesssim \bar{\mathcal{E}}^0_{N+2}(t) + \bar{\mathcal{E}}^+_{N+2}(t), \tag{7.33}$$

and

$$\mathcal{E}_{N+2} \lesssim (\mathcal{D}_{N+2})^{(4N-8)/(4N-7)}(\mathcal{E}_{2N})^{1/(4N-7)}. \tag{7.34}$$

Proof. The bound $\mathcal{G}_{2N}(T) \le \delta$ and Theorems 6.1 and 6.2 imply that

$$\mathcal{D}_{N+2} \le C(\bar{\mathcal{D}}^0_{N+2}(t) + \bar{\mathcal{D}}^+_{N+2}(t) + \bar{\mathcal{D}}^-_{N+2}(t)) + C\mathcal{E}^\theta_{2N}\mathcal{D}_{N+2} \le C(\bar{\mathcal{D}}^0_{N+2}(t) + \bar{\mathcal{D}}^+_{N+2}(t) + \bar{\mathcal{D}}^-_{N+2}(t)) + C\delta^\theta\mathcal{D}_{N+2} \tag{7.35}$$

and

$$\mathcal{E}_{N+2} \le C(\bar{\mathcal{E}}^0_{N+2}(t) + \bar{\mathcal{E}}^+_{N+2}(t)) + C\mathcal{E}^\theta_{2N}\mathcal{E}_{N+2} \le C(\bar{\mathcal{E}}^0_{N+2}(t) + \bar{\mathcal{E}}^+_{N+2}(t)) + C\delta^\theta\mathcal{E}_{N+2} \tag{7.36}$$

for constants $C>0$ and $\theta>0$. Then if δ is small enough so that $C\delta^\theta \le 1/2$, we may absorb the second term on the right side of (7.35) and (7.36) into the left to deduce the bounds in (7.33).

We now turn to the proof of (7.34), which is based on the standard Sobolev interpolation inequality:

$$\|f\|_s \lesssim \|f\|_{s-r}^{q/(r+q)}\|f\|_{s+q}^{r/(r+q)} \tag{7.37}$$

for $s,q>0$ and $0\le r\le s$. Applying this for $0\le j\le N+2$ with $s = 2(N+2)-2j$, $r=1/2$, and $q = 2N-4$ shows that

$$\left\|\partial_t^j\eta\right\|_{2(N+2)-2j} \lesssim \left\|\partial_t^j\eta\right\|^\theta_{2(N+2)-2j-1/2}\left\|\partial_t^j\eta\right\|^{1-\theta}_{4N-2j} \lesssim (\sqrt{\mathcal{D}_{N+2}})^\theta(\sqrt{\mathcal{E}_{2N}})^{1-\theta}, \tag{7.38}$$

$$\left\|\partial_t^j u\right\|_{2(N+2)-2j} \lesssim \left\|\partial_t^j u\right\|_{2(N+2)-2j-1/2}^{\theta} \left\|\partial_t^j u\right\|_{4N-2j}^{1-\theta} \lesssim (\sqrt{\mathcal{D}_{N+2}})^{\theta}(\sqrt{\mathcal{E}_{2N}})^{1-\theta}, \tag{7.39}$$

where

$$\theta = \frac{4N-8}{4N-7} \text{ and } 1-\theta = \frac{1}{4N-7}. \tag{7.40}$$

Similarly, we may use $0 \le j \le N+1$ with $s = 2(N+2) - 2j - 1$, $r = 1/2$, and $q = 2N-4$

$$\left\|\partial_t^j p\right\|_{2(N+2)-2j-1} \lesssim \left\|\partial_t^j p\right\|_{2(N+2)-2j-3/2}^{\theta} \left\|\partial_t^j p\right\|_{4N-2j-1}^{1-\theta} \lesssim (\sqrt{\mathcal{D}_{N+2}})^{\theta}(\sqrt{\mathcal{E}_{2N}})^{1-\theta}. \tag{7.41}$$

We may then sum the squares of these interpolation inequalities to deduce (7.34). □

Now we show that the extra integral term appearing in Proposition 4.4 can essentially be absorbed into $\bar{\mathcal{E}}_{N+2}^0 + \bar{\mathcal{E}}_{N+2}^+$.

Lemma 7.6. *Let F^2 be defined by (2.19) with $\partial^\alpha = \partial_t^{N+2}$. There exists a universal $0 < \delta < 1$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\frac{2}{3}(\bar{\mathcal{E}}_{N+2}^0(t) + \bar{\mathcal{E}}_{N+2}^+(t)) \le \bar{\mathcal{E}}_{N+2}^0(t) + \bar{\mathcal{E}}_{N+2}^+(t) - 2\int_\Omega J(t)\partial_t^{N+1}p(t)F^2(t) \le \frac{4}{3}(\bar{\mathcal{E}}_{N+2}^0(t) + \bar{\mathcal{E}}_{N+2}^+(t)) \tag{7.42}$$

for all $0 \le t \le T$.

Proof. Suppose that δ is as small as in Proposition 7.5. Then we combine estimate (4.5) of Theorem 4.2, Lemma 2.3, and estimate (7.33) of Proposition 7.5 to see that

$$\begin{aligned} \|J\|_{L^\infty} \left\|\partial_t^{N+1} p\right\|_0 \left\|F^2\right\|_0 &\lesssim \sqrt{\mathcal{E}_{N+2}}\sqrt{\mathcal{E}_{2N}^{\theta}\mathcal{E}_{N+2}} \\ &= \mathcal{E}_{2N}^{\theta/2}\mathcal{E}_{N+2} \lesssim \mathcal{E}_{2N}^{\theta/2}(\bar{\mathcal{E}}_{N+2}^0 + \bar{\mathcal{E}}_{N+2}^+) \lesssim \delta^{\theta/2}(\bar{\mathcal{E}}_{N+2}^0 + \bar{\mathcal{E}}_{N+2}^+) \end{aligned} \tag{7.43}$$

for some $\theta > 0$. This estimate and Cauchy-Schwarz then imply that

$$\left|2\int_\Omega J\partial_t^{N+1}pF^2\right| \le 2\|J\|_{L^\infty}\left\|\partial_t^{N+1}p\right\|_0 \left\|F^2\right\|_0 \le C\delta^{\theta/2}(\bar{\mathcal{E}}_{N+2}^0 + \bar{\mathcal{E}}_{N+2}^+) \le \frac{1}{3}(\bar{\mathcal{E}}_{N+2}^0 + \bar{\mathcal{E}}_{N+2}^+) \tag{7.44}$$

if δ is small enough. The bound (7.42) follows easily from (7.44). □

Now we prove decay at the $N+2$ level.

Theorem 7.7. *There exists a universal constant $0 < \delta < 1$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$\sup_{0 \le r \le t}(1+r)^{4N-8}\mathcal{E}_{N+2}(r) \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) \tag{7.45}$$

for all $0 \le t \le T$.

Proof. Fix $0 \le t \le T$. According to Propositions 5.3, 5.5, there exist constants $C_1 > 0$ so that for any $\varepsilon \in (0,1)$,

$$\partial_t(\bar{\mathcal{E}}_{N+2}^+ + \bar{\mathcal{E}}_{N+2}^-) + \bar{\mathcal{D}}_{N+2}^+ + \bar{\mathcal{D}}_{N+2}^- \le C_1(\mathcal{E}_{2N}^{\theta}\mathcal{D}_{N+2} + \varepsilon\mathcal{D}_{N+2} + \varepsilon^{-4N-9}\bar{\mathcal{D}}_{N+2}^0). \tag{7.46}$$

On the other hand, Proposition 4.4 provides a constant $C_2 > 0$ so that

$$\partial_t\left(\bar{\mathcal{E}}_{N+2}^0 - \int_\Omega 2J\partial_t^{N+1}pF^2\right) + \bar{\mathcal{D}}_{N+2}^0 \le C_2\sqrt{\mathcal{E}_{2N}}\mathcal{D}_{N+2}. \tag{7.47}$$

We multiply inequality (7.47) by $1 + C_*$ for $C_* > 0$ a constant to be chosen later and add the resulting inequality to (7.46) to find

$$\begin{aligned} &\partial_t\left(\bar{\mathcal{E}}_{N+2}^+ + \bar{\mathcal{E}}_{N+2}^- + (1+C_*)\bar{\mathcal{E}}_{N+2}^0 - \int_\Omega 2J\partial_t^{N+1}pF^2\right) + (\bar{\mathcal{D}}_{N+2}^+ + \bar{\mathcal{D}}_{N+2}^- + \bar{\mathcal{D}}_{N+2}^0) + C_*\bar{\mathcal{D}}_{N+2}^0 \\ &\qquad \le (C_1 + C_2C_*)(\mathcal{E}_{2N})^{\psi}\mathcal{D}_{N+2} + C_1(\varepsilon\mathcal{D}_{N+2} + \varepsilon^{-4N-9}\bar{\mathcal{D}}_{N+2}^0), \end{aligned} \tag{7.48}$$

where $\psi = \min\{1/2, \theta\}$.

Let $\delta \in (0,1)$ be as small as in both Proposition 7.5 and Lemma 7.6. Then

$$\mathcal{D}_{N+2} \le C_3(\bar{\mathcal{D}}^0_{n+2} + \bar{\mathcal{D}}^-_{n+2} + \bar{\mathcal{D}}^+_{n+2}) \tag{7.49}$$

for $C_3 \ge 1$ (we know that $C_3 > 0$, but we may further assume this). Then (7.49) and (7.48) imply that

$$\begin{aligned}\partial_t \left(\bar{\mathcal{E}}^+_{N+2} + \bar{\mathcal{E}}^-_{N+2} + (1+C_*)\bar{\mathcal{E}}^0_{N+2} - \int_\Omega 2J\partial_t^{N+1} pF^2 \right) + \frac{1}{C_3}\mathcal{D}_{n+2} + C_*\bar{\mathcal{D}}^0_{2n} \\ \le (C_1 + C_2 C_*)(\mathcal{E}_{2N})^\psi \mathcal{D}_{N+2} + C_1(\varepsilon \mathcal{D}_{N+2} + \varepsilon^{-4N-9}\bar{\mathcal{D}}^0_{N+2}).\end{aligned} \tag{7.50}$$

Now we choose

$$\varepsilon = \min\left\{ \frac{1}{2}, \frac{1}{4C_1C_3} \right\} \Rightarrow \frac{3}{4C_3} \le \frac{1}{C_3} - C_1\varepsilon \tag{7.51}$$

and $C_* = C_1 \varepsilon^{-4N-9}$. Further, we assume δ is sufficiently small so that

$$(C_1 + C_2C_*)\delta^\psi \le \frac{1}{4C_3}. \tag{7.52}$$

With this choice of ε, C_* and the bound $\mathcal{E}_{2N} \le \mathcal{G}_{2N}(T) \le \delta$, the inequality (7.50) implies

$$\partial_t \left(\bar{\mathcal{E}}^+_{N+2} + \bar{\mathcal{E}}^-_{N+2} + (1+C_*)\bar{\mathcal{E}}^0_{N+2} - \int_\Omega 2J\partial_t^{N+1} pF^2 \right) + \frac{1}{2C_3}\mathcal{D}_{N+2} \le 0. \tag{7.53}$$

Let δ be as small as in Theorem 7.4, Proposition 7.5, and Lemma 7.6. Then Theorem 7.4, (7.34) of Proposition 7.5, and (7.42) of Lemma 7.6 imply that

$$\begin{aligned} 0 &\le \frac{2}{3}(\bar{\mathcal{E}}^+_{N+2} + \bar{\mathcal{E}}^-_{N+2}) + (1+C_*)\bar{\mathcal{E}}^-_{N+2} \le \bar{\mathcal{E}}^+_{N+2} + \bar{\mathcal{E}}^-_{N+2} + (1+C_*)\bar{\mathcal{E}}^0_{N+2} - \int_\Omega 2J\partial_t^{N+1} pF^2 \\ &\le \frac{4}{3}(\bar{\mathcal{E}}^+_{N+2} + \bar{\mathcal{E}}^-_{N+2}) + (1+C_*)\bar{\mathcal{E}}^-_{N+2} \le C_4 \mathcal{E}_{N+2} \le C C_4 (\mathcal{E}_{2N})^{1/(4N-7)} (\mathcal{D}_{N+2})^{(4N-8)/(4N-7)} \\ &\le C_5 \mathcal{Z}_0^{1/(4N-7)} (\mathcal{D}_{N+2})^{(4N-8)/(4N-7)} \end{aligned} \tag{7.54}$$

for all $0 \le t \le T$, where we have written $\mathcal{Z}_0 := \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)$, and C_4, C_5 are universal constants which we may assume satisfy $C_5 \ge C_4 \ge 1$. Let us write

$$h(t) = \bar{\mathcal{E}}^+_{N+2}(t) + \bar{\mathcal{E}}^-_{N+2}(t) + (1+C_*)\bar{\mathcal{E}}^0_{N+2}(t) - \int_\Omega 2J(t)\partial_t^{N+1} p(t)F^2(t) \ge 0, \tag{7.55}$$

as well as

$$s = \frac{1}{4N-8} \text{ and } C_6 = \frac{1}{2C_3 C_5^{1+s}\mathcal{Z}_0^s}. \tag{7.56}$$

We may then combine (7.53) with (7.54) and use our new notation to derive the differential inequality

$$\partial_t h(t) + C_6 (h(t))^{1+s} \le 0 \tag{7.57}$$

for $0 \le t \le T$.

Since (7.54) says that $h(t) \ge 0$, we may integrate (7.57) to find that for any $0 \le r \le T$,

$$h(r) \le \frac{h(0)}{[1 + sC_6(h(0))^s r]^{1/s}}. \tag{7.58}$$

Then (7.54) implies that $h(0) \le C_4 \mathcal{E}_{N+2}(0) \le C_4 \mathcal{E}_{2N}(0) \le C_4 \mathcal{Z}_0$, which in turn implies that

$$sC_6(h(0))^s = \frac{s}{2C_3C_5^{1+s}}\left(\frac{h(0)}{\mathcal{Z}_0}\right)^s \le \frac{s}{2C_3C_5^{1+s}} C_4^s = \frac{s}{2C_3C_5}\left(\frac{C_4}{C_5}\right)^s \le 1 \tag{7.59}$$

since $0 < s < 1$, $C_5 \ge C_4 \ge 1$, and $C_3 \ge 1$. A simple computation shows that

$$\sup_{r \ge 0} \frac{(1+r)^{1/s}}{(1+Mr)^{1/s}} = \frac{1}{M^{1/s}} \tag{7.60}$$

when $0 \le M \le 1$ and $s > 0$. This, (7.58), and (7.59) then imply that

$$(7.61)\quad (1+r)^{1/s} h(r) \le h(0) \frac{(1+r)^{1/s}}{[1+sC_6(h(0))^s r]^{1/s}} \le h(0) \left(\frac{2C_3 C_5^{1+s}}{s}\right)^{1/s} \frac{\mathcal{Z}_0}{h(0)} = \left(\frac{2C_3 C_5^{1+s}}{s}\right)^{1/s} \mathcal{Z}_0.$$

Now we use (7.33) of Proposition 7.5 together with (7.54) to bound

$$(7.62)\quad \mathcal{E}_{N+2}(r) \lesssim \bar{\mathcal{E}}^0_{N+2}(r) + \bar{\mathcal{E}}^+_{N+2}(r) \lesssim h(r) \text{ for } 0 \le r \le T.$$

The estimate (7.45) then follows from (7.61), (7.62), and the fact that $s = 1/(4N-8)$ and $\mathcal{Z}_0 = \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)$. □

7.4. **A priori estimates for $\mathcal{G}_{2N}$.** We now collect the results of Theorems 7.4 and 7.7 into a single bound on $\mathcal{G}_{2N}$. The estimate recorded specifically names the constant in the inequality with $C_1 > 0$ so that it can be referenced later.

Theorem 7.8. *There exists a universal $0 < \delta < 1$ so that if $\mathcal{G}_{2N}(T) \le \delta$, then*

$$(7.63)\quad \mathcal{G}_{2N}(t) \le C_1(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0))$$

for all $0 \le t \le T$, where $C_1 > 0$ is a universal constant.

Proof. Let δ be as small as in Theorems 7.4 and 7.7. Then the conclusions of the theorems hold, and we may sum them to deduce (7.63). □

8. Specialized local well-posedness

We now record a specialized version of the local well-posedness theorem.

Theorem 8.1. *Suppose the initial data satisfy the compatibility conditions of Theorem 1.1 and $\|u(0)\|^2_{4N} + \|\eta(0)\|^2_{4N+1/2} < \infty$. Let $\varepsilon > 0$. There exists a $\delta_0 = \delta_0(\varepsilon) > 0$ and a*

$$(8.1)\quad T_0 = C(\varepsilon) \min\left\{1, \frac{1}{\|\eta(0)\|^2_{4N+1/2}}\right\} > 0,$$

where $C(\varepsilon) > 0$ is a constant depending on ε, so that if $0 < T \le T_0$ and $\|u(0)\|^2_{4N} + \|\eta(0)\|^2_{4N} \le \delta_0$, then there exists a unique solution (u, p, η) to (1.12) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimates

$$(8.2)\quad \sup_{0 \le t \le T} \mathcal{E}_{2N}(t) + \int_0^T \mathcal{D}_{2N}(t) dt + \int_0^T \left(\left\| \partial_t^{2N+1} u(t) \right\|^2_{({}_0H^1)^*} + \left\| \partial_t^{2N} p(t) \right\|^2_0 \right) dt \le C_2 \varepsilon$$

and

$$(8.3)\quad \sup_{0 \le t \le T} \mathcal{F}_{2N}(t) \le C_2 \mathcal{F}_{2N}(0) + \varepsilon$$

for $C_2 > 0$ a universal constant. If η_0 satisfies the zero average condition

$$(8.4)\quad \int_\Sigma \eta_0 = 0, \text{ then } \int_\Sigma \eta(t) = 0$$

for all $t \in [0, T]$.

Proof. The existence, uniqueness, and estimates follow directly from Theorem 1.1. Then (8.4) follows from (1.5) and the fact that η_0 satisfies the zero average condition. □

Remark 8.2. *The finiteness of the terms on the left of (8.2)–(8.3) justify all of the computations leading to Theorem 7.8.*

9. Global well-posedness and decay: proof of Theorem 1.3

In order to combine the local existence result, Theorem 8.1, with the a priori estimates of Theorem 7.8, we must be able to estimate $\mathcal{G}_{2N}$ in terms of the estimates given in (8.2)–(8.3). We record this estimate now.

Proposition 9.1. *Suppose that $N \geq 3$. Then there exists a universal constant $C_3 > 0$ with the following properties. If $0 \leq T$, then we have the estimate*

$$\mathcal{G}_{2N}(T) \leq \sup_{0\leq t\leq T} \mathcal{E}_{2N}(t) + \int_0^{T_2} \mathcal{D}_{2N}(t)dt + \sup_{0\leq t\leq T} \mathcal{F}_{2N}(t) + C_3(1+T)^{4N-8} \sup_{0\leq t\leq T} \mathcal{E}_{2N}(t). \tag{9.1}$$

If $0 < T_1 \leq T_2$, then we have the estimate

$$\begin{aligned}\mathcal{G}_{2N}(T_2) \leq C_3 \mathcal{G}_{2N}(T_1) + \sup_{T_1\leq t\leq T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)dt \\ + \frac{1}{(1+T_1)} \sup_{T_1\leq t\leq T_2} \mathcal{F}_{2N}(t) + C_3(T_2-T_1)^2(1+T_2)^{4N-8} \sup_{T_1\leq t\leq T_2} \mathcal{E}_{2N}(t).\end{aligned} \tag{9.2}$$

Proof. We will only prove the estimate (9.2); the bound (9.1) follows from a similar, but easier argument. The definition of $\mathcal{G}_{2N}(T_2)$ allows us to estimate

$$\begin{aligned}\mathcal{G}_{2N}(T_2) \leq \mathcal{G}_{2N}(T_1) + \sup_{T_1\leq t\leq T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t)dt \\ + \sup_{T_1\leq t\leq T_2} \frac{\mathcal{F}_{2N}(t)}{(1+t)} + \sup_{T_1\leq t\leq T_2} \left((1+t)^{4N-8}\mathcal{E}_{N+2}(t)\right).\end{aligned} \tag{9.3}$$

Since $N \geq 3$ it is easy to verify that

$$\sum_{j=0}^{N+2} \left\|\partial_t^{j+1} u\right\|^2_{2(N+2)-2j} + \left\|\partial_t^j u\right\|^2_{2(N+2)-2j} + \left\|\partial_t^{j+1} \eta\right\|^2_{2(N+2)-2j} + \left\|\partial_t^j \eta\right\|^2_{2(N+2)-2j} \lesssim \mathcal{E}_{2N} \tag{9.4}$$

and

$$\sum_{j=0}^{N+1} \left\|\partial_t^{j+1} p\right\|^2_{2(N+2)-2j-1} + \left\|\partial_t^j p\right\|^2_{2(N+2)-2j-1} \lesssim \mathcal{E}_{2N}. \tag{9.5}$$

For $j = 0, \dots, 2N$, we may then integrate $\partial_t \left[(1+t)^{(4N-8)/2}\partial_t^j u(t)\right]$ in time from T_1 to $T_1 \leq t \leq T_2$ to deduce the bound

$$\begin{aligned}\left\|(1+t)^{(4N-8)/2}\partial_t^j u(t)\right\|_{2N+4-2j} \leq \left\|(1+T_1)^{(4N-8)/2}\partial_t^j u(T_1)\right\|_{2N+4-2j} \\ + \int_{T_1}^{T_2} (1+s)^{(4N-8)/2} \left\|\partial_t^{j+1} u(s)\right\|_{2N+4-2j} + \frac{(4N-8)}{2}(1+s)^{(4N-10)/2} \left\|\partial_t^j u(s)\right\|_{2N+4-2j} \\ \lesssim \sqrt{\mathcal{G}_{2N}(T_1)} + (T_2-T_1)(1+T_2)^{(4N-8)/2} \sqrt{\sup_{T_1\leq t\leq T_2} \mathcal{E}_{2N}(t)}.\end{aligned} \tag{9.6}$$

Squaring both sides of this then yields, for $j = 0, \dots, N+2$,

$$\sup_{T_1\leq t\leq T_2} \left((1+t)^{4N-8} \left\|\partial_t^j u(t)\right\|^2_{2(N+2)-2j}\right) \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{4N-8} \sup_{T_1\leq t\leq T_2} \mathcal{E}_{2N}(t). \tag{9.7}$$

Similar estimates hold for $j = 0, \dots, N+2$ with $\partial_t^j u$ replaced by $\partial_t^j \eta$ and for $j = 0, \dots, N+1$ with $\left\|\partial_t^j u(t)\right\|^2_{2(N+2)-2j}$ replaced by $\left\|\partial_t^j p(t)\right\|^2_{2(N+2)-2j-1}$. From these we may then estimate

$$\sup_{T_1\leq t\leq T_2} \left((1+t)^{4N-8}\mathcal{E}_{N+2}(t)\right) \lesssim \mathcal{G}_{2N}(T_1) + (T_2-T_1)^2(1+T_2)^{4N-8} \sup_{T_1\leq t\leq T_2} \mathcal{E}_{2N}(t). \tag{9.8}$$

Then (9.2) follows from (9.3), (9.8), and the trivial bound

$$\sup_{T_1 \le t \le T_2} \frac{\mathcal{F}_{2N}(t)}{(1+t)} \le \frac{1}{(1+T_1)} \sup_{T_1 \le t \le T_2} \mathcal{F}_{2N}(t). \tag{9.9}$$

□

We now turn to our main result.

Proof of Theorem 1.3. Let $0 < \delta < 1$ and $C_1 > 0$ be the constants from Theorem 7.8, $C_2 > 0$ be the constant from Theorem 8.1, and $C_3 > 0$ be the constant from Proposition 9.1. According to (9.1) of Proposition 9.1, if a solution exists on the interval $[0, T]$ with $T < 1$ and obeys the estimates (8.2)–(8.3), then

$$\mathcal{G}_{2N}(T) \le C_2 \kappa + \varepsilon \left[C_2 + 1 + C_3 (2)^{4N-8} \right]. \tag{9.10}$$

If ε is chosen so that the latter term in (9.10) equals $\delta/2$, then we may choose κ sufficiently small so that $C_2\kappa < \delta/2$ and $\kappa < \delta_0(\varepsilon)$ (with $\delta_0(\varepsilon)$ given by Theorem 8.1); then Theorem 8.1 provides a unique solution on $[0, T]$ obeying the estimates (8.2)–(8.3), and hence $\mathcal{G}_{2N}(T) \le \delta$. According to Remark 8.2, all of the computations leading to Theorem 7.8 are justified by the estimates (8.2)–(8.3).

Let us now define

$$T_*(\kappa) = \sup\{T > 0 \mid \text{for every choice of initial data satisfying the compatibility conditions and } \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa \text{ there exists a unique solution on } [0,T] \text{ that achieves the data and satisfies } \mathcal{G}_{2N}(T) \le \delta\}. \tag{9.11}$$

By the above analysis, $T_*(\kappa)$ is well-defined and satisfies $T_*(\kappa) > 0$ if κ is small enough, i.e. there is a $\kappa_1 > 0$ so that $T_* : (0, \kappa_1] \to (0, \infty]$. It is easily verified that T_* is non-increasing on $(0, \kappa_1]$. Let us now set

$$\varepsilon = \frac{\delta}{3} \min\left\{ \frac{1}{1 + C_2}, \frac{1}{C_3} \right\} \tag{9.12}$$

and then define $\kappa_0 \in (0, \kappa_1]$ by

$$\kappa_0 = \min\left\{ \frac{\delta}{3C_1(C_3 + 2C_2)}, \frac{\delta_0(\varepsilon)}{C_1}, \kappa_1 \right\}, \tag{9.13}$$

where $\delta_0(\varepsilon)$ is given by Theorem 8.1 with ε given by (9.12). We claim that $T_*(\kappa_0) = \infty$. Once the claim is established, the proof of the theorem is complete since then $T_*(\kappa) = \infty$ for all $0 < \kappa \le \kappa_0$.

Suppose, by way of contradiction, that $T_*(\kappa_0) < \infty$. We will show that solutions can actually be extended past $T_*(\kappa_0)$ and that these solutions satisfy $\mathcal{G}_{2N}(T_2) \le \delta$ for $T_2 > T_*(\kappa_0)$, contradicting the definition of $T_*(\kappa_0)$. We begin by extending the solutions. By the definition of $T_*(\kappa_0)$, we know that for every $0 < T_1 < T_*(\kappa_0)$ and for any choice of data satisfying the compatibility conditions and the bound $\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) < \kappa_0$, there exists a unique solution on $[0, T_1]$ that achieves the initial data and satisfies $\mathcal{G}_{2N}(T_1) \le \delta$. Then by Theorem 7.8, we know that actually

$$\mathcal{G}_{2N}(T_1) \le C_1(\mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0)) < C_1 \kappa_0. \tag{9.14}$$

In particular, this and (9.13) imply that

$$\mathcal{E}_{2N}(T_1) + \frac{\mathcal{F}_{2N}(T_1)}{(1+T_1)} < C_1\kappa_0 \le \delta_0(\varepsilon) \text{ for all } 0 < T_1 < T_*(\kappa_0), \tag{9.15}$$

where ε is given by (9.12). We view $(u(T_1), p(T_1), \eta(T_1))$ as initial data for a new problem; since (u, p, η) are already solutions, they satisfy the compatibility conditions needed to use them as data. Then since $\mathcal{E}_{2N}(T_1) < \delta_0(\varepsilon)$, we can use Theorem 8.1 with ε given by (9.12) to extend solutions to $[T_1, T_2]$ for any T_2 satisfying

$$0 < T_2 - T_1 \le T_0 = C(\varepsilon) \min\{1, \mathcal{F}_{2N}(T_1)^{-1}\}. \tag{9.16}$$

In light of (9.15), we may bound

$$\bar{T} := C(\varepsilon) \min \left\{ 1, \frac{1}{\delta_0(\varepsilon)(1+T_*(\kappa_0))} \right\} \le T_0. \tag{9.17}$$

Notice that $\bar{T}$ depends on ε (given by (9.12)) and $T_*(\kappa_0)$, but is independent of T_1. Let

$$\gamma = \min \left\{ \bar{T}, T_*(\kappa_0), \frac{1}{(1+2T_*(\kappa_0))^{(4N-8)/2}} \right\}, \tag{9.18}$$

and then let us choose $T_1 = T_*(\kappa_0) - \gamma/2$ and $T_2 = T_*(\kappa_0) + \gamma/2$. The choice of γ implies that

$$0 < T_1 < T_*(\kappa_0) < T_2 < 2T_*(\kappa_0) \text{ and } 0 < \gamma = T_2 - T_1 \le \bar{T} \le T_0. \tag{9.19}$$

Then Theorem 8.1 allows us to extend solutions to the interval $[0, T_2]$, and it provides estimates on the extended interval $[T_1, T_2]$:

$$\sup_{T_1 \le t \le T_2} \mathcal{E}_{2N}(t) + \int_{T_1}^{T_2} \mathcal{D}_{2N}(t) dt + \int_{T_1}^{T_2} \left(\left\| \partial_t^{2N+1} u(t) \right\|_{({}_0H^1)^*}^2 + \left\| \partial_t^{2N} p(t) \right\|_0^2 \right) dt \le C_2 \varepsilon, \tag{9.20}$$

$$\sup_{T_1 \le t \le T_2} \mathcal{F}_{2N}(t) \le C_2 \mathcal{F}_{2N}(T_1) + \varepsilon. \tag{9.21}$$

Having extended the existence interval, we will now show that $\mathcal{G}_{2N}(T_2) \le \delta$. We combine the estimates (9.20)–(9.21) with (9.14)–(9.15) and the bound (9.2) of Proposition 9.1 to see that

$$\begin{aligned} \mathcal{G}_{2N}(T_2) &< C_1 C_3 \kappa_0 + C_2(\varepsilon + C_1 \kappa_0) + \frac{C_1 C_2 \kappa_0 (1+T_1) + \varepsilon}{(1+T_1)} + \varepsilon C_3 (T_2 - T_1)^2 (1+T_2)^{4N-8} \\ &\le \kappa_0 C_1 (C_3 + 2C_2) + \varepsilon(1 + C_2) + \varepsilon C_3 \gamma^2 (1 + 2T_*(\kappa_0))^{4N-8} \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta, \end{aligned} \tag{9.22}$$

where the second inequality follows from (9.19) and the third follows from the choice of ε, κ_0, and γ given in (9.12), (9.13), and (9.18), respectively. Hence $\mathcal{G}_{2N}(T_2) \le \delta$, contradicting the definition of $T_*(\kappa_0)$. We deduce then that $T_*(\kappa_0) = \infty$, which completes the proof of the claim and the theorem. □

Appendix A. Analytic tools

A.1. **Products in Sobolev spaces.** We will need some estimates of the product of functions in Sobolev spaces.

Lemma A.1. *The following hold on Σ and on sufficiently smooth subsets of $\mathbb{R}^n$.*

(1) *Let $0 \le r \le s_1 \le s_2$ be such that $s_1 > n/2$. Let $f \in H^{s_1}$, $g \in H^{s_2}$. Then $fg \in H^r$ and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A.1}$$

(2) *Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{s_1}$, $g \in H^{s_2}$. Then $fg \in H^r$ and*

$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{A.2}$$

(3) *Let $0 \le r \le s_1 \le s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. Then $fg \in H^{-s_1}(\Sigma)$ and*

$$\|fg\|_{-s_1} \lesssim \|f\|_{-r} \|g\|_{s_2}. \tag{A.3}$$

Proof. The proofs of (A.1) and (A.2) are standard; the bounds are first proved in $\mathbb{R}^n$ with the Fourier transform, and then the bounds in sufficiently nice subsets of $\mathbb{R}^n$ are deduced by use of an extension operator. To prove (A.3) we argue by duality. For $\varphi \in H^{s_1}$ we use (A.2)bound

$$\int_\Sigma \varphi f g \lesssim \|\varphi g\|_r \|f\|_{-r} \lesssim \|\varphi\|_{s_1} \|g\|_{s_2} \|f\|_{-r}, \tag{A.4}$$

so that taking the supremum over φ with $\|\varphi\|_{s_1} \le 1$ we get (A.3). □

We will also need the following variant.

Lemma A.2. *Suppose that $f \in C^1(\Sigma)$ and $g \in H^{1/2}(\Sigma)$. Then*

$$\|fg\|_{1/2} \lesssim \|f\|_{C^1} \|g\|_{1/2} . \tag{A.5}$$

Proof. Consider the operator $F : H^k \to H^k$ given by $F(g) = fg$ for $k = 0, 1$. It is a bounded operator for $k = 0, 1$ since

$$\|fg\|_0 \le \|f\|_{C^1} \|g\|_0 \text{ and } \|fg\|_1 \lesssim \|f\|_{C^1} \|g\|_1 . \tag{A.6}$$

Then the theory of interpolation of operators implies that F is bounded from $H^{1/2}$ to itself, with operator norm less than a constant times $\sqrt{\|f\|_{C^1}}\sqrt{\|f\|_{C^1}} = \|f\|_{C^1}$, which is the desired result. □

A.2. **Poisson integral.** Suppose that $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$. We define the Poisson integral in $\Omega_- = \Sigma \times (-\infty, 0)$ by

$$\mathcal{P}f(x) = \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} e^{2\pi i n\cdot x'} e^{2\pi|n|x_3} \hat{f}(n), \tag{A.7}$$

where for $n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})$ we have written

$$\hat{f}(n) = \int_\Sigma f(x') \frac{e^{-2\pi i n\cdot x'}}{L_1 L_2} dx'. \tag{A.8}$$

It is well known that $\mathcal{P} : H^s(\Sigma) \to H^{s+1/2}(\Omega_-)$ is a bounded linear operator for $s > 0$. We now show that how derivatives of $\mathcal{P}f$ can be estimated in the smaller domain Ω.

Lemma A.3. *Let $\mathcal{P}f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then*

$$\|\nabla^q \mathcal{P}f\|_0^2 \lesssim \|f\|^2_{\dot{H}^{q-1/2}(\Sigma)} \text{ and } \|\nabla^q \mathcal{P}f\|_0^2 \lesssim \|f\|^2_{\dot{H}^q(\Sigma)} . \tag{A.9}$$

Proof. Since $\mathcal{P}f$ is defined on $\Sigma\times(-\infty,0)$, it suffices to prove the estimates on $\tilde{\Omega} := \Sigma\times(-b_+,0)$ since $\Omega \subset \tilde{\Omega}$. By Fubini and Parseval,

$$\begin{aligned} \|\nabla^q \mathcal{P}f\|^2_{H^0(\tilde{\Omega})} &\lesssim \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \int_{-b_+}^0 |n|^{2q} \left|\hat{f}(n)\right|^2 e^{4\pi|n|x_3} dx_3 \\ &\lesssim \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} |n|^{2q} \left|\hat{f}(n)\right|^2 \left(\frac{1-e^{-4\pi b_+|n|}}{|n|}\right). \end{aligned} \tag{A.10}$$

However,

$$\frac{1-e^{-4\pi b_+|n|}}{|n|} \le \min\left\{4\pi b_+, \frac{1}{|n|}\right\}, \tag{A.11}$$

which means we are free to bound the right hand side of (A.10) by either $\|f\|^2_{\dot{H}^{q-1/2}(\Sigma)}$ or $\|f\|^2_{\dot{H}^q(\Sigma)}$. □

We will also need L^∞ estimates.

Lemma A.4. *Let $\mathcal{P}f$ be the Poisson integral of a function f that is in $\dot{H}^{q+s}(\Sigma)$ for $q \ge 1$ an integer and $s > 1$. Then*

$$\|\nabla^q \mathcal{P}f\|^2_{L^\infty} \lesssim \|f\|^2_{\dot{H}^{q+s}} . \tag{A.12}$$

The same estimate holds for $q = 0$ if f satisfies $\int_\Sigma f = 0$.

Proof. We estimate

$$\|\nabla^q \mathcal{P} f\|_{L^\infty} \lesssim \sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})} \left|\hat{f}(n)\right| |n|^q \lesssim \|f\|_{\dot{H}^{q+s}} \left(\sum_{n\in(L_1^{-1}\mathbb{Z})\times(L_2^{-1}\mathbb{Z})\setminus\{0\}} |n|^{-2s} \right)^{1/2} \lesssim \|f\|_{\dot{H}^{q+s}} \tag{A.13}$$

if $s > 1$. The same estimate works with $q = 0$ if $\hat{f}(0) = 0$. □

A.3. **Transport estimate.** Let Σ be either periodic or non-periodic. Consider the equation

$$\begin{cases} \partial_t \eta + u \cdot D\eta = g & \text{in } \Sigma \times (0, T) \\ \eta(t = 0) = \eta_0 \end{cases} \tag{A.14}$$

with $T \in (0, \infty]$. We have the following estimate of the transport of regularity for solutions to (A.14), which is a particular case of a more general result proved in [9]. Note that the result in [9] is stated for $\Sigma = \mathbb{R}^2$, but the same result holds in the periodic setting $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$, as described in [10].

Lemma A.5 (Proposition 2.1 of [9])**.** *Let η be a solution to* (A.14)*. Then there is a universal constant $C > 0$ so that for any $0 \le s < 2$*

$$\sup_{0\le r\le t} \|\eta(r)\|_{H^s} \le \exp\left(C \int_0^t \|Du(r)\|_{H^{3/2}} \, dr \right) \left(\|\eta_0\|_{H^s} + \int_0^t \|g(r)\|_{H^s} \, dr \right). \tag{A.15}$$

Proof. Use $p = p_2 = 2$, $N = 2$, and $\sigma = s$ in Proposition 2.1 of [9] along with the embedding $H^{3/2} \hookrightarrow B^1_{2,\infty} \cap L^\infty$. □

A.4. **Poincaré-type inequalities.** Let Σ and Ω be as above.

Lemma A.6. *It holds that*

$$\|f\|^2_{L^2(\Omega)} \lesssim \|f\|^2_{L^2(\Sigma)} + \|\partial_3 f\|^2_{L^2(\Omega)} \tag{A.16}$$

for all $f \in H^1(\Omega)$. Also, if $f \in W^{1,\infty}(\Omega)$, then

$$\|f\|^2_{L^\infty(\Omega)} \lesssim \|f\|^2_{L^\infty(\Sigma)} + \|\partial_3 f\|^2_{L^\infty(\Omega)}. \tag{A.17}$$

Proof. By density we may assume that f is smooth. Writing $x = (x', x_3)$ for $x' \in \Sigma$ and $x_3 \in (-b(x'), 0)$, we have

$$\left|f(x', x_3)\right|^2 = \left|f(x', 0)\right|^2 - 2 \int_{x_3}^0 f(x', z)\partial_3 f(x', z) dz \le \left|f(x', 0)\right|^2 + 2 \int_{-b(x')}^0 \left|f(x', z)\right| \left|\partial_3 f(x', z)\right| dz. \tag{A.18}$$

We may integrate this with respect to $x_3 \in (-b(x'), 0)$ to get

$$\int_{-b(x')}^0 \left|f(x', x_3)\right|^2 dx_3 \lesssim \left|f(x', 0)\right|^2 + 2 \int_{-b(x')}^0 \left|f(x', z)\right| \left|\partial_3 f(x', z)\right| dz. \tag{A.19}$$

Now we integrate over $x' \in \Sigma$ to find

$$\int_\Omega |f(x)|^2 \, dx \lesssim \|f\|^2_{L^2(\Sigma)} + 2 \int_\Omega |f(x)| \, |\partial_3 f(x)| \, dx \le \|f\|^2_{L^2(\Sigma)} + \varepsilon \|f\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon} \|\partial_3 f\|^2_{L^2(\Omega)} \tag{A.20}$$

for any $\varepsilon > 0$. Choosing $\varepsilon > 0$ sufficiently small then yields (A.16). The estimate (A.17) follows similarly, taking suprema rather than integrating. □

We will need a version of Korn's inequality, which is proved, for instance, in Lemma 2.7 [3].

Lemma A.7. *It holds that* $\|u\|_1 \lesssim \|\mathbb{D}u\|_0$ *for all* $u \in H^1(\Omega;\mathbb{R}^3)$ *so that* $u = 0$ *on* Σ_b.

A.5. **An elliptic estimate.** The proof of the following estimate may be found in [3] for horizontally infinite domains. The same proof holds in the periodic case with obvious modification.

Lemma A.8. *Suppose* (u, p) *solve*

$$\begin{cases} -\Delta u + \nabla p = \phi \in H^{r-2}(\Omega) \\ \operatorname{div} u = \psi \in H^{r-1}(\Omega) \\ (pI - \mathbb{D}(u))e_3 = \alpha \in H^{r-3/2}(\Sigma) \\ u|_{\Sigma_b} = 0. \end{cases} \tag{A.21}$$

Then for $r \geq 2$,

$$\|u\|_{H^r}^2 + \|p\|_{H^{r-1}}^2 \lesssim \|\phi\|_{H^{r-2}}^2 + \|\psi\|_{H^{r-1}}^2 + \|\alpha\|_{H^{r-3/2}}^2 . \tag{A.22}$$

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