Almost Hamiltonian Cubic Graphs

Narong Punnim, Varaporn Saenpholphat, and Sermsri Thaithae

Department of Mathematics, Srinakharinwirot University, Sukhumvit 23, Bangkok 10110, Thailand

Abstract

A Hamiltonian walk in a connected graph G of order n is a closed spanning walk of minimum length in G. For a connected graph G, let $h(G)$ be the length of a Hamiltonian walk in G and call it the *Hamiltonian number* of G. Let i be a non-negative integer. A connected graph G of order n is called an *i-Hamiltonian* if $h(G) = n+i$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an *almost Hamiltonian graph.* We prove in this paper that for an even integer $n \geq 10$ there exists an almost Hamiltonian cubic graph of order n. Let $P(k, m)$ be the generalized Petersen graph of order $2k$. We show that $P(k, m)$ is an almost Hamiltonian graph if and only if $m = 2$ and $k \equiv 5 (\text{mod } 6)$. For a cubic graph G, we define G^* to be the graph obtained from G by replacing each vertex of G to a triangle, matching the vertices of the triangle to the former neighbors of the replaced vertex. We show that G is Hamiltonian if and only if G^* is Hamiltonian and if G is almost Hamiltonian then G^* is 2-Hamiltonian.

Key words:
Hamiltonian walk, Hamiltonian number, and cubic graph.

1. Introduction

While certainly not every connected graph of order at least 3 contains a Hamiltonian cycle, every connected graph does contain a closed spanning walk (in which all vertices are encountered, possibly more than once). If G is a connected graph of size m, there is always a closed spanning walk of length at most $2m$. In [6, 7] Goodman and Hedetniemi introduced the concept of a *Hamiltonian walk* in a connected graph G, defined as a closed spanning walk of minimum length in G. They denoted the length of a Hamiltonian walk in G by $h(G)$. Therefore, for a connected graph G of order $n \geq 3$, it follows that $h(G) = n$ if and only if G is Hamiltonian. Hamiltonain walks were studied further by T. Asano, T. Nishizeki, and T. Watanabe [2, 3], J. C. Bermond [4], and P. Vacek [9]. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian.

In [5] an alternative way to define the length $h(G)$ of a Hamiltonian walk in a connected graph G was presented. A Hamiltonian graph G contains a spanning cycle $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$, where then $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n$. Thus Hamiltonian graphs of order $n \geq 3$ are those graphs for which there is a cyclic ordering $C : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ of $V(G)$ such that $\sum_{i=1}^{n} d(v_i, v_{i+1}) = n$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1} for $1 \leq i \leq n$. For a connected graph G of order $n \geq 3$ and a cyclic ordering $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of the elements of $V(G)$, the number $d(s)$ is defined as $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$. Therefore, $d(s) \geq n$ for each cyclic ordering s of the elements of $V(G)$. The *Hamiltonian number* $h(G)$ of G is defined in [5] by $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of elements of $V(G)$. It was shown in [5] that the Hamiltonian number of a connected graph G is, in fact, the length of a Hamiltonian walk in G.

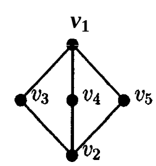

Figure 1: A graph G with $h(G) = 6$

To illustrate these concepts, consider the graph $G = K_{2,3}$ of Figure 1. For the cyclic orderings $s_1 : v_1, v_2, v_3, v_4, v_5, v_1$ and $s_2 : v_1, v_3, v_2, v_4, v_5, v_1$ of $V(G)$, we see that $d(s_1) = 8$ and $d(s_2) = 6$. Since G is a non-Hamiltonian graph of order 5 and $d(s_2) = 6$, it follows that $h(G) = 6$.

Let i be a non-negative integer. A connected graph G of order n is called an *i-Hamiltonian* if $h(G) = n + i$. Thus a 0-Hamiltonian graph is Hamiltonian. An *almost Hamiltonian* graph is a graph G of order n and $h(G) = n + 1$. Thus $K_{2,3}$ is an example of an almost Hamiltonian graph.

The following results are known (see [5, 7]).

Manuscript received January 5, 2007
Manuscript revised January 25, 2007

Theorem A *For every connected graph G of order* $n \geq 2$,

$$n \leq h(G) \leq 2n - 2.$$

Moreover,

1. $h(G) = 2n - 2$ *if and only if G is a tree and*
2. *for every pair* n, k *of integers with* $3 \leq n \leq k \leq 2n-2$, *there exists a connected graph G of order n having* $h(G) = k$.

+

Thus for a connected graph G of order n, G is an i-Hamiltonian graph for some $i = 0, 1, 2, \cdots, n-2$. Moreover, for integers n and i with $n \geq 3$ and $0 \leq i \leq n-2$, there is an i-Hamiltonian graph G of order n.

Let $P(k, m)$ be a generalized Petersen graph such that $V(P(k, m)) = \{u_i, v_i : i = 0, 1, \cdots, k-1\}$ and $E(P(k, m)) = \{u_iu_{i+1}, v_iv_{i+m}, u_iv_i : i = 0, 1, 2, \cdots, k-1\}$ where addition is taken modulo k and $m \leq \frac{k}{2}$. In [1] Alspach completed the determination of the parameters k, m for which $P(k, m)$ is Hamiltonian as stated in the following theorem.

Theorem B *The generalized Petersen graph* $P(k, m)$ *is non-Hamiltonian if and only if* $m = 2$ *and* $k \equiv 5 (mod\ 6)$.

+

2. Almost Hamiltonian cubic graphs

We have seen that a generalized Petersen graph is not a Hamiltonian graph if and only if $m = 2$ and $k \equiv 5$(mod 6). Thus in this case $h(P(k, m)) \geq 2k + 1$. We will show in the next theorem that $P(k, m)$ is an almost Hamiltonian graph if and only if $m = 2$ and $k \equiv 5$ (mod 6).

Theorem 2.1 *Let* $P(k, m)$ *be a generalized Petersen graph. Then*

$$h(P(k,m)) = \begin{cases} 2k+1 & \text{if } m = 2 \text{ and } k \equiv 5 (\text{mod } 6), \\ 2k & \text{otherwise.} \end{cases}$$

Proof. Let $m = 2$ and $k \equiv 5$ (mod 6). By Theorem B, it is suffice to show that $h(P(k,2)) = 2k+1$. Consider a closed spanning walk $W : v_0, v_2, \cdots, v_{k-1}, v_1, v_3, \cdots, v_{k-2}, u_{k-2}, u_{k-3}, u_{k-4}, \ldots, u_1, u_0, u_{k-1}, u_0, v_0$ of $P(k,2)$. It is clear that W has length $2k+1$. Thus $h((P(k, m)) = 2k+1$.

+

It was shown in [8] that all connected cubic graphs of order n, where $4 \leq n \leq 8$, are Hamiltonian. It was also shown in [8] that the Petersen graph $P(5,2)$ and the Tietze graph (denoted by T_{12}) are the only 2-connected cubic graph of order 10 and 12, respectively, that are not Hamiltonian. They are, in fact, almost Hamiltonian cubic graphs of respective order. Note that the T_{12} is obtained from $P(5,2)$ by replacing one vertex of $P(5,2)$ to a triangle, matching the vertices of the triangle to the former neighbors of the replaced vertex. Thus T_{12} contains a triangle. Let G be a cubic graph and $v \in V(G)$. We denote $G * v$ to be the graph obtained from G by replacing v to a triangle, matching the vertices of the triangle to the former neighbors of v. Thus $G * v$ is also a cubic graph containing a triangle.

Theorem 2.2 *Let G be a cubic graph of order* $n \geq 4$ *and* $v \in V(G)$. *Then G is Hamiltonian if and only if* $G * v$ *is Hamiltonian.*

Proof. Let G be a cubic graph and $V(G) = \{v_1, v_2, \cdots, v_n\}$. Put $v = v_1$. Thus $G * v$ is the graph with $V(G * v) = (V(G) - v) \cup \{x_1, y_1, z_1\}$, $\{x_1, y_1, z_1\}$ induced a triangle in $G * v$ and y_1v_2, $v_nz_1 \in E(G * v)$.

Suppose that G is Hamiltonian. Without loss of generality we may assume that $C : v_1, v_2, \cdots, v_n, v_1$ is a Hamiltonian cycle of G. Thus

$$C_v : z_1, x_1, y_1, v_2, v_3, \cdots, v_n, z_1$$

is a Hamiltonian cycle of $G * v$.

Conversely, suppose that $G * v$ is Hamiltonian and let

$$C_v : u_1, u_2, \cdots, u_{n+2}, u_1$$

be a Hamiltonian cycle of $G * v$. If x_1 is not a neighbor of y_1 and z_1 in C_v, then $d_{G*v}(x_1) \geq 4$. Thus x_1 is a neighbor of y_1 or z_1 in C_v. It is also true for y_1 and z_1. Thus x_1, y_1, z_1 must appear as consecutive vertices in C_v. Deleting the three vertices and replacing by v_1, we obtain a Hamiltonian cycle of G.

+

Let G be a cubic graph of order n with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Put $G^1 = G * v_1$ and for $1 \leq i \leq n-1$, put $G^{i+1} = G^i * v_{i+1}$. Thus from Theorem 2.2 we have the following corollary.

Corollary 2.3 *Let G be a cubic graph of order n. Then G is Hamiltonian if and only if* G^i *is Hamiltonian for all* $1 \leq i \leq n$.

+

Now we consider when G is not Hamiltonian cubic graph. Thus $G * v$ is not Hamiltonian by Theorem 2.2. Let K'_4 be the graph obtained from K_4 and a subdivision to an edge of K_4 (see Figure 2).

K'_4 =

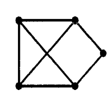

Figure 2 : Graph K'_4

Let G be a graph obtained from three copies of K'_4 and connecting three vertices of degree two to a new vertex v.

Thus G is cubic of order 16 with $h(G) = 21$ but $h(G * v) = 24$. That is G is a 5-Hamiltonian while $G * v$ is 6-Hamiltonian. Note that $G * w$ is 5-Hamiltonian, for each vertex w of G difference from v.

Theorem 2.4 *For an even integer $n \geq 10$, there exists an almost Hamiltonian cubic graph of order n.*

Proof. The Petersen graph $P(5,2)$ is the unique almost Hamiltonian cubic graph of order 10 and the Tietze graph T_{12} is also the unique almost Hamiltonian cubic graph of order 12 and $T_{12} = G * v$, where $G = P(5,2)$ and $v \in V(P(5, 2))$. Let u_1, v_1, w_1 be the induced triangle of T_{12}. Let $T_{14} = T_{12} * v_1$. Thus for an integer $i \geq 1$, let u_i, v_i, w_i be the induced triangle of $T_{12+2(i-1)}$ and $T_{12+2i} = T_{12+2(i-1)} * v_i$. By assuming that the graph $T_{12+2(i-1)}$ is almost hamiltonian, we have that $h(T_{12+2i}) \leq 12 + 2i + 1$. By Theorem 2.2 we have that T_{12+2i} is not Hamiltonian. Therefore $h(T_{12+2i}) = 12 + 2i + 1$ and T_{12+2i} is almost Hamiltonian.

+

A Hamiltonian graph is necessary 2-connected. The same result is also hold in the class of almost Hamiltonian cubic graphs. The following results can be considered as a characterization of cubic graphs for being almost Hamiltonian.

Theorem 2.5 *Let G be a connected cubic graph of order $n \geq 10$. If G is almost Hamiltonian, then G is 2-connected.*

Proof. Suppose G is not 2-connected and v is a cut vertex of G. Since G is cubic, there exists a vertex u such that u is also a cut vertex of G and u is adjacent to v. Furthermore, uv is a cut edge of G. Let $G - e = G_1 \cup G_2$. It follows that $h(G) \geq h(G_1)+h(G_2)+2 \geq n + 2$. The proof is complete.

+

Theorem 2.6 *Let G be a connected non-Hamiltonian cubic graph of order $n \geq 10$. Then G is an almost Hamiltonian graph if and only if for every Hamiltonian walk W of G, W contains a cycle of order n -1.*

Proof. Suppose $h(G) = n + 1$. Let $v_1, v_2, \cdots, v_{n+2} = v_1$ be a Hamiltonian walk of length $n + 1$. Thus there exist v_i and v_j with $1 \leq i < j \leq n$ and $v_i = v_j$ and all other vertices are distinct. Without loss of generality we may assume that $i = 1$. If $j \geq 4$, then $d(v_1) \geq 4$. Thus $j = 3$ and $v_3, v_4, \cdots, v_{n+2} = v_3$ is a cycle in G of length $n - 1$. Suppose G contains a cycle $v_1, v_2, \cdots, v_n = v_1$ of length $n - 1$. Let $v \in V(G) - \{v_1 = v_n, v_2, v_3, \cdots, v_{n-1}\}$. Thus there exists an integer k with $1 \leq k \leq n - 1$ such that v_k is adjacent to v. We now form a Hamiltonian walk $v_1, v_2, \cdots, v_k, v, v_k, \cdots, v_n = v_1$ and this walk has length $n + 1$. Therefore $h(G) = n + 1$.

+

Let G be a Hamiltonian cubic graph. We have shown in Theorem 2.2 that for every v $V(G)$, $G * v$ is Hamiltonian and vise versa. We have also mentioned that there is a 5-Hamiltonian graph G and v $V(G)$ such that $G * v$ is 6-Hamiltonian.

Let G be a connected cubic graph of order n with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let $G^* = G^n$. Figure 3 shows the graphs $P(5,2)$ and $P^*(5,2)$.

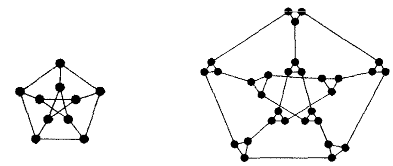

Figure 3: Graphs $P(5,2)$ and $P^*(5, 2)$

Theorem 2.7 *If G is an almost Hamiltonian cubic graph of order n, then $h(G^*) = 3n + 2$.*

Proof. By Theorem 2.2, it follows that $h(G^*) \geq 3n + 1$. Assume, to the contrary, that $h(G^*) = 3n + 1$. By Theorem 2.6, let $C : x_1, x_2, \cdots, x_{3n-1}, x_1$ be a cycle of length $3n - 1$ of G^*, where $V(G^*) = \{ x_1, x_2, \cdots, x_{3n}\}$. Without loss of generality we may assume that x_{3n} is adjacent to x_1. Since G^* is non-Hamiltonian, $x_{3n}x_2, x_{3n}x_{3n-1} \notin E(G^*)$. Since G^* is cubic, there exist i, j with $1 < i < j < 3n - 1$ such that $\{x_{3n}, x_i, x_j\}$ induced a triangle in G^*. Since G^* is cubic, $j = i + 1$ and G^* is Hamiltonian. This is a contradiction.

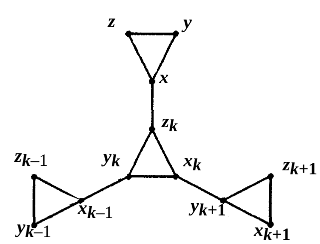

Figure 4: Part of G^*

In order to show that $h(G^*) = 3n + 2$, we will construct a Hamiltonian walk of G^* of length $3n + 2$. In the proof of Theorem 2.6, let

$$W : v_1, v_2, \cdots, v_k, v, v_k, \cdots, v_{n+1} = v_1$$

be a Hamiltonian walk of G.

For each i, $1 \leq i \leq n$, we replace vertices v_i and v in W by triangles x_i, y_i, z_i and x, y, z respectively, and then arrange them in such a way that x_i is adjacent to y_{i+1}, for all $i = 1, 2, \cdots, n - 1$. Without loss of generality we may assume that z_k is adjacent to x as shown in Figure 4. Thus the Hamiltonian walk

$W : z_1, x_1, y_2, z_2, x_2, \cdots, y_{k-1}, z_{k-1}, x_{k-1}, y_k, z_k, x, y, z, x, z_k, x_k, y_{k+1}, z_{k+1}, x_{k+1}, \cdots, y_{n-1}, z_{n-1}, x_{n-1}, y_n, z_n = z_1$

has length $3n + 2$.

+

The following result can be obtained as a direct consequence of Theorem 2.7.

Corollary 2.8 *$h(P^*(k, 2)) = 6k + 2$, for every positive integer k with k* 5 (mod 6).

+

3. Conclusion

A Hamiltonian walk in a connected graph G of order n is a closed opening walk of minimum length in G. Let $h(G)$ be the length of a Hamiltonian walk in G. The graph parameter h is called the Hamiltonian number of G. Thus $h(G)$ may be considered as a measure of how far the graph G is from being Hamiltonian. A connected graph G of order n is called an i-Hamiltonian if $h(G) = n + i$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an almost Hamiltonian graph. Some characterizations of almost Hamiltonian cubic graphs are obtained in this paper. In other words, we proved that a cubic graph G of order n is almost Hamiltonian if and only if G is 2-connected containing a cycle of length $n - 1$. In particular, we proved that the generalized Petersen graph $P(k, m)$ is almost Hamiltonian if and only if $m = 2$ and k 5(mod 6). Let G be a cubic graph of order n. we denote G^* the graph obtained from G by replacing each vertex of G to a triangle, matching the vertices of the triangle to the former neighbors. We proved that G is Hamiltonian if and only if G^* is Hamiltonian and if G is almost Hamiltonian, then G^* is 2-Hamiltonian.

Acknowledgments

This research was supported by the Thailand Research Fund for the first author and was supported in part by Srinakharinwirot University, the Thailand Research Fund and the Commission on Higher Education, Thailand under the grant number MRG 5080075 for the second author. The authors wish to express their gratitude to these institutions.

References

[1] B. R. Alspach, The classification of Hamiltonian gerneralized Petersen graphs. *J. Comb. Th.* B, **34** (1983) 293-312

[2] T. Asano, T. Nishizeki, and T. Watanabe, An upper bound on the length of a Hamiltonian walk of a maximal planar graph. *J. Graph Theory* **4** (1980) 315-336.

[3] T. Asano, T. Nishizeki, and T. Watanabe, An approximation algorithm for the Hamiltonian walk problems on maximal planar graphs. *Discrete Appl. Math.* **5** (1983) 211-222.

[4] J. C. Bermond, On Hamiltonian walks. *Congr. Numer.* **15** (1976) 41-51.

[5] G. Chartrand, T. Thomas, V. Saenpholphat, and P. Zhang, A new look at Hamiltonian walks. *Bull. Inst. Combin. Appl.* **42** (2004) 37-52.

[6] S. E. Goodman and S. T. Hedetniemi, On Hamiltonian walks in graphs. *Congr. Numer.* (1973) 335-342.

[7] S. E. Goodman and S. T. Hedetniemi, On Hamiltonian walks in graphs. *SIAM J. Comput.* **3** (1974) 214-221.

[8] P. Stcinbach, "field guide to SIMPLE GRAPHS 1" 2nd revised edition 1999, Educational Ideas & Materials Albuquerque.

[9] P. Vacek, On open Hamiltonian walks in graphs. *Arch Math.* (*Brno*) **27A** (1991) 105-111.