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ALMOST HERMITIAN MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

SUMIO SAWAKI & KOUEI SEKIGAWA

1. Introduction

B. Smyth proved in his thesis [6] the following

Theorem. *Let M be a complex hypersurface of a Kählerian manifold $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$. If M is of complex dimension ≥ 2, the following statements are equivalent:*

(i) *M is totally geodesic in $\tilde{M}$,*

(ii) *M is of constant holomorphic sectional curvature,*

(iii) *M is an Einstein manifold and at one point of M all sectional curvatures of M are $\geq \frac{1}{4}\tilde{c}$ (resp. $\leq \frac{1}{4}\tilde{c}$) when $\tilde{c} \geq 0$ (resp. ≤ 0).*

One of the purposes of the present paper is to generalize this theorem to almost Hermitian manifolds, and another is to prove that an F-space of constant holomorphic sectional curvature is Kählerian. Here by an F-space we mean an almost Hermitian manifold M satisfying $R(X, Y) \cdot F = 0$ for any vector fields X and Y on M, where the endomorphism $R(X, Y)$ operates on the almost complex structure tensor F as a derivation at each point of M.

In § 2, we shall state the differential-geometric properties of a complex hypersurface of an almost Hermitian manifold satisfying a certain condition and a generalization of the equivalence of the first two statements of Smyth's result. We proceed in § 3 to study the same properties of $*O$-spaces and K-spaces, and to state a generalization of the result of Smyth. In § 4 we shall prove some theorems for F-spaces of constant holomorphic sectional curvature. In §§ 2 and 3, by a complex hypersurface we mean a connected almost complex hypersurface.

2. Complex hypersurfaces of an almost Hermitian manifold

Let $\tilde{M}$ be an almost Hermitian manifold of complex dimension $n + 1$, and denote the almost complex structure and the Hermitian metric of $\tilde{M}$ by F and g respectively. Moreover, let M be a complex hypersurface of $\tilde{M}$, i.e., suppose that there exists a complex analytic mapping $f: M \to \tilde{M}$. Then for each $x \in M$ we identify the tangent space $T_x(M)$ with $f_*(T_x(M)) \subset T_{f(x)}(\tilde{M})$ by means of f_*. Since $f^* \circ g = g'$ and $F \circ f_* = f_* \circ F'$ where g' and F' are the Hermitian

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metric and the almost complex structure of M respectively, g' and F' are respectively identified with the restrictions of the structures g and F to the subspace $f_*(T_x(M))$.

As is well known, we can choose the following special neighborhood $U(x)$ of x for a neighborhood $\tilde{U}(f(x))$ of $f(x)$. Let $\{\tilde{U}\,;\tilde{x}^i\}$ $(i = 1, \cdots, 2n+2)$ be a system of coordinate neighborhoods of $\tilde{M}$. Then $\{U\,; x^i\}$ is a system of coordinate neighborhoods of M such that $x^{2n+1} = x^{2n+2} = 0$ where $x^i = \tilde{x}^i \circ f$.

By $\tilde{\nabla}$ we always mean the Riemannian covariant differentiation on $\tilde{M}$ and by ξ a differentiable unit vector field normal to M at each point of $U(x)$.

If X and Y are vector fields on the neighborhood $U(x)$, we may write

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi + k(X, Y)F\xi \,, \tag{2.1}$$

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to M.

Lemma 2.1. (i) *∇ is the covariant differentiation of the almost Hermitian manifold M.*

(ii) *h and k are symmetric covariant tensor fields of degree 2 on $U(x)$.*

Proof. Making use of (2.1), we have

$$\begin{aligned}\tilde{\nabla}_{f_1X}(f_2Y) &= f_1\tilde{\nabla}_X(f_2Y) = f_1(Xf_2)Y + f_1f_2\tilde{\nabla}_XY \\ &= f_1(Xf_2)Y + f_1f_2\nabla_XY + f_1f_2h(X,Y)\xi + f_1f_2k(X,Y)F\xi \,, \\ \tilde{\nabla}_{f_1X}(f_2Y) &= \nabla_{f_1X}(f_2Y) + h(f_1X, f_2Y)\xi + k(f_1X, f_2Y)F\xi \,,\end{aligned}$$

where X and Y are vector fields on $U(x)$, and f_1 and f_2 are differentiable functions on $U(x)$. From the above two equations, we have

$$h(f_1X, f_2Y) = f_1f_2h(X, Y) \,, \qquad k(f_1X, f_2Y) = f_1f_2k(X, Y) \,,$$

which show that h and k are tensor fields on $U(x)$.

Thus, since ∇_XY becomes a vector fields, from (2.1) it follows that ∇ is a covariant differentiation on $U(x)$.

Next, from

$$\begin{aligned}\tilde{\nabla}_XY &= \nabla_XY + h(X, Y)\xi + k(X, Y)F\xi \,, \\ \tilde{\nabla}_YX &= \nabla_YX + h(Y, X)\xi + k(Y, X)F\xi \,, \\ [X, Y]_{\tilde{M}} &= [X, Y]_M \,,\end{aligned}$$

we have

$$\tilde{T}(X, Y) = T(X, Y) + \{h(X, Y) - h(Y, X)\}\xi + \{k(X, Y) - k(Y, X)\}F\xi \,,$$

where $\tilde{T}$ (resp. T) is the torsion tensor of the connection on $\tilde{M}$ (resp. $U(x)$) with respect to $\tilde{\nabla}$ (resp. ∇). Since $\tilde{T} = 0$, it follows that $T = 0$ and h and k are symmetric.

From $\tilde{\nabla} g = 0$ we have easily $\nabla g = 0$. Hence the proof is completed.

The identities $g(\xi, \xi) = 1$ and $g(F\xi, F\xi) = 1$ imply $g(\tilde{\nabla}_X \xi, \xi) = 0$ and $g(\tilde{\nabla}_X(F\xi), F\xi) = 0$ respectively. Therefore we may put

$$\tilde{\nabla}_X \xi = -A(X) + s(X)F\xi \ , \tag{2.2}$$

$$\tilde{\nabla}_X(F\xi) = -B(X) + t(X)\xi \ , \tag{2.3}$$

where $A(X)$ and $B(X)$ are tangent to M.

Lemma 2.2. (i) *A, B and s, t are tensor fields on $U(x)$ of type* (1.1) *and* (0,1) *respectively.*

(ii) *A and B are symmetric with respect to g, and satisfy*

$$h(X, Y) = g(AX, Y) \ , \tag{2.4}$$

$$k(X, Y) = g(BX, Y) \tag{2.5}$$

for any vector fields X and Y.

Proof. For any vector field X and any differentiable function f on $U(x)$, we have

$$f\tilde{\nabla}_X \xi = \tilde{\nabla}_{fX} \xi = -A(fX) + s(fX)F\xi = -fA(X) + fs(X)F\xi \ ,$$

from which it follows that $A(fX) = fA(X)$, $s(fX) = fs(X)$. Thus A and s are tensor fields on $U(x)$. For ξ and any vector field Y on $U(x)$, we have $g(Y, \xi) = 0$ and therefore

$$g(\tilde{\nabla}_X Y, \xi) + g(Y, \tilde{\nabla}_X \xi) = 0 \ ,$$

in which substitution of (2.1) and (2.2) gives (2.4). However, since h is symmetric, from (2.4) it follows that $g(AX, Y) = g(X, AY)$ which shows that A is symmetric. Similarly the properties of B are verified.

Now let M be a complex hypersurface satisfying the condition

$$h(X, Y) = k(X, FY) \tag{2.6}$$

for any vector fields X and Y on $U(x)$ at every point $x \in M$. It is easily verified that the condition (2.6) is independent of the choice of mutually orthogonal unit vectors ξ and $F\xi$ normal to M.

Lemma 2.3. *In a complex hypersurface M of $\tilde{M}$ satisfying* (2.6), *we have*

(i) $FA = -AF \ , \quad FB = -BF \ ,$

(ii) *FA and FB are symmetric with respect to g,*

(iii) $B = FA \ .$

Proof. By virture of (2.4) and (2.6), for any vector fields X and Y we have

(2.7) $$g(AFX, Y) = h(FX, Y) = k(FX, FY) ,$$

(2.8) $$g(FAX, Y) = -g(AX, FY) = -h(X, FY) = -k(FX, FY) ,$$

which imply that $g(AFX, Y) = -g(FAX, Y)$, so that $AF = -FA$. Since A is symmetric, by (i) we thus have

$$g(AFX, Y) = g(FX, AY) = g(X, AFY) ,$$

which shows that AF is symmetric. Similarly the properties of B are verified.

Finally, by (2.6) and (2.5) we have

$$h(X, FY) = -k(X, Y) = -g(BX, Y) = -g(FBX, FY) .$$

On the other hand, we have $h(X, FY) = g(AX, FY)$ by (2.4) and therefore $g(AX, FY) = -g(FBX, FY)$, from which it follows that $A = -FB$, i.e., $B = FA$.

Remark. In a complex hypersurface M of $\tilde{M}$, $h(X, Y) = k(X, FY)$ is equivalent to $B = FA$.

Since A is symmetric and $FA = -AF$ in a complex hypersurface M of $\tilde{M}$ satisfying (2.6), we have the following well-known

Lemma 2.4. *In a complex hypersurface M of $\tilde{M}$ satisfying* (2.6), *at any point $y \in U(x)$ there exists an orthonormal basis $\{e_1, \cdots, e_n, Fe_1, \cdots, Fe_n\}$ of $T_y(M)$ with respect to which the matrix A is diagonal of the form*

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix} ,$$

where $Ae_i = \lambda_i e_i$, and $AFe_i = -\lambda_i Fe_i$, $i = 1, \cdots, n$.

Lemma 2.5. *If $\tilde{R}$ and R are the Riemannian curvature tensors of $\tilde{M}$ and a complex hypersurface M of $\tilde{M}$ satisfying* (2.6) *respectively, then for any vector fields X, Y, Z and W on $U(x)$ we have the following Gauss equation*:

(2.9) $$\begin{aligned} \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) & \\ & - \{g(AX, Z)g(AY, W) - g(AX, W)g(AY, Z)\} \\ & - \{g(FAX, Z)g(FAY, W) - g(FAX, W)g(FAY, Z)\} . \end{aligned}$$

Proof. From (2.1) it follows that

$$\tilde{\nabla}_Y W = \nabla_Y W + h(Y, W)\xi + k(Y, W)F\xi .$$

Applying $\tilde{\nabla}_X$ to this equation and making use of (2.2) and (2.3), we obtain

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y W = \nabla_X\nabla_Y W - h(Y, W)A(X) - k(Y, W)B(X) \\ + \{h(X, \nabla_Y W) + X(h(Y, W)) + k(Y, W)t(X)\}\xi \\ + \{k(X, \nabla_Y W) + X(k(Y, W)) + h(Y, W)s(X)\}F\xi ,\end{aligned} \tag{2.10}$$

$$\tilde{\nabla}_{[X,Y]}W = \nabla_{[X,Y]}W + h([X, Y], W)\xi + k([X, Y], W)F\xi .$$

Substitution of (2.10) in

$$\begin{aligned}&\tilde{R}(X, Y)W - R(X, Y)W \\ &\quad = \tilde{\nabla}_X\tilde{\nabla}_Y W - \tilde{\nabla}_Y\tilde{\nabla}_X W - \tilde{\nabla}_{[X,Y]}W - (\nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W)\end{aligned}$$

gives easily

$$\begin{aligned}\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - \{g(AX, Z)h(Y, W) - g(AY, Z)h(X, W)\} \\ - \{g(BX, Z)k(Y, W) - g(BY, Z)k(X, W)\} ,\end{aligned}$$

or (2.9) by (2.4), (2.5) and (2.6).

Lemma 2.6. *Let M be a complex hypersurface of $\tilde{M}$ and satisfy the condition* (2.6).

(i) *If p is 2-plane tangent to M at a point of $U(x)$, then*

$$\begin{aligned}\tilde{K}(p) = K(p) - \{g(AX, X)g(AY, Y) - g(AX, Y)^2\} \\ - \{g(FAX, X)g(FAY, Y) - g(FAX, Y)^2\} ,\end{aligned} \tag{2.11}$$

where X, Y form an orthonormal basis of p, and $\tilde{K}(p)$ (resp. $K(p)$) is the sectional curvature of p considered as a 2-plane tangent to $\tilde{M}$ (resp. M).

(ii) *If X is a unit vector tangent to M at a point of $U(x)$, then*

$$\tilde{H}(X) = H(X) + 2\{g(AX, X)^2 + g(FAX, X)^2\} , \tag{2.12}$$

where $\tilde{H}(X)$ (resp. $H(X)$) is the holomorphic sectional curvature in $\tilde{M}$ (resp. M).

Proof. (i) is immediate on replacing Z and W in the Gauss equation by X and Y respectively, and making use of the fact that A and FA are symmetric. (ii) is also immediate on replacing Y by FX in (2.11) and making use of the fact that $FA = -AF$.

Proposition 2.7. *Let M be a complex hypersurface of $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$. If M is of complex dimension ≥ 2 and satisfies the condition* (2.6), *then at each point of M there exists a holomorphic plane whose sectional curvature in M is $\tilde{c}$, and therefore if M is of constant holomorphic sectional curvature c, then $c = \tilde{c}$.*

Proof. Let $\{e_1, \cdots, e_n, Fe_1, \cdots, Fe_n\}$ be an orthonormal basis in Lemma 2.4. Since $n \geq 2$, there exist λ_i and λ_j $(i \neq j)$ defined in Lemma 2.4.

In the case where $\lambda_i > 0$ and $\lambda_j > 0$, we set

$$X = (\lambda_i + \lambda_j)^{-\frac{1}{2}}(\sqrt{\lambda_j}e_i + \sqrt{\lambda_i}Fe_j) .$$

Then

$$AX = \frac{\sqrt{\lambda_j}\lambda_i e_i - \sqrt{\lambda_i}\lambda_j Fe_j}{(\lambda_i + \lambda_j)^{\frac{1}{2}}} , \qquad FAX = \frac{\sqrt{\lambda_j}\lambda_i Fe_i + \sqrt{\lambda_i}\lambda_j e_j}{(\lambda_i + \lambda_j)^{\frac{1}{2}}} ,$$

so that

$$g(AX, X) = 0 , \qquad g(FAX, X) = 0 . \tag{2.13}$$

In the case where $\lambda_i < 0$ and $\lambda_j > 0$, and in the case where $\lambda_i < 0$ and $\lambda_j < 0$, we set, respectively,

$$X = \frac{\sqrt{\lambda_j}e_i + \sqrt{-\lambda_i}e_j}{(\lambda_j - \lambda_i)^{\frac{1}{2}}} , \qquad X = \frac{\sqrt{-\lambda_j}e_i + \sqrt{-\lambda_i}Fe_j}{(-\lambda_j - \lambda_i)^{\frac{1}{2}}} ,$$

so that we can also obtain (2.13).

Consequently, from (2.12) and (2.13) we have $\tilde{c} = H(\tilde{X}) = H(X)$ which proves the proposition.

Theorem 2.8. *Let M be a complex hypersurface of $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$. If M is of complex dimension ≥ 2 and satisfies the condition* (2.6), *then the following statements are equivalent*:

(i) *M is totally geodesic in $\tilde{M}$,*

(ii) *M is of constant holomorphic sectional curvature.*

Proof. If M is totally geodesic, then A vanishes on M, and therefore from (2.12) it follows that M is of constant holomorphic sectional curvature $\tilde{c}$. Conversely, if M is of constant holomorphic sectional curvature c, then by virtue of Proposition 2.7 we have, for any unit vector X tangent to M, $\tilde{c} = H(\tilde{X}) = H(X)$, which reduces (2.12) to $g(AX, X)^2 + g(FAX, X)^2 = 0$, so that $A = 0$, that is, M is totally geodesic.

3. *O-spaces and K-spaces

An almost Hermitian manifold $\tilde{M}$ is called an *O-space (or quasi-Kählerian manifold) [3] or a K-space (or Tachibana space or nearly Kähler manifolds) [7] according as

$$\tilde{\nabla}_X(F)Y + \tilde{\nabla}_{FX}(F)FY = 0 , \tag{3.1}$$

or

$$\tilde{\nabla}_X(F)Y + \tilde{\nabla}_Y(F)X = 0 \qquad \text{(or equivalently } \tilde{\nabla}_X(F)X = 0) \tag{3.2}$$

holds for any vector fields X and Y on $\tilde{M}$. It is well-known that a K-space is an *O-space.

First of all, let M be a complex hypersurface of an $*O$-space $\tilde{M}$. Then for any vector fields X and Y on $U(x) \subset M$ we have

$$\tilde{\nabla}_X(FY) = F\tilde{\nabla}_X Y + \tilde{\nabla}_X(F)Y , \qquad \tilde{\nabla}_{FX}(FFY) = F\tilde{\nabla}_{FX}(FY) + \tilde{\nabla}_{FX}(F)FY .$$

Adding these equations and making use of (3.1) we obtain

$$\tilde{\nabla}_X(FY) - \tilde{\nabla}_{FX}Y = F(\tilde{\nabla}_X Y + \tilde{\nabla}_{FX}(FY)) . \tag{3.3}$$

Substituting (2.1) in (3.3) gives immediately

$$\nabla_X(FY) - \nabla_{FX}Y - F\nabla_X Y - F\nabla_{FX}(FY) = 0 , \tag{3.4}$$

$$h(X, FY) - h(FX, Y) = -k(X, Y) - k(FX, FY) , \tag{3.5}$$

$$k(X, FY) - k(FX, Y) = h(X, Y) + h(FX, FY) . \tag{3.6}$$

In consequence of

$$\nabla_X(F)FY = -F\nabla_X(F)Y , \tag{3.7}$$

(3.4) reduces to

$$\nabla_X(F)Y + \nabla_{FX}(F)FY = 0 ,$$

which shows that M is also an $*O$-space.

Since the left hand side of (3.5) is skew-symmetric in X, Y and the right hand side is symmetric in X, Y due to the symmetry of h and k, we have

$$h(X, FY) = h(FX, Y) , \qquad k(X, Y) + k(FX, FY) = 0 .$$

Similarly, from (3.6) follow

$$k(X, FY) = k(FX, Y) , \qquad h(X, Y) + h(FX, FY) = 0 ,$$

which are equivalent to the above two equations.

Hence we have

Lemma 3.1. *A complex hypersurface M of an $*O$-space $\tilde{M}$ is also an $*O$-space, and satisfies*

$$h(X, FY) = h(FX, Y) , \tag{3.8}$$

$$k(X, FY) = k(FX, Y) . \tag{3.9}$$

Next, let M be a complex hypersurface of a K-space $\tilde{M}$. Then for vector fields X and Y on $U(x) \subset M$ we have

$$\tilde{\nabla}_X(FY) = F\tilde{\nabla}_X Y + \tilde{\nabla}_X(F)Y , \qquad \tilde{\nabla}_Y(FY) = F\tilde{\nabla}_Y X + \tilde{\nabla}_Y(F)X .$$

Adding these equations and making use of (3.2) we obtain

$$\tilde{\nabla}_X(FY) + \tilde{\nabla}_Y(FX) = F(\tilde{\nabla}_X Y + \tilde{\nabla}_Y X) . \tag{3.10}$$

Substituting (2.1) in (3.10) gives readily

$$\nabla_X(FY) + \nabla_Y(FX) = F\nabla_X Y + F\nabla_Y X , \tag{3.11}$$

$$h(X, FY) + h(FX, Y) = -2k(X, Y) , \tag{3.12}$$

$$k(X, FY) + k(FX, Y) = -2h(X, Y) . \tag{3.13}$$

(3.11) reduces to

$$\nabla_X(F)Y + \nabla_Y(F)X = 0 ,$$

which shows that M is also a K-space. (3.12) and (3.8) imply $h(X, FY) = -k(X, Y)$, i.e., $h(X, Y) = k(X, FY)$, which is equivalent to $B = FA$ by the remark in § 2. From (3.13) we shall get the same result.

Consequently, we have

Lemma 3.2. *A complex hypersurface M of a K-space $\tilde{M}$ is also a K-space, and satisfies*

$$h(X, Y) = k(X, FY) \qquad (\textit{or equivalently } B = FA) .$$

Recently, Gray [1] proved

Lemma 3.3. *In a K-space $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$ at a point $x \in \tilde{M}$, we have*

$$\tilde{K}(p) = \tfrac{1}{4}\tilde{c}\{1 + 3g(FX, Y)^2\} + \tfrac{3}{4}\|\tilde{\nabla}_X(F)Y\|^2 , \tag{3.15}$$

where p is a 2-plane spanned by any two orthonormal vectors $X, Y \in T_x(\tilde{M})$.

Making use of these Lemmas, we can prove

Theorem 3.4. *Let M be a complex hypersurface of a K-space $\tilde{M}$ with constant holomorphic sectional curvature $\tilde{c}$. If M is of complex dimension ≥ 3, then the following statements are equivalent:*

(i) *M is totally geodesic in $\tilde{M}$,*
(ii) *M is of constant holomorphic sectional curvature,*
(iii) *at every point $x \in M$, all the sectional curvatures of M satisfy*

$$K(p) \geq \tfrac{1}{4}\tilde{c}\{1 + 3g(FX, Y)^2\} , \tag{3.16}$$

where p is a 2-plane spanned by any two orthonormal vectors $X, Y \in T_x(M)$.

Proof. Since, by Lemma 3.2, K-space satisfies (2.6), the fact that (i) is equivalent to (ii) is nothing but Theorem 2.8 (i). Next, if M is of constant holomorphic sectional curvature c, then $c = \tilde{c}$ by Proposition 2.7, and therefore by Lemma 3.3 we have, for any orthonormal vectors $X, Y \in T_x(M)$ at every point $x \in M$,

$$K(p) = \tfrac{1}{4}\tilde{c}\{1 + 3g(FX, Y)^2\} + \tfrac{3}{4}\|\nabla_X(F)Y\|^2\ , \tag{3.17}$$

which implies (3.16).

Finally, we shall prove that (iii) implies (i). Substituting (3.15) in (2.11), and making use of (3.16) we can easily obtain

$$\begin{aligned}&\tfrac{3}{4}\|\tilde{\nabla}_X(F)Y\|^2 + \{g(AX, X)g(AY, Y) - g(AX, Y)^2\}\\ &\qquad + \{g(FAX, X)g(FAY, Y) - g(FAX, Y)^2\} \geq 0\ .\end{aligned} \tag{3.18}$$

Now let $\{e_1, \cdots, e_n, Fe_1, \cdots, Fe_n\}$ be an orthonormal basis given in Lemma 2.4, and set

$$X = (e_i + Fe_i)/\sqrt{2}\ , \qquad Y = (e_i - Fe_i)/\sqrt{2}\ .$$

Since

$$\begin{aligned}AX &= \lambda_i(e_i - Fe_i)/\sqrt{2}\ , &\quad AY &= \lambda_i(e_i + Fe_i)/\sqrt{2}\ ,\\ FAX &= \lambda_i(Fe_i + e_i)/\sqrt{2}\ , &\quad FAY &= \lambda_i(Fe_i - e_i)/\sqrt{2}\ ,\end{aligned}$$

we have

$$\begin{aligned}&g(AX, X) = 0\ , \qquad g(FAX, X) = \lambda_i\\ &g(FAY, Y) = -\lambda_i\ , \qquad g(FAX, Y) = 0\ , \qquad g(AX, Y) = \lambda_i\ .\end{aligned}$$

Moreover, from $Y = -FX$, (3.2) and (3.7) we have

$$\tilde{\nabla}_X(F)Y = -\tilde{\nabla}_X(F)FX = F\tilde{\nabla}_X(F)X = 0\ .$$

Thus (3.18) reduces to $\lambda_i = 0$ $(i = 1, \cdots, n)$, which together with Lemma 2.4 implies that A is identically zero at each point of M, so that M is totally geodesic in $\tilde{M}$.

Remark. It is well-known that in a K-space M of constant holomorphic sectional curvature $\tilde{c}$, $\tilde{c} > 0$ [8]. Hence from (3.17) we have

$$K(p) \geq \tfrac{1}{4}\tilde{c}\ . \tag{3.19}$$

However, the authors do not know whether M is totally geodesic or not if (3.19) holds.

4. F-spaces

Recall that an almost Hermitian manifold M of dimension $2n$ is called an F-space if $R(X, Y)\cdot F = 0$ holds for any vector fields X and Y on M. Of course, a Kählerian manifold is an F-space, and an almost Kählerian manifold or a K-space satisfying $R(X, Y)\cdot F = 0$ is Kählerian [5]. However, an example of a nonkählerian *O-space satisfying $R(X, Y)\cdot F = 0$ has been recently given by Yanamoto [9].

Now for an F-space M of constant holomorphic sectional curvature c we have (cf. [2, pp. 165–166])

$$(4.1)\qquad \begin{aligned} R(X,Y,Z,W) = \tfrac{1}{4}c\{&g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\ &+ g(X,FZ)g(Y,FW) - g(X,FW)g(Y,FZ) \\ &+ 2g(X,FY)g(Z,FW)\}\ , \end{aligned}$$

where X, Y, Z and W are any tangent vectors at a point of M, since $R(X, Y)\cdot F = 0$ means that

$$R(X,Y,Z,W) = R(X,Y,FZ,FW) = R(FX,FY,Z,W)\ .$$

On replacing Z and W in (4.1) by mutually orthogonal unit vectors X and Y respectively, we obtain

$$K(p) = \tfrac{1}{4}c\{1 + 3g(X,FY)^2\}\ .$$

Hence we have the following theorem which is a generalization of the corresponding result in a Kählerian manifold [10].

Theorem 4.1. *An F-space M of constant holomorphic sectional curvature c is an Einstein space. When $c \neq 0$, the sectional curvature $K(p)$ of a 2-plane p spanned by any two orthonormal vectors X and Y in M satisfies the inequalities*:

$$\tfrac{1}{4}c \leq K(p) \leq c \quad \text{for } c > 0\ , \qquad \tfrac{1}{4}c \geq K(p) \geq c \quad \text{for } c < 0\ ,$$

where the equality $\frac{1}{4}c = K(p)$ occurs when $g(X, FY) = 0$, and $K(p) = c$ occurs when $g(X, FY) = \pm 1$.

Proof. It is sufficient to prove the first assertion of the theorem. Let $R_{jih}{}^k$, g_{ji} and $F_j{}^i$ be the local components of R, g and F respectively, and put $R_{jihk} = g_{ka}R_{jih}{}^a$ and $F_{ji} = g_{ia}F_j{}^a$. Then (4.1) can be written as

$$(4.2)\qquad R_{jihk} = -\tfrac{1}{4}c(g_{jh}g_{ik} - g_{jk}g_{ih} + F_{hj}F_{ki} - F_{kj}F_{hi} + 2F_{ij}F_{kh})\ .$$

Transvecting (4.2) with g^{ih} we have

$$(4.3)\qquad R_{jk} = \tfrac{1}{2}(n+1)cg_{jk}\ ,$$

so that our space is Einsteinian. q.e.d.

Applying $\nabla_b\nabla_a$ to (4.2), we have

$$\begin{aligned} \nabla_b\nabla_a R_{jihk} = -\tfrac{1}{4}c\{&(\nabla_b\nabla_a F_{hj})F_{ki} + F_{hj}\nabla_b\nabla_a F_{ki} - (\nabla_b\nabla_a F_{kj})F_{hi} \\ &- F_{kj}\nabla_b\nabla_a F_{hi} + 2(\nabla_b\nabla_a F_{ij})F_{kh} + 2F_{ij}\nabla_b\nabla_a F_{kh}\} \end{aligned}$$

$$
\begin{aligned}
(4.4) \qquad & -\tfrac{1}{4}c\{(\nabla_a F_{hj})\nabla_b F_{ki} + (\nabla_b F_{hj})\nabla_a F_{ki}\} \\
& + \tfrac{1}{4}c\{(\nabla_a F_{kj})\nabla_b F_{hi} + (\nabla_b F_{kj})\nabla_a F_{hi}\} \\
& - \tfrac{1}{2}c\{(\nabla_a F_{ij})\nabla_b F_{kh} + (\nabla_b F_{ij})\nabla_b F_{kh}\} .
\end{aligned}
$$

Since $R(X, Y)\cdot F = 0$ means that $\nabla_b \nabla_a F_{hj}$ is symmetric in a, b, the right hand side of (4.4) is symmetric in a, b. Thus from (4.4) we have

Lemma 4.2. *In an F-space of constant holomorphic sectional curvature, we have*

$$\nabla_b \nabla_a R_{jihk} - \nabla_a \nabla_b R_{jihk} = 0 , \quad \text{i.e.,} \quad R(X, Y)\cdot R = 0 .$$

Next, calculating the square of both sides of (4.2) we have

$$R_{jihk}R^{jihk} = 2c^2 n(n+1)$$

and therefore

$$(4.5) \qquad R_{jihk}R^{jihk} = 2R^2/[n(n+1)] ,$$

since $C = R/[n(n+1)]$ from (4.3). Hence we obtain

Lemma 4.3. *In an F-space of constant holomorphic sectional curvature, the length of the tensor R_{jihk} is constant.*

On the other hand, the following two lemmas are known.

Lemma 4.4 (Lichnerowicz [4], Yano [10]). *In a Riemannian manifold, we have*

$$
\begin{aligned}
\Delta(R_{jihk}R^{jihk}) = {} & 2(\nabla_s R_{jihk})\nabla^s R^{jihk} - 4R^{jihk}\nabla_j(\nabla_h R_{ik} - \nabla_k R_{ih}) \\
& - 4R^{jihk}H^s{}_{ihk,sj} ,
\end{aligned}
$$

where Δ and $H_{jih}{}^k{}_{,st}X^sY^t$ are the Laplacian and the components of $R(X, Y)\cdot R$ respectively.

Lemma 4.5 (Sawaki [5]). *An almost Hermitian manifold M is Kählerian if it satisfies:*

(i) $R(X, Y)\cdot F = 0 , \quad \nabla_Z R(X, Y)\cdot F = 0$

for any vector fields X, Y and Z on M,

(ii) *the rank of the Ricci form is maximum.*

Making use of the above results, we can prove

Theorem 4.6. *If M is an F-space of nonzero constant holomorphic sectional curvature, then M is Kählerian.*

Proof. By virtue of Theorem 4.1, Lemma 4.2 and Lemma 4.3, from Lemma 4.4 we have $\nabla_s R_{jihk} = 0$, so that M is locally symmetric. Thus from Lemma 4.5 it follows that M is Kählerian.

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NIIGATA UNIVERSITY, NIIGATA, JAPAN