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ALMOST HOMOGENEOUS TORUS ACTIONS ON VARIETIES WITH AMPLE TANGENT BUNDLE

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0. Introduction. The main purpose of the present paper is to prove:

THEOREM 4.1. *Let V be an n-dimensional irreducible non-singular projective variety on which an n-dimensional algebraic torus $(\boldsymbol{C}^*)^n$ acts regularly and effectively. Assume that every irreducible non-singular hypersurface in V has ample normal bundle. Then V is isomorphic to $\boldsymbol{P}^n(\boldsymbol{C})$.*

In recent years, systematic studies of torus embeddings have been made by several authors, (cf. Demazure [1], Miyake-Oda [5], Mumford et al. [6]). Their method is, in its spirit, to translate geometric problems of varieties with almost homogeneous torus actions into purely combinatorial problems of the complexes constructed from the group actions. This very spirit is also the main motivation that has led us to re-examine the following conjecture of Hartshorne from a group theoretical point of view:

CONJECTURE. *An n-dimensional irreducible non-singular projective variety with ample tangent bundle is isomorphic to $\boldsymbol{P}^n(\boldsymbol{C})$.*

Note that every non-singular hypersurface sitting in a non-singular projective variety with ample tangent bundle has ample normal bundle, since a quotient bundle of an ample bundle is itself ample, (cf. [3, 4]).

We now quickly discuss how Theorem 4.1 can be proven for $n = 2$. (Although the same formulation generalizes for an arbitrary n, in the actual proof we adopt the standard notation and complexes of Mumford which are somewhat different from those used in this introduction.):

Let $B = (\beta_{ij}) = (\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_m)$ be a $2 \times m$ integral matrix consisting of m column vectors

$$\boldsymbol{b}_i = \begin{pmatrix} \beta_{1i} \\ \beta_{2i} \end{pmatrix} \in \boldsymbol{Z}^2 , \qquad i = 1, 2, \cdots, m ,$$

such that

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$$(\sharp)\qquad \begin{cases} \boldsymbol{b}_i \neq \boldsymbol{b}_j \quad \text{if} \quad i \neq j\,, \\ \boldsymbol{b}_1 - \boldsymbol{b}_m,\, \boldsymbol{b}_2 - \boldsymbol{b}_m,\, \cdots,\, \boldsymbol{b}_{m-1} - \boldsymbol{b}_m \ \text{generate}\ \boldsymbol{Z}^2\,. \end{cases}$$

To such an integral matrix B, we associate a polygonal complex K_B and a projective algebraic surface S_B as follows:

(1) Let $|K_B|$ be the (compact polygonal) convex hull of the integral points $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_m$ in $\boldsymbol{R}^2$. Denoting by $\{v_1, v_2, \cdots, v_h\}$ (resp. $\{e_1, e_2, \cdots, e_h\}$) the set of vertices (resp. edges) of the polygon $|K_B|$, we define a complex K_B by

$$K_B = \{|K_B|, e_1, e_2, \cdots, e_h, v_1, v_2, \cdots, v_h\}\,.$$

(2) Consider the $(\boldsymbol{C}^*)^2$-action on $\boldsymbol{P}^{m-1}(\boldsymbol{C})$ which is induced from the monomorphism

$$(\boldsymbol{C}^*)^2 \hookrightarrow PGL\,(m;\, \boldsymbol{C})$$
$$(s,\, t) \mapsto \begin{pmatrix} s^{\beta_{11}}\cdot t^{\beta_{21}} & & & 0 \\ & s^{\beta_{12}}\cdot t^{\beta_{22}} & & \\ & & \ddots & \\ 0 & & & s^{\beta_{1m}}\cdot t^{\beta_{2m}} \end{pmatrix}.$$

Putting $p_0 = (1\colon 1\colon \cdots \colon 1) \in \boldsymbol{P}^{m-1}(\boldsymbol{C})$, we define an irreducible algebraic surface S_B in $\boldsymbol{P}^{m-1}(\boldsymbol{C})$ by

$$S_B = \text{the Zariski-closure of the orbit } (\boldsymbol{C}^*)^2\cdot p_0 \text{ in } \boldsymbol{P}^{m-1}(\boldsymbol{C})\,.$$

Then we have:

FACT 1 (cf. 1.5.1). To each $\sigma \in K_B$, we can associate a $(\boldsymbol{C}^*)^2$-stable closed irreducible subvariety, denoted by $F(\sigma)$, of S_B in such a way that

$$\begin{aligned} &\dim_C F(\sigma) = \dim_R \sigma && \text{for every } \sigma \in K_B\,, \\ &F(\sigma) \subseteqq F(\sigma') && \text{if and only if } \sigma \leqq \sigma'. \end{aligned}$$

Furthermore, $\sigma \leftrightarrow F(\sigma)$ defines a 1-1 correspondence between K_B and the set of all $(\boldsymbol{C}^*)^2$-stable closed irreducible subvarieties of S_B.

FACT 2 (cf. 1.5.2). Let V be an irreducible non-singular projective algebraic surface on which a 2-dimensional algebraic torus $(\boldsymbol{C}^*)^2$ acts regularly and effectively. Then for some positive integer m, there exists a $2 \times m$ integral matrix B with the property ($\sharp$) above, such that V is $(\boldsymbol{C}^*)^2$-equivariantly isomorphic to S_B.

FACT 3 (cf. 1.5.4). Assume that S_B (where B is a $2 \times m$ integral matrix satisfying ($\sharp$) above) is a non-singular surface. Then for every $\sigma \in K_B$, $F(\sigma)$ is also non-singular. Let $[F(\sigma)] \in H^*(S_B;\, \boldsymbol{Q})$ be the Poincaré dual of the homology class carried by the algebraic cycle $F(\sigma)$ in S_B, and let $c(S_B) = 1 + c_1(S_B) + c_2(S_B)$ denote the total Chern class of the tangent bundle of S_B. Then

$$c(S_B) = \sum_{\sigma \in K_B} [F(\sigma)] .$$

In particular, the Euler number of S_B is exactly the number of vertices of the polygon $|K_B|$. (This fact is interpreted as the Gauss-Bonnet formula for S_B.)

In the above Fact 3, if we assume $S_B \cong \boldsymbol{P}^2(\boldsymbol{C})$, then the number of vertices of $|K_B|$ is 3, i.e., $|K_B|$ is a triangle. It can also be shown that the converse is true:

FACT 4 (cf. 2.1). Assume that S_B (where B is the same as in Fact 3) is a non-singular surface. Then $S_B \cong \boldsymbol{P}^2(\boldsymbol{C})$ if and only if $|K_B|$ is a triangle:

We now explain how the ampleness of normal bundles of the hypersurfaces in S_B affects the shape of the polygon $|K_B|$. The key fact is the following ampleness criterion.

FACT 5 (cf. 3.2). Assume that S_B (where B is the same as in Fact 3) is a non-singular surface. We number the vertices of the polygon $|K_B|$ as in Figure 1, and put $v_{h+1} = v_1$ for simplicity. For each $i \in \{1, 2, \cdots, h\}$, let σ_i denote the edge $\overline{v_i v_{i+1}}$. Then the corresponding irreducible non-singular $(\boldsymbol{C}^*)^2$-stable subvariety $F(\sigma_i)$ of S_B has ample normal bundle if and only if $\angle v_i + \angle v_{i+1} < 180°$.

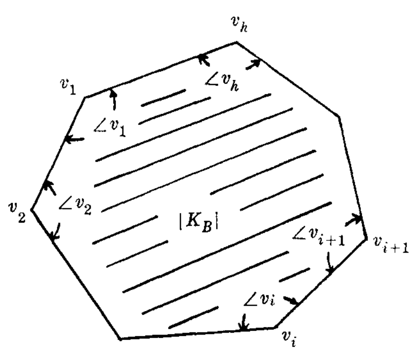

FIGURE 1

In the above Fact 5, if each $F(\sigma_i)$ in S_B has ample normal bundle, then $\angle v_i + \angle v_{i+1} < 180°$, $i = 1, 2, \cdots, h$, and therefore $|K_B|$ is a triangle. Theorem 4.1 (for $n = 2$) is now straightforward from this and Facts 2 and 4.

Although the above complex K_B (which can generally be defined for an arbitrary dimension n) is slightly different from those constructed by Demazure [1], Miyake-Oda [5], and Mumford et al. [6], it has the advantage that an intuitive interpretation of the ampleness of normal bundles of G-stable subvarieties can be obtained easily. However, to avoid the tedious routine work of reproving and reformulating well-known facts, we shall use the standard notation and complexes of Mumford, etc. As a result, the actual proof of Theorem 4.1 somewhat differs from the one described above, but still the main idea of the proof does not change so much. (Although we work over the field $\boldsymbol{C}$ of complex numbers, our argument is valid for every algebraically closed field.)

In concluding this introduction, I wish to thank all those people who encouraged me and gave me suggestions, and in particular Professors S. Kobayashi, S. S. Roan, and I. Satake, who helped me again and again during the preparation of this paper. Thanks go also to Professors T. Oda and H. Sumihiro for their useful comments by correspondence.

1. Notation, Conventions, and Preliminaries. We often use the same notation as in Miyake-Oda [5] and Mumford et al. [6].

1.1. $\boldsymbol{Z}_+$(resp. $\boldsymbol{Z}_{\text{non-neg}}$) denotes the set of positive (resp. non-negative) integers, and $\boldsymbol{R}_+$(resp. $\boldsymbol{R}_{\text{non-neg}}$) denotes the set of positive (resp. non-negative) real numbers.

1.2. All varieties and algebraic groups are defined over $\boldsymbol{C}$.

1.3. We denote by G the n-dimensional algebraic torus

$$(\boldsymbol{C}^*)^n = \boldsymbol{C}^* \times \boldsymbol{C}^* \times \cdots \times \boldsymbol{C}^* = \begin{pmatrix}\text{the direct product of } n \text{ copies of} \\ \boldsymbol{C}^* = GL(1; \boldsymbol{C}) \end{pmatrix} .$$

Let $N = \boldsymbol{Z}^n$ = the set of n-dimensional integral *row* vectors,
$N^* = \boldsymbol{Z}^n$ = the set of n-dimensional integral *column* vectors,
$N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R} = \boldsymbol{R}^n$ = the set of n-dimentional real *row* vectors,
$N^*_{\boldsymbol{R}} = N^* \otimes_{\boldsymbol{Z}} \boldsymbol{R} = \boldsymbol{R}^n$ = the set of n-dimensional real *column* vectors.
To each $\boldsymbol{a} = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in N$, we associate a 1-parameter subgroup $\lambda_a \in \mathrm{Hom}_{\text{alg gp}}(\boldsymbol{C}^*, G)$ of G by

$$\lambda_a \colon \boldsymbol{C}^* \to G = (\boldsymbol{C}^*)^n$$
$$r \mapsto \lambda_a(r) = (r^{\alpha_1}, r^{\alpha_2}, \cdots, r^{\alpha_n}) .$$

To each $\boldsymbol{b} = {}^t(\beta_1, \beta_2, \cdots, \beta_n) \in N^*$, we associate a character $\chi^b \in \mathrm{Hom}_{\text{alg gp}}(G, \boldsymbol{C}^*)$ of G by

$$\chi^b \colon G = (\boldsymbol{C}^*)^n \to \boldsymbol{C}^*$$
$$g = (r_1, r_2, \cdots, r_n) \mapsto \chi^b(g) = (r_1)^{\beta_1} \cdot (r_2)^{\beta_2} \cdot \cdots \cdot (r_n)^{\beta_n} .$$

The correspondence $a \mapsto \lambda_a$(resp. $b \mapsto \chi^b$) canonically induces an isomorphism between the additive group N(resp. N^*) and the multiplicative group $\mathrm{Hom}_{\mathrm{alg\ gp}}(C^*, G)$(resp. $\mathrm{Hom}_{\mathrm{alg\ gp}}(G, C^*)$). Note that

$$\chi^b(\lambda_a(r)) = r^{(a \cdot b)} \qquad \text{for all} \quad r \in C^* ,$$

where $a \cdot b$ (or sometimes denoted by $\langle a, b \rangle$) is the product of a and b by matrix multiplication.

1.4 (1) A non-empty subset σ of $N(=Z^n)$ is said to be a *cone* if the following three conditions are satisfied:

i) If $a \in N$ satisfies $\alpha \cdot a \in \sigma$ for some $\alpha \in Z_+$, then $a \in \sigma$.

ii) If $0 \neq a \in \sigma$, then $-a \notin \sigma$.

iii) There exists a finite subset $\{a_i; i = 1, 2, \cdots, m\}$ of σ such that $\sigma = \sum_{i=1}^m Z_{\text{non-neg}} \cdot a_i$.

One may always assume that the finite set $\{a_i; i = 1, 2, \cdots, m\}$ is irredundant and that no a_i can be written as a positive integral multiple of an element of N except for itself. In this case we call $a_1, a_2, \cdots, a_m$ the *fundamental generators* of the cone σ. Setting $\sigma_R = \sum_{i=1}^m R_{\text{non-neg}} \cdot a_i$, we define:

$$\dim \sigma = \dim_R \sigma_R .$$

In other words, $\dim \sigma$ denotes the rank of the Z-submodule of N generated by σ. For σ above with the fundamental generators $a_1, \cdots, a_m$, we call $\sum_{i=1}^m Z_+ \cdot a_i$(resp. $\sum_{i=1}^m R_+ \cdot a_i$) the interior of σ(resp. σ_R).

(2) A non-empty subset σ' of a cone σ is called a *face* of σ, denoted by $\sigma' \leqq \sigma$, if there exists an element b of N^* such that $\langle a, b \rangle \geqq 0$ for all a in σ and that $\sigma' = \{a \in \sigma; \langle a, b \rangle = 0\}$. A *finite polyhedral decomposition* of N is a finite set S of cones in N such that

i) If $\sigma \leqq \sigma' \in S$, then $\sigma \in S$.

ii) If $\sigma, \sigma' \in S$, then $\sigma \cap \sigma' \leqq \sigma$ and $\sigma \cap \sigma' \leqq \sigma'$.

iii) $N = \bigcup_{\sigma \in S} \sigma$.

For every finite polyhedral decomposition S of N, we define:

$$S_q = \{\sigma \in S; \dim \sigma = q\} ,$$

each element of which is called a *q-cone*. Then

$S_0 = \{0\}$, $S_n =$ (the set of maximal cones in S), $N = \bigcup_{\sigma \in S_n} \sigma$. Note that S_n completely determines the original S.

(3) A finite polyhedral decomposition S of N is said to be *non-singular* if for each $\sigma \in S_n$, the set of fundamental generators of σ consists of n elements and forms a Z-basis for N. For every non-singular S, the set of fundamental generators of each q-cone consists of exactly q elements and is completed to a Z-basis for N.

1.5. In concluding §1, we shall quote the following fundamental results (cf. 1.5.1, 1.5.2) due to Demazure [1], Miyake-Oda [5], and Mumford et al. [6]. Recall that $N = \boldsymbol{Z}^n$ (as a set of row vectors), $N^* = \boldsymbol{Z}^n$ (as a set of column vectors), $G = (\boldsymbol{C}^*)^n$, where n is an arbitrary positive integer.

1.5.1 THEOREM. *To every non-singular finite polyhedral decomposition S of N, we can uniquely associate an n-dimensional irreducible non-singular G-equivariant compactification G_S of G possessing the following two properties:*

(i) *To each $\sigma \in S_q$, $q = 0, 1, \cdots, n$, there corresponds a unique $(n-q)$-dimensional G-orbit in G_S, denoted by $\boldsymbol{O}^\sigma$, such that G_S is expressible as*

$$G_S = \bigcup_{\sigma \in S} \boldsymbol{O}^\sigma , \qquad (\textit{disjoint union}) .$$

Furthermore, the closure $F(\sigma)$ of $\boldsymbol{O}^\sigma$ in G_S is an irreducible non-singular $(n-q)$-dimesional G-stable subvariety of G_S written in the form

$$F(\sigma) = \bigcup_{\tau \geqq \sigma} \boldsymbol{O}^\tau \qquad (\textit{disjoint union}) .$$

(ii) *For each $\sigma \in S_n$, $U_\sigma (\overset{\text{defn}}{=\!=} \bigcup_{\tau \leqq \sigma} \boldsymbol{O}^\tau)$ forms an affine open G-stable neighborhood of $\boldsymbol{O}^\sigma$ in G_S satisfying the conditions*

$$G \subsetneqq U_\sigma \cong A^n(\boldsymbol{C})$$

and

$$G_S = \bigcup_{\sigma \in S_n} U_\sigma .$$

Let $\{\boldsymbol{a}_{\sigma;1}, \boldsymbol{a}_{\sigma;2}, \cdots, \boldsymbol{a}_{\sigma;n}\}$ be the set of fundamental generators of σ (which forms a $\boldsymbol{Z}$-basis for N), and let $\{\boldsymbol{a}^*_{\sigma;1}, \boldsymbol{a}^*_{\sigma;2}, \cdots, \boldsymbol{a}^*_{\sigma;n}\}$ be the dual basis for N^* defined by the relation

$$\langle \boldsymbol{a}_{\sigma;i}, \boldsymbol{a}^*_{\sigma;j} \rangle = \delta_{ij} , \qquad i, j = 1, 2, \cdots, n .$$

Then the corresponding characters (cf. 1.3)

$$\chi_{\sigma;i} (\overset{\text{defn}}{=\!=} \chi^{\boldsymbol{a}^*_{\sigma;i}}) \colon G \to \boldsymbol{C}^* , \qquad i = 1, 2, \cdots, n ,$$

extend to rational functions on G_S, which are all regular on U_σ, forming a system of coordinate functions on U_σ by the isomorphism

$$U_\sigma \cong A^n(\boldsymbol{C})$$
$$u \mapsto (\chi_{\sigma;1}(u), \chi_{\sigma;2}(u), \cdots, \chi_{\sigma;n}(u)) .$$

In terms of these coordinates, the G-action on U_σ is described by

$$(\chi_{\sigma;1}(g\cdot u), \chi_{\sigma;2}(g\cdot u), \cdots, \chi_{\sigma;n}(g\cdot u))$$
$$= (\chi_{\sigma;1}(g)\cdot\chi_{\sigma;1}(u), \chi_{\sigma;2}(g)\cdot\chi_{\sigma;2}(u), \cdots, \chi_{\sigma;n}(g)\cdot\chi_{\sigma;n}(u)) ,$$

where both $g \in G$ and $u \in U_\sigma$ are arbitrary.

1.5.2 THEOREM. *Let V be an n-dimensional irreducible non-singular complete variety on which $G = (\boldsymbol{C}^*)^n$ acts regularly and effectively. Then there exists a non-singular finite polyhedral decomposition S of N such that V is G-equivariantly isomorphic to G_S.*

1.5.3 REMARK. Fix an arbitrary $\boldsymbol{Z}$-basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ for N and set $\boldsymbol{a}_0 = -(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \cdots + \boldsymbol{a}_n)$. Let $\{\boldsymbol{a}_1^*, \boldsymbol{a}_2^*, \cdots, \boldsymbol{a}_n^*\}$ be the dual basis for N^* defined by the relation

$$\langle \boldsymbol{a}_i, \boldsymbol{a}_j^* \rangle = \delta_{ij}\ , \qquad i, j = 1, 2, \cdots, n\ .$$

For each $\boldsymbol{a}_j^*$, let χ_j denote the corresponding character of G (which is written as $\chi^{\boldsymbol{a}_j^*}$ in terms of the notation of 1.3).

We now consider the following example: Let S be the non-singular finite polyhedral decomposition of N defined by

$$S_q = \{\sigma_{i_1 i_2 \cdots i_q};\ 0 \leqq i_1 < i_2 < \cdots < i_q \leqq n\}\ , \qquad q = 1, 2, \cdots, n\ ,$$

where $\sigma_{i_1 i_2 \cdots i_q}$ denotes the cone $\sum_{k=1}^{q} \boldsymbol{Z}_{\text{non-neg}} \cdot \boldsymbol{a}_{i_k}$ in N, (cf. Figure 2 which illustrates the case $n = 2$). Then one immediately verifies that G_S is G-equivariantly isomorphic to the projective space $\boldsymbol{P}^n(\boldsymbol{C})$ on which $G = (\boldsymbol{C}^*)^n$ acts via the representation

$$\begin{aligned} G &\to PGL\,(n+1; \boldsymbol{C}) \\ g &\mapsto \varDelta(1; \chi_1(g); \cdots; \chi_n(g))\ , \end{aligned}$$

where $\varDelta(1; \chi_1(g); \cdots; \chi_n(g))$ denotes the $(n+1) \times (n+1)$ diagonal matrix

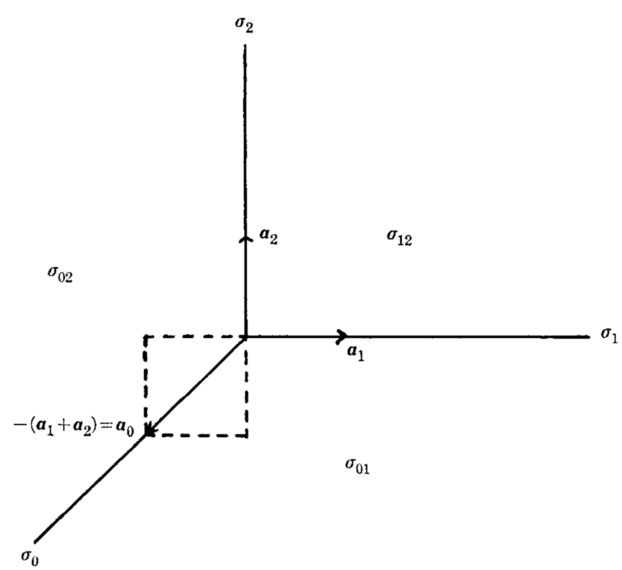

FIGURE 2

with the j-th diagonal element equal to

$$\begin{cases} 1 & \text{if } j = 1 , \\ \chi_{j-1}(g) & \text{if } 2 \leqq j \leqq n+1 . \end{cases}$$

1.5.4 REMARK. Let S be an arbitrary non-singular finite polyhedral decomposition of N, and we consider the corresponding G-equivariant compactification G_S of G. Then one can show that the total Chern class $c(G_S)(=\sum_{k=0}^{n} c_k(G_S))$ of the tangent bundle $T(G_S)$ of G_S is expressible as

$$c(G_S) = \sum_{\sigma \in S} [F(\sigma)] ,$$

where $[F(\sigma)] \in H^*(G_S; \boldsymbol{Q})$ denotes the Poincaré dual of the homology class carried by the subvariety $F(\sigma)$ of G_S, (cf. (i) of 1.5.1). In particular, the Euler number of G_S equals the number of elements in S_n, (see for instance, Iversen [2]).

2. Characterization of $\boldsymbol{P}^n(\boldsymbol{C})$.

We now give the following characterization of $\boldsymbol{P}^n(\boldsymbol{C})$.

2.1 THEOREM. *Let S be a non-singular finite polyhedral decomposition of N and G_S be the corresponding G-equivariant compactification of $G = (\boldsymbol{C}^*)^n$, (cf. 1.5.1). Then $G_S \cong \boldsymbol{P}^n(\boldsymbol{C})$ if and only if the number of elements in S_n (which we shall denote by $\#(S_n)$) is $n+1$.*

PROOF OF 2.1. i) If $G_S \cong \boldsymbol{P}^n(\boldsymbol{C})$, then

$$\#(S_n) = \text{Euler number of } G_S(\cong \boldsymbol{P}^n(\boldsymbol{C})) = n+1 , \qquad \text{(cf. 1.5.4)} .$$

ii) Assume that $\#(S_n) = n+1$. Then one immediately checks that $\#(S_1) = n+1$. Since S is non-singular, we may set $S_1 = \{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ so that $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ forms the set of fundamental generators of some $\sigma_0 \in S_n$. In particular, $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ is a $\boldsymbol{Z}$-basis for N, and hence $\boldsymbol{a}_0$ is expressible as

$$\boldsymbol{a}_0 = \sum_{j=1}^{n} \gamma_j \cdot \boldsymbol{a}_j \qquad \gamma_j \in \boldsymbol{Z} .$$

On the other hand, for each $i \neq 0$, $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \hat{\boldsymbol{a}}_i, \cdots, \boldsymbol{a}_n\}$ forms the set of fundamental generators of the n-cone which is adjacent to σ_0, sitting on the opposite side of σ_0 across from the 1-codimensional face $\sum_{j\neq 0,i} \boldsymbol{Z}_{\text{non-neg}} \cdot \boldsymbol{a}_j$ of σ_0. Hence $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \hat{\boldsymbol{a}}_i, \cdots, \boldsymbol{a}_n\}$ is a $\boldsymbol{Z}$-basis for N, and in particular we obtain $\gamma_i = \pm 1$. Since the interior parts of the above adjacent cones have empty intersection, i.e., $(\sum_{j\neq i} \boldsymbol{Z}_+ \cdot \boldsymbol{a}_j) \cap (\sum_{j\neq 0} \boldsymbol{Z}_+ \cdot \boldsymbol{a}_j) = \varnothing$, we conclude that only $\gamma_i = -1$ can happen. This holds for all $i \in \{1, 2, \cdots, n\}$, and hence $\boldsymbol{a}_0 = -(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \cdots + \boldsymbol{a}_n)$. In view of Remark 1.5.3, $G_S \cong \boldsymbol{P}^n(\boldsymbol{C})$, which finishes the proof of 2.1.

3. Ampleness criterion. We shall briefly discuss what property of S corresponds to the ampleness of normal bundles of G-stable hypersurfaces in G_S. Throughout this section, we fix a non-singular finite polyhedral decomposition S of N, and keep the notation defined in §1: For each $\sigma \in S_n$, $\{\boldsymbol{a}_{\sigma;i} \in N;\ i = 1, 2, \cdots, n\}$ denotes the set of fundamental generators of σ (which is at the same time a $\boldsymbol{Z}$-basis for N), and $\{\boldsymbol{a}^*_{\sigma;i} \in N^*;\ i = 1, 2, \cdots, n\}$ denotes the corresponding dual $\boldsymbol{Z}$-basis for N^* defined by the relation

$$\langle \boldsymbol{a}_{\sigma;i}, \boldsymbol{a}^*_{\sigma;j} \rangle = \delta_{ij}\ , \qquad i, j = 1, 2, \cdots, n\ .$$

Furthermore, we adopt a new notation: For every $\tau \in S_1$, we denote its unique fundamental generator by $\boldsymbol{a}_\tau$. In other words, $\tau = \boldsymbol{Z}_{\text{non-neg}} \cdot \boldsymbol{a}_\tau$.

3.1 Definition. i) Two elements $\tau, \tau' \in S_1$ are called *adjacent*, denoted by $\tau \leftrightarrow \tau'$, if $\tau \neq \tau'$ and simultaneously there exists an element $\sigma \in S_n$ such that $\sigma \geqq \tau$ and $\sigma \geqq \tau'$, i.e., the 2-cone spanned by τ and τ' belongs to S.

ii) For each triple $(\sigma, \tau, \tau') \in S_n \times S_1 \times S_1$ with $\sigma \geqq \tau$, we define an integer $m(\sigma, \tau, \tau')$ as follows: Since $\sigma \geqq \tau$, there exists an $i \in \{1, 2, \cdots, n\}$ such that $\boldsymbol{a}_\tau = \boldsymbol{a}_{\sigma;i}$. We now set

$$m(\sigma, \tau, \tau') \overset{\text{defn}}{=\!=} -\langle \boldsymbol{a}_{\tau'}, \boldsymbol{a}^*_{\sigma;i} \rangle\ .$$

In particular, if $\tau' \leqq \sigma$, then

$$m(\sigma, \tau, \tau') = 0 \qquad \text{for} \quad \tau' \neq \tau\ ,$$
$$m(\sigma, \tau, \tau') = -1 \qquad \text{for} \quad \tau' = \tau\ .$$

The main purpose of this section is to prove the following analogue of Demazure's ampleness criterion, (cf. Demazure [1]). (I here thank Prof. T. Oda for his valuable comments which largely improve both the statement and proof of Theorem 3.2.)

3.2 Theorem. *Let S be a non-singular finite polyhedral decomposition of N. Fix an arbitrary $\tau \in S_1$ and let $F(\tau)$ be the corresponding G-stable non-singular hypersurface* (cf. (i) of 1.5.1) *in G_S. Then the following are equivalent*:

(a) $m(\sigma, \tau, \tau') > 0$ *for every pair $(\sigma, \tau') \in S_n \times S_1$ with $\sigma \geqq \tau \leftrightarrow \tau' \not\leqq \sigma$.*

(b) *The normal bundle $N(G_S: F(\tau))$ of $F(\tau)$ in G_S is ample.*

Recall (cf. Demazure [1]) that the finite set $\{F(\tau);\ \tau \in S_1\}$ forms a $\boldsymbol{Z}$-basis of the G-invariant divisors on G_S and that we have an exact sequence

$$0 \longrightarrow N^* \xrightarrow{\text{div}} \bigoplus_{\tau \in S_1} \boldsymbol{Z} \cdot F(\tau) \xrightarrow{\xi} \operatorname{Pic}(G_S) \longrightarrow 0\ ,$$

where for each $b \in N^*$, $\operatorname{div}(b) = \sum_{\tau \in S_1} \langle a_\tau, b\rangle \cdot F(\tau)$ is the divisor of the rational function χ^b, and ξ maps the divisor $D = \sum_{\tau \in S_1} m_\tau \cdot F(\tau)$ to the corresponding line bundle $\mathcal{O}(D)$ on G_S.

3.3.1 THEOREM (Demazure [1]). *Let S be a non-singular finite polyhedral decomposition of N. Then for every G-invariant divisor $D = \sum_{\tau \in S_1} m_\tau \cdot F(\tau)$, the following are equivalent:*

i) *$\mathcal{O}(D)$ is ample.*

ii) *$\mathcal{O}(D)$ is very ample.*

iii) *For each $\sigma \in S_n$, the unique element $b_\sigma \in N^*$ defined by the equalities $\langle a_{\tau'}, b_\sigma\rangle = -m_{\tau'}$ for all $\tau' \in S_1$ with $\tau' \leqq \sigma$ satisfies $\langle a_{\tau'}, b_\sigma\rangle > -m_{\tau'}$ for all $\tau' \in S_1$ with $\tau' \not\leqq \sigma$.*

For τ in S_1, $F(\tau)$ is a non-singular $(n-1)$-dimensional torus embedding corresponding to

$$\begin{cases} \bar{N} = N/\, \mathbf{Z}\cdot a_\tau, \quad \bar{N}^* = \{b \in N^*;\ \langle a_\tau, b\rangle = 0\} \hookrightarrow N^*\,, \\ \text{non-singular polyhedral decomposition } \bar{S} \text{ of } \bar{N} \text{ defined by} \\ \qquad \bar{S} = \{\bar{\sigma};\ S \ni \sigma \geqq \tau\}\,, \\ \text{where } \bar{\sigma} \text{ is the image of } \sigma \text{ by } N \to \bar{N}\,. \end{cases}$$

Noting that $\bar{S}_1 = \{\bar{\tau}';\ S_1 \ni \tau' \leftrightarrow \tau\}$, we apply Theorem 3.3.1 to prove Theorem 3.2.

LEMMA. *Let f_τ be an element of N^* such that $\langle a_\tau, f_\tau\rangle = 1$. Then the normal bundle $N(G_S{:}\, F(\tau))$ is the line bundle on $F(\tau)$ associated to the divisor*

$$\sum_{\bar{\tau}' \in \bar{S}_1} (-\langle a_{\tau'}, f_\tau\rangle)\cdot \bar{F}(\bar{\tau}')\,,$$

where τ' is the unique element of S_1 mapped to $\bar{\tau}'$ and $\bar{F}(\bar{\tau}')$ is the codimension 1 orbit in $F(\tau)$ corresponding to $\bar{\tau}'$.

PROOF OF LEMMA. $F(\tau)$ as a divisor on G_S is linearly equivalent to $F(\tau) - \operatorname{div}(f_\tau)$ which is a linear combination of $F(\tau')$'s with $S_1 \ni \tau' \neq \tau$. Since $F(\tau')$ and $F(\tau)$ intersect transversally, the divisor on $F(\tau)$ induced by $F(\tau) - \operatorname{div}(f_\tau)$ is equal to $\sum_{\bar{\tau}' \in \bar{S}_1} (-\langle a_{\tau'}, f_\tau\rangle)\cdot \bar{F}(\bar{\tau}')$, which finishes the proof of lemma.

PROOF OF THEOREM 3.2. For every $\bar{\sigma} \in \bar{S}_{n-1}$, let σ be the unique element of S_n mapped to $\bar{\sigma}$ by $N \to \bar{N}$. Then by 3.3.1, $N(G_S{:}\, F(\tau))$ is ample if and only if, for every $\bar{\sigma} \in \bar{S}_{n-1}$, the unique element $b_\sigma \in \bar{N}^* \subseteq N^*$ defined by

$$\langle \boldsymbol{a}_{\tau'}, \boldsymbol{b}_\sigma \rangle = \langle \boldsymbol{a}_{\tau'}, \boldsymbol{f}_\tau \rangle \quad \text{for all} \quad \tau' \in S_1 \quad \text{with} \quad \tau \neq \tau' \leqq \sigma$$

satisfies

$$\langle \boldsymbol{a}_{\tau'}, \boldsymbol{b}_\sigma \rangle > \langle \boldsymbol{a}_{\tau'}, \boldsymbol{f}_\tau \rangle \quad \text{for all} \quad \tau' \in S_1 \quad \text{with} \quad \tau \leftrightarrow \tau' \nleqq \sigma ,$$

i.e.,

$$m(\sigma, \tau, \tau')(= \langle \boldsymbol{a}_{\tau'}, \boldsymbol{b}_\sigma - \boldsymbol{f}_\tau \rangle) > 0 \quad \text{for all} \quad \tau' \in S_1 \quad \text{with} \quad \tau \leftrightarrow \tau' \nleqq \sigma .$$

q.e.d.

4. **Main Theorem.** We finally prove the following main theorem:

4.1 THEOREM. *Let V be an n-dimensional irreducible non-singular complete (therefore as a special case, projective) variety on which an n-dimensional algebraic torus $G = (\boldsymbol{C}^*)^n$ acts regularly and effectively. Assume that every irreducible non-singular hypersurface in V has ample normal bundle. Then V is isomorphic to $\boldsymbol{P}^n(\boldsymbol{C})$.*

PROOF OF 4.1. *Step* 1: By Theorem 1.5.2, there exists a non-singular finite polyhedral decomposition S of N such that V is G-equivariantly isomorphic to G_S. Since for every $\tau \in S_1$, the corresponding G-stable irreducible non-singular hypersurface $F(\tau)$ in $G_S(=V)$ has ample normal bundle, Theorem 3.2 asserts that

$$m(\sigma, \tau, \tau') > 0 \text{ for every triple } (\sigma, \tau, \tau') \in S_n \times S_1 \times S_1 \text{ with } \sigma \geqq \tau \leftrightarrow \tau' \nleqq \sigma .$$

We now fix an arbitrary $\sigma \in S_n$ with its fundamental generators $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$. Since $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ forms a $\boldsymbol{Z}$-basis for N, we define its dual basis $\{\boldsymbol{a}_1^*, \cdots, \boldsymbol{a}_n^*\}$ for N^* by the relation $\langle \boldsymbol{a}_i, \boldsymbol{a}_j^* \rangle = \delta_{ij}$. For each $i \in \{1, 2, \cdots, n\}$, let $\sigma_i \in S_n$ denote the n-cone which is adjacent to σ, sitting on the opposite side of σ in N across from the 1-codimensional face $\sum_{j \neq i} \boldsymbol{Z}_{\text{non-neg}} \cdot \boldsymbol{a}_j$ of σ. Then the set of fundamental generators of σ_i is written as $\{\boldsymbol{a}_1, \cdots, \hat{\boldsymbol{a}}_i, \cdots, \boldsymbol{a}_n, \boldsymbol{a}_i'\}$ for some $\boldsymbol{a}_i' \in N$. Let τ_i(resp. τ_i') be the 1-cone $\boldsymbol{Z}_{\text{non-neg}} \cdot \boldsymbol{a}_i$ (resp. $\boldsymbol{Z}_{\text{non-neg}} \cdot \boldsymbol{a}_i'$). Now if $i, j \in \{1, 2, \cdots, n\}$ are distinct, then $\sigma \geqq \tau_j \leftrightarrow \tau_i' \nleqq \sigma$, and therefore $0 < m(\sigma, \tau_j, \tau_i') = -\langle \boldsymbol{a}_i', \boldsymbol{a}_j^* \rangle$, i.e.,

$$\boldsymbol{a}_i' = \sum_{j=1}^n c_{ij} \cdot \boldsymbol{a}_j , \qquad c_{ij} \in \boldsymbol{Z}, \text{ where } c_{ij} \leqq -1 \text{ whenever } i \neq j .$$

On the other hand, since both $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$ and $\{\boldsymbol{a}_1, \cdots, \hat{\boldsymbol{a}}_i, \cdots, \boldsymbol{a}_n, \boldsymbol{a}_i'\}$ are $\boldsymbol{Z}$-bases for N, one immediately verifies that

$$c_{ii} = -1 , \qquad \text{for all} \quad i \in \{1, 2, \cdots, n\} .$$

Step 2. For contradiction, we now assume $c_{i_0 j_0} < -1$ for some i_0

and j_0 in $\{1, 2, \cdots, n\}(i_0 \neq j_0)$. Putting $\boldsymbol{a}_0 = -(\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n)$, we define cones C_i(resp. $(C_i)_{\boldsymbol{R}}$), $i = 1, \cdots, n$, in N(resp. $N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$) by

$$C_i = \sum_{0 \leq j \neq i}^{n} \boldsymbol{Z}_{\text{non-neg}} \cdot \boldsymbol{a}_j \,, \qquad \left(\text{resp. } (C_i)_{\boldsymbol{R}} = \sum_{0 \leq j \neq i}^{n} \boldsymbol{R}_{\text{non-neg}} \cdot \boldsymbol{a}_j\right).$$

Since for each i,

$$\boldsymbol{a}_0 = -(\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_n) = \boldsymbol{a}'_i + \sum_{1 \leq j \neq i} (-1 - c_{ij}) \cdot \boldsymbol{a}_j \in \sigma_i \,,$$

it follows that

$$(1) \qquad \sigma_i \supseteqq C_i \quad ((\sigma_i)_{\boldsymbol{R}} \supseteqq (C_i)_{\boldsymbol{R}}) \,, \qquad i = 1, 2, \cdots, n \,.$$

We now define a unit ball D in $N_{\boldsymbol{R}}$ by

$$D = \left\{\sum_{i=1}^{n} r_i \cdot \boldsymbol{a}_i ;\, r_i \in \boldsymbol{R},\, \sum_{i=1}^{n} r_i^2 \leqq 1\right\}.$$

Since $N_{\boldsymbol{R}} = \sigma_{\boldsymbol{R}} \cup (\bigcup_{i=1}^{n} (C_i)_{\boldsymbol{R}})$ (union with measure 0 overlappings), we have

$$(2) \qquad \operatorname{Vol}(D) = \operatorname{Vol}(D \cap \sigma_{\boldsymbol{R}}) + \sum_{i=1}^{n} \operatorname{Vol}(D \cap (C_i)_{\boldsymbol{R}}) \,.$$

Consider the (non-empty) open n-cone

$$C_{\boldsymbol{R}}(\varepsilon) = \left\{(r \cdot \cos\theta) \cdot \boldsymbol{a}_0 - (r \cdot \sin\theta) \cdot \boldsymbol{a}_{j_0} + \sum_{1 \leq j \neq i_0, j_0} r_j \cdot \boldsymbol{a}_j ;\; \begin{array}{l} 0 < \theta < \varepsilon,\, 0 < r \in \boldsymbol{R} \\ 0 < r_j \in \boldsymbol{R} \end{array}\right\}$$

with ε sufficiently small (denoted as $\varepsilon \ll 1$). From the definition, we immediately see that

$$(3) \qquad C_{\boldsymbol{R}}(\varepsilon) \cap (C_{i_0})_{\boldsymbol{R}} = \phi \,.$$

On the other hand,

$$\begin{aligned} &(r \cdot \cos\theta) \cdot \boldsymbol{a}_0 - (r \cdot \sin\theta) \cdot \boldsymbol{a}_{j_0} + \sum_{1 \leq j \neq i_0, j_0} r_j \cdot \boldsymbol{a}_j \\ &= (r \cdot \cos\theta) \cdot \Big(\boldsymbol{a}'_{i_0} + \sum_{1 \leq j \neq i_0} (-1 - c_{i_0 j}) \cdot \boldsymbol{a}_j\Big) - (r \cdot \sin\theta) \cdot \boldsymbol{a}_{j_0} + \sum_{1 \leq j \neq i_0, j_0} r_j \cdot \boldsymbol{a}_j \\ &= (r \cdot \cos\theta) \cdot \boldsymbol{a}'_{i_0} + r \cdot \{(-1 - c_{i_0 j_0}) \cdot \cos\theta - \sin\theta\} \cdot \boldsymbol{a}_{j_0} \\ &\quad + \sum_{1 \leq j \neq i_0, j_0}^{n} \{(r \cdot \cos\theta) \cdot (-1 - c_{i_0 j}) + r_j\} \cdot \boldsymbol{a}_j \,. \end{aligned}$$

Since $-1 - c_{ij} \geqq 0$, $i, j = 1, 2, \cdots, n$, and since $-1 - c_{i_0 j_0} > 0$, letting $0 < \theta < \varepsilon$, $0 < r \in \boldsymbol{R}$, and $0 < r_j \in \boldsymbol{R}$ with $\varepsilon \ll 1$, we obtain:

$$\begin{cases} r \cdot \cos\theta > 0 \,, \qquad r \cdot \{(-1 - c_{i_0 j_0}) \cdot \cos\theta - \sin\theta\} > 0 \,, \\ (r \cdot \cos\theta) \cdot (-1 - c_{i_0 j}) + r_j > 0 \,, \qquad \text{for } 1 \leqq j \leqq n \text{ with } j \neq i_0, j_0 \,. \end{cases}$$

Thus, for $0 < \varepsilon \ll 1$,

(4) $$C_R(\varepsilon) \subseteqq (\sigma_{i_0})_R .$$

Now by (1), (3), and (4),

$$\mathrm{Vol}\,(D \cap (\sigma_{i_0})_R) - \mathrm{Vol}\,(D \cap (C_{i_0})_R) \geqq \mathrm{Vol}\,(D \cap C_R(\varepsilon)) > 0 ,$$
$$\mathrm{Vol}\,(D \cap (\sigma_i)_R) \geqq \mathrm{Vol}\,(D \cap (C_i)_R) , \qquad i = 1, 2, \cdots, n .$$

Therefore,

$$\mathrm{Vol}\,(D) \geqq \mathrm{Vol}\,(D \cap \sigma_R) + \sum_{i=1}^{n} \mathrm{Vol}\,(D \cap (\sigma_i)_R)$$
$$> \mathrm{Vol}\,(D \cap \sigma_R) + \sum_{i=1}^{n} \mathrm{Vol}\,(D \cap (C_i)_R) ,$$

which is in contradiction to (2). Thus,

$$c_{ij} = -1 , \qquad \text{for} \quad i, j = 1, 2, \cdots, n ,$$

i.e.,

$$\sigma_i = C_i , \qquad \text{for} \quad i = 1, 2, \cdots, n .$$

Since $N = \sigma \cup (\bigcup_{i=1}^{n} C_i)$, we immediately obtain:

$$S_n = \{\sigma, C_1, C_2, \cdots, C_n\} .$$

Hence by Theorem 2.1,

$$G_S(=V) \cong \boldsymbol{P}^n(C) .$$

q.e.d.

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