Research Article

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Almost Kenmotsu 3-h-manifolds with transversely Killing-type Ricci operators

Abstract: In this paper, it is proved that the Ricci operator of an almost Kenmotsu 3-h-manifold M is of transversely Killing-type if and only if M is locally isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)$ or a non-unimodular Lie group endowed with a left invariant non-Kenmotsu almost Kenmotsu structure. This result extends those results obtained by Cho [*Local symmetry on almost Kenmotsu three-manifolds*, Hokkaido Math. J. **45** (2016), no. 3, 435–442] and Wang [*Three-dimensional locally symmetric almost Kenmotsu manifolds*, Ann. Polon. Math. **116** (2016), no. 1, 79–86; *Three-dimensional almost Kenmotsu manifolds with η-parallel Ricci tensor*, J. Korean Math. Soc. **54** (2017), no. 3, 793–805].

1 Introduction

The classifications of symmetric and homogeneous almost contact manifolds are one of the most important problems in differential geometry of almost contact manifolds. With regard to fruitful symmetry classification results in the framework of contact Riemannian manifolds, we refer the reader to the study of D. E. Blair [1]. In this paper, we try to study the symmetry classifications on the other kind of almost contact manifolds which are named (almost) Kenmotsu manifolds. Note that a Riemannian manifold is locally symmetric if and only if the curvature tensor is parallel with respect to the Levi-Civita connection. The study of locally symmetric Kenmotsu manifolds was initiated by K. Kenmotsu in [2], who proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature −1. This result has attracted many researcher's interests. After K. Kenmotsu's result, many results concerning symmetry classification problems on Kenmotsu manifolds have emerged (see, for example, the studies of U. C. De [3] and G. Pitis [4]).

G. Dileo and A. M. Pastore in [5] initiated the symmetry classification problem on almost Kenmotsu manifolds. Locally symmetric almost Kenmotsu manifolds under $R(X, Y)\xi = 0$ for any contact vector fields X, Y (see Section 2 for the notation), some nullity conditions and CR-integrability have been classified in [5–7], respectively. In particular, J. T. Cho [8] and Y. Wang [9] independently completed the classification problem for almost Kenmotsu 3-manifolds. They proved that an almost Kenmotsu 3-manifold is locally symmetric if and only if the manifold is locally isometric to either the hyperbolic 3-space $\mathbb{H}^3(-1)$ or

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a Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. Some generalizations of their results were obtained in [10,11]. Very recently, Y. Wang in [12] studied locally symmetric almost Kenmotsu manifolds for general dimensions under some addition conditions. As far as we know, the problem of "classify completely locally symmetric almost Kenmotsu manifolds of dimension >3" is still open.

In this paper, we aim to give some local classification results for almost Kenmotsu manifolds in terms of the Ricci tensor. D. E. Blair in [13] introduced the so-called Killing tensor (on a Riemannian manifold) which is defined by

$$(\nabla_X T)X = 0,$$

where ∇ is the Levi-Civita connection, T is a tensor field of type (1,1) and X denotes an arbitrary vector field. By employing the Killing tensor, D. E. Blair in [13] presented a new characterization for an almost contact metric manifold to be a cosymplectic manifold. In this paper, we say that the Ricci operator Q on an almost Kenmotsu manifold is transversely Killing if it satisfies

$$(\nabla_X Q)X = 0 \tag{1.1}$$

for any vector field X orthogonal to the Reeb vector field ξ. It is known that on a Riemannian 3-manifold, the Ricci tensor is parallel if and only if the curvature tensor is parallel. Therefore, relation (1.1) is much weaker than local symmetry condition. Y. Wang in [10] considered $\nabla_\xi h = 0$ on an almost Kenmotsu 3-manifold and constructed many examples satisfying this condition (see Section 2 for the notion of h). For simplicity, we say that an almost Kenmotsu 3-manifold is an almost Kenmotsu 3-*h*-manifold if it satisfies $\nabla_\xi h = 0$. Applying the aforementioned two notions, we obtain the following.

Theorem 1.1. *The Ricci operator of an almost Kenmotsu 3-h-manifold M is transversely Killing if and only if M is locally isometric to either the hyperbolic 3-space $\mathbb{H}^3(-1)$ or a non-unimodular Lie group endowed with a left invariant almost Kenmotsu structure.*

On the hyperbolic 3-space $\mathbb{H}^3(-1)$ there exists a Kenmotsu structure (see [14]) and on any non-unimodular Lie group there exists a left invariant non-Kenmotsu almost Kenmotsu structure (see [6, Theorem 5.2]).

Remark 1.1. Since those conditions we have employed are much weaker than local symmetry, our results are generalizations of Cho and Wang's results (see [8,9]).

Remark 1.2. Replacing X by $X + Y$ in (1.1), we see that this relation is also equivalent to $(\nabla_X Q)Y + (\nabla_Y Q)X = 0$ for any vector fields X and Y orthogonal to the Reeb vector field ξ. Such a condition is much weaker than Ricci η-parallel (i.e., $g((\nabla_X Q)Y, Z) = 0$ for any vector fields X, Y, Z orthogonal to the Reeb vector field ξ). Therefore, our Theorem 1.1 can also be viewed as an extension of results in [10].

2 Almost Kenmotsu manifolds

By an almost contact metric manifold, we mean a Riemannian manifold (M, g) of dimension $2n + 1$ on which there exists a quadruple (ϕ, ξ, η, g) satisfying

$$\phi^2 = -id + \eta \otimes \xi, \eta \circ \phi = 0, \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y, where ϕ is a (1,1)-type tensor field, ξ is a vector field called the Reeb vector field, η is a global 1-form called the almost contact 1-form and g is a Riemannian metric (see the study of Blair [1]).

A vector field orthogonal to ξ is called a contact vector field. By an almost Kenmotsu manifold we mean an almost contact metric manifold on which there holds $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, where Φ is the fundamental 2-form defined by $\Phi(X, Y) = g(X, \phi Y)$ (see [5,6]). We consider the product $M^{2n+1} \times \mathbb{R}$ of an almost contact metric manifold M^{2n+1} and $\mathbb{R}$ and define on it an almost complex structure J by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a C^∞-function on $M^{2n+1} \times \mathbb{R}$. The almost contact metric manifold is said to be normal if J is integrable, or equivalently,

$$[\phi, \phi] = -2d\eta \otimes \xi,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold (cf. [2,5,6]). An almost Kenmotsu manifold is a Kenmotsu manifold if and only if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$$

for any vector fields X, Y.

Let M^{2n+1} be an almost Kenmotsu manifold. We consider three tensor fields $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on M^{2n+1}, where R is the Riemannian curvature tensor of g and $\mathcal{L}$ is the Lie differentiation. From [5,6], we know that the three (1,1)-type tensor fields l, h' and h are symmetric and satisfy

$$h\xi = 0, \ l\xi = 0, \ \mathrm{tr}h = 0, \ \mathrm{tr}(h') = 0, \ h\phi + \phi h = 0,$$

and

$$\nabla\xi = id - \eta \otimes \xi + h'. \tag{2.3}$$

3 Main results and proofs

It is well known that an almost Kenmotsu 3-manifold becomes a Kenmotsu 3-manifold if and only if $h = 0$ (see [5]). Therefore, we discuss the proof of Theorem 1.1 by two main situations. First, we consider a non-Kenmotsu almost Kenmotsu case. Let $\mathcal{U}_1$ be the maximal open subset of a 3-dimensional almost Kenmotsu manifold M^3 on which $h \neq 0$, and $\mathcal{U}_2$ the maximal open subset on which $h = 0$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we may set $he = \lambda e$ and hence $h\phi e = -\lambda\phi e$, where λ is a positive eigenfunction of h on $\mathcal{U}_1$.

Applying (2.3), the following lemma was obtained by Cho and Kimura in [15, Lemma 6].

Lemma 3.1. *On a non-Kenmotsu almost Kenmotsu 3-manifold we have*

$$\begin{aligned}
&\nabla_\xi \xi = 0, \ \nabla_\xi e = a\phi e, \ \nabla_\xi \phi e = -ae, \\
&\nabla_e \xi = e - \lambda\phi e, \ \nabla_e e = -\xi - b\phi e, \ \nabla_e \phi e = \lambda\xi + be, \\
&\nabla_{\phi e} \xi = -\lambda e + \phi e, \ \nabla_{\phi e} e = \lambda\xi + c\phi e, \ \nabla_{\phi e} \phi e = -\xi - ce,
\end{aligned} \tag{3.1}$$

where a, b, c are smooth functions.

Applying Lemma 3.1, the Ricci operator Q of M^3 is written by

$$\begin{aligned}
Q\xi &= -2(\lambda^2 + 1)\xi - \sigma(e)e - \sigma(\phi e)\phi e, \\
Qe &= -\sigma(e)\xi - (A + 2\lambda a)e + (\xi(\lambda) + 2\lambda)\phi e, \\
Q\phi e &= -\sigma(\phi e)\xi + (\xi(\lambda) + 2\lambda)e - (A - 2\lambda a)\phi e,
\end{aligned} \tag{3.2}$$

with respect to the local basis $\{\xi, e, \phi e\}$, where for simplicity we set

$$A = e(c) + \phi e(b) + b^2 + c^2 + 2 \tag{3.3}$$

and

$$\sigma(e) := -g(Q\xi, e) = \phi e(\lambda) + 2\lambda b, \quad \sigma(\phi e) := -g(Q\xi, \phi e) = e(\lambda) + 2\lambda c. \tag{3.4}$$

From (3.2), we see that the scalar curvature of M^3 is given by

$$r = -2(A + \lambda^2 + 1). \tag{3.5}$$

Theorem 3.1. *The Ricci operator of a non-Kenmotsu almost Kenmotsu 3-h-manifold is transversely Killing if and only if the manifold is locally isometric to a non-unimodular Lie group endowed with a left invariant almost Kenmotsu structure.*

Proof. Applying Lemma 3.1, by a direct calculation we obtain

$$(\nabla_\xi h)e = \xi(\lambda)e + 2a\lambda\phi e \quad \text{and} \quad (\nabla_\xi h)\phi e = -\xi(\lambda)\phi e + 2a\lambda e.$$

By the aforementioned two relations, it is easily seen that $\nabla_\xi h = 0$ if and only if

$$a = 0 \quad \text{and} \quad \xi(\lambda) = 0, \tag{3.6}$$

where we have used the assumption $\lambda > 0$. By means of (3.2), (3.6) and Lemma 3.1, we have

$$(\nabla_e Q)e = (A - 2 - e(\sigma(e)) - b\sigma(\phi e))\xi + (4\lambda b - 2\sigma(e) - e(A))e + (\lambda\sigma(e) - \sigma(\phi e) + 2e(\lambda))\phi e, \tag{3.7}$$

$$(\nabla_e Q)\phi e = (2\lambda^3 - A\lambda - e(\sigma(\phi e)) + b\sigma(e))\xi + (\lambda\sigma(e) - \sigma(\phi e) + 2e(\lambda))e + (2\lambda\sigma(\phi e) - 4\lambda b - e(A))\phi e, \tag{3.8}$$

$$(\nabla_{\phi e} Q)e = (2\lambda^3 - A\lambda - \phi e(\sigma(e)) + c\sigma(\phi e))\xi + (2\lambda\sigma(e) - 4\lambda c - \phi e(A))e + (\lambda\sigma(\phi e) + 2\phi e(\lambda) - \sigma(e))\phi e, \tag{3.9}$$

$$(\nabla_{\phi e} Q)\phi e = (A - 2 - \phi e(\sigma(\phi e)) - c\sigma(e))\xi + (\lambda\sigma(\phi e) + 2\phi e(\lambda) - \sigma(e))e + (4\lambda c - 2\sigma(\phi e) - \phi e(A))\phi e. \tag{3.10}$$

Suppose that the Ricci operator is transversely Killing, setting $X = Y = e$ and $X = Y = \phi e$ in (1.1) we obtain $(\nabla_e Q)e = 0$ and $(\nabla_{\phi e} Q)\phi e = 0$, respectively, which are compared with (3.7) and (3.10) implying

$$\begin{aligned} A - 2 - e(\sigma(e)) - b\sigma(\phi e) &= 0, \\ 4\lambda b - 2\sigma(e) - e(A) &= 0, \\ \lambda\sigma(e) - \sigma(\phi e) + 2e(\lambda) &= 0, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} A - 2 - \phi e(\sigma(\phi e)) - c\sigma(e) &= 0, \\ \lambda\sigma(\phi e) + 2\phi e(\lambda) - \sigma(e) &= 0, \\ 4\lambda c - 2\sigma(\phi e) - \phi e(A) &= 0. \end{aligned} \tag{3.12}$$

Applying (3.4) in the last term of (3.11), we obtain that

$$e(\lambda) + \lambda\phi e(\lambda) = 2\lambda c - 2\lambda^2 b. \tag{3.13}$$

Taking the covariant derivative of (3.13) along the Reeb flow we obtain

$$(3\lambda^2 - 1)e(\lambda) - 2\lambda\phi e(\lambda) = 4\lambda^2 b - 2\lambda c - 2\lambda^3 c. \tag{3.14}$$

The addition of (3.14) to (3.13) multiplied by 2 gives

$$(3\lambda^2 + 1)e(\lambda) = 2\lambda c(1 - \lambda^2). \tag{3.15}$$

Similarly, applying (3.4) in the second term of (3.12) we obtain that

$$\lambda e(\lambda) + \phi e(\lambda) = 2\lambda b - 2\lambda^2 c. \tag{3.16}$$

Taking the covariant derivative of (3.16) along the Reeb flow we obtain

$$(3\lambda^2 - 1)\phi e(\lambda) - 2\lambda e(\lambda) = 4\lambda^2 c - 2\lambda b - 2\lambda^3 b. \tag{3.17}$$

The addition of (3.17) to (3.16) multiplied by 2 gives

$$(3\lambda^2 + 1)\phi e(\lambda) = 2\lambda b(1 - \lambda^2). \tag{3.18}$$

Consequently, the subtraction of (3.15) multiplied by b from (3.18) multiplied by c implies that

$$be(\lambda) = c\phi e(\lambda), \tag{3.19}$$

where we have used $3\lambda^2 + 1 \neq 0$.

In view of (3.6), from Lemma 3.1, by a direct calculation we have

$$[\xi, e] = -e + \lambda\phi e, \quad [\xi, \phi e] = \lambda e - \phi e \ \text{ and } \ [e, \phi e] = be - c\phi e. \tag{3.20}$$

Applying again the first term of (3.6), according to the first two terms of (3.20) we get

$$\xi(e(\lambda)) = -e(\lambda) + \lambda\phi e(\lambda) \ \text{ and } \ \xi(\phi e(\lambda)) = \lambda e(\lambda) - \phi e(\lambda). \tag{3.21}$$

Moreover, applying again (3.20), with the help of (3.6), the well-known Jacobi identity for tangent vector fields $\{\xi, e, \phi e\}$ becomes

$$\begin{aligned} \xi(b) &= e(\lambda) + \lambda c - b, \\ \xi(c) &= \phi e(\lambda) + \lambda b - c. \end{aligned} \tag{3.22}$$

Taking the covariant derivative of (3.19) gives $\xi(b)e(\lambda) + b\xi(e(\lambda)) = \xi(c)\phi e(\lambda) + c\xi(\phi e(\lambda))$, which is simplified by (3.21) and (3.22) implying $(e(\lambda))^2 = (\phi e(\lambda))^2$. According to this, we have to consider the following two cases. □

Case 1. $e(\lambda) = \phi e(\lambda)$. In this situation, relations (3.13) and (3.16) become $(1 + \lambda)e(\lambda) = 2\lambda(c - \lambda b)$ and $(1 + \lambda)e(\lambda) = 2\lambda(b - \lambda c)$, respectively. Because λ is the positive eigenfunction of h, it follows that

$$b = c. \tag{3.23}$$

Now with the help of the second term of (3.4), the last term of (3.12) becomes $\phi e(A) = -2e(\lambda)$, which is simplified by (3.5) implying

$$4e(\lambda)(\lambda - 1) + \phi e(r) = 0, \tag{3.24}$$

where we have used $e(\lambda) = \phi e(\lambda)$. With the aid of (3.6), from Lemma 3.1 and the first term of (3.2) we have

$$(\nabla_\xi Q)\xi = -\xi(\sigma(e))e - \xi(\sigma(\phi e))\phi e. \tag{3.25}$$

Recall that on any Riemannian manifold (M, g) of dimension m there holds

$$\frac{1}{2}X(r) = g\left(\sum_{i=1}^{m} (\nabla_{e_i}Q)e_i, X\right), \tag{3.26}$$

where $\{e_1, e_2, \ldots, e_m\}$ is an orthonormal basis for the tangent space of the manifold and X is an arbitrary vector field. Thus, setting $X = \phi e$ in (3.26), with the help of (3.7), (3.10), (3.11), (3.12) and (3.25), we obtain $\phi e(r) = -2\xi(\sigma(\phi e))$, which is simplified by (3.21), (3.22) and $e(\lambda) = \phi e(\lambda)$ implying

$$\phi e(r) = 2(1 - 3\lambda)e(\lambda) + 4\lambda b(1 - \lambda). \tag{3.27}$$

Now putting (3.27) into (3.24) we obtain

$$(\lambda + 1)e(\lambda) + 2\lambda(\lambda - 1)b = 0. \tag{3.28}$$

Taking the covariant derivative of (3.28) along the Reeb flow, with the help of (3.6), (3.21), (3.22), we obtain

$$(\lambda - 1)((3\lambda + 1)e(\lambda) + 2\lambda(\lambda - 1)b) = 0. \tag{3.29}$$

Next the proof is divided into the following two subcases.

If $\lambda \neq 1$, it follows that $(3\lambda + 1)e(\lambda) + 2\lambda(\lambda - 1)b = 0$, which is compared with (3.28) implying $e(\lambda) = 0$. Therefore, in terms of (3.6) and $e(\lambda) = \phi e(\lambda)$ we see that λ is a positive constant $\neq 1$. Using this in (3.28) we obtain $b = c = 0$ because of (3.23) and $\lambda \neq 1$. In this context, (3.20) becomes

$$[\xi, e] = -e + \lambda\phi e, \quad [\xi, \phi e] = \lambda e - \phi e \text{ and } [e, \phi e] = 0. \tag{3.30}$$

The so-called adjoint operator ad_X is defined by $\mathrm{ad}_X : Y \to [X, Y]$ for any vector fields X, Y on tangent space. Applying (3.30), it is easily seen that

$$\mathrm{trace}(\mathrm{ad}_\xi) = -2, \mathrm{trace}(\mathrm{ad}_e) = 0, \mathrm{trace}(\mathrm{ad}_e) = 0.$$

Thus, according to J. Milnor [16] we see that the manifold is locally isometric to a non-unimodular Lie group endowed with a left invariant non-Kenmotsu almost Kenmotsu structure. For the constructions of almost Kenmotsu structures on this kind of Lie groups we refer the reader to [6, Theorem 5.2].

Otherwise, let us consider the case $\lambda = 1$. Applying (3.25), (3.7) and (3.10) in (3.26), with the aid of (3.11) and (3.12) we observe that the scalar curvature is a constant. Therefore, using this and $\lambda = 1$ in (3.5) we know that A is also a constant. Moreover, applying $\lambda = 1$ in relation (3.14) or (3.17) we obtain $b = c$. Finally, applying $\lambda = 1$, $b = c$ and (3.4), according to (3.7)–(3.10) we see that the Ricci tensor is η-parallel, i.e., $g((\nabla_X Q)Y, Z) = 0$ for any vector fields X, Y, Z orthogonal to ξ. Following Y. Wang [10], we know that the manifold is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. Note that such a product corresponds to the non-unimodular Lie group whose Lie algebra is given by

$$[\xi, e] = -e + \phi e, [\xi, \phi e] = e - \phi e \text{ and } [e, \phi e] = 0,$$

and this is in fact a special case of (3.30) for $\lambda = 1$.

Case 2. $e(\lambda) + \phi e(\lambda) = 0$. In this situation, the addition of (3.15) to (3.18) implies that either $\lambda = 1$ or $b + c = 0$. This makes us to have to consider the following two subcases.

If $\lambda = 1$, from (3.13) or (3.16) we obtain $b = c$, which is compared with the previous relation giving $b = c = 0$. Now (3.20) becomes

$$[\xi, e] = -e + \phi e, \quad [\xi, \phi e] = e - \phi e \text{ and } [e, \phi e] = 0.$$

As seen before, now the manifold is locally isometric to the product $\mathbb{H}^2(-4) \times \mathbb{R}$ which can be realized as a non-unimodular Lie group whose Lie algebra is given by (3.30) for $\lambda = 1$.

If $\lambda \neq 1$, then $b + c = 0$. Applying this equation and $e(\lambda) + \phi e(\lambda) = 0$ in (3.13) we obtain

$$(\lambda - 1)e(\lambda) = 2\lambda(\lambda + 1)b. \tag{3.31}$$

Similarly, applying $b + c = 0$ and $e(\lambda) + \phi e(\lambda) = 0$ in (3.14) we obtain

$$(3\lambda^2 + 2\lambda - 1)e(\lambda) = 2\lambda(\lambda + 1)^2 b. \tag{3.32}$$

Finally, multiplying (3.31) by $-(\lambda + 1)$ gives an equation, and the addition of the resulting relation to (3.32) implies that $2\lambda(\lambda + 1)e(\lambda) = 0$. Recall that λ is assumed to be the positive eigenfunction of the operator h, it follows that $e(\lambda) = 0$ and hence together this with (3.6) and $e(\lambda) + \phi e(\lambda) = 0$ we see that λ is a constant. Applying this in (3.32) we obtain $b = c = 0$. Therefore, it is easily seen that (3.30) holds in this context and the manifold is locally isometric to a non-unimodular Lie group endowed with a left invariant almost Kenmotsu structure.

Conversely, as the product $\mathbb{H}^2(-4) \times \mathbb{R}$ is locally symmetric, the Ricci operator of this product is necessarily parallel and hence transversely Killing. On a left invariant almost Kenmotsu structure defined on a Lie group whose Lie algebra is given by (3.30), following Wang [17,18] we have

$$\nabla_\xi \xi = 0, \nabla_{e_2} \xi = e_2 - \lambda e_3, \nabla_{e_3} \xi = -\lambda e_2 + e_3,$$
$$\nabla_\xi e_2 = 0, \nabla_{e_2} e_2 = -\xi, \nabla_{e_3} e_2 = \lambda\xi,$$
$$\nabla_\xi e_3 = 0, \nabla_{e_2} e_3 = \lambda\xi, \nabla_{e_3} e_3 = -\xi$$

with $\lambda \in \mathbb{R}^+$. The Ricci operator is given by

$$Q\xi = -2(1 + \lambda^2)\xi, \; Qe_2 = -2e_2 + 2\lambda e_3, \; Qe_3 = 2\lambda e_2 - 2e_3.$$

From the aforementioned relations, it is easily seen that (1.1) is true for $X \in \{e_2, e_3\}$. □

Theorem 3.1. in fact is the non-Kenmotsu version of Theorem 1.1 and the corresponding Kenmotsu version of Theorem 1.1 is given as follows.

Theorem 3.2. *The Ricci operator of a Kenmotsu 3-manifold is transversely Killing if and only if it is locally isometric to the hyperbolic 3-space* $\mathbb{H}^3(-1)$.

Proof. On a Kenmotsu 3-manifold, $h = 0$ and hence from (2.3) we have

$$\nabla_X \xi = X - \eta(X)\xi,$$

for any vector field X. By a direct calculation, from the aforementioned equation we get

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{3.33}$$

for any vector field X. And hence from (3.33) we obtain

$$Q\xi = -2\xi.$$

In view of $\nabla_\xi \xi = 0$, it follows from the aforementioned relation that $(\nabla_\xi Q)\xi = 0$. Therefore, if the Ricci operator is of transversely Killing-type (that is, $(\nabla_X Q)X = 0$ for any vector field X orthogonal to ξ), it is easily seen that the scalar curvature is a constant because of (3.26).

It is well known that on a Riemannian 3-manifold, the curvature tensor R is given by

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y) \tag{3.34}$$

for any vector fields X, Y, Z, where r is the scalar curvature. Replacing Z by ξ in (3.34), with the aid of (3.33), we obtain

$$QX = \left(\frac{r}{2} + 1\right)X - \left(\frac{r}{2} + 3\right)\eta(X)\xi \tag{3.35}$$

for any vector field X. On the other hand, because the scalar curvature r is a constant, applying (3.26) on (3.35) we have $r = -6$. Putting this in (3.35) we see that $QX = -2X$, which is used in (3.34) giving that

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X$$

for any vector fields X, Y, Z. Therefore, the manifold is of constant sectional curvature -1. The converse is also true as $\mathbb{H}^3(-1)$ is locally symmetric. □

Proof of Theorem 1.1. The proof of Theorem 1.1 follows immediately from Theorems 3.1 and 3.2. □

It was proved by Y. Wang in [19, Theorem 3.7] that if an almost Kenmotsu 3-manifold is a Ricci soliton with the potential vector field orthogonal to the Reeb vector field ξ and ξ is an eigenvector field for the Ricci operator, then the manifold is locally isometric to either $\mathbb{H}^3(-1)$ or a non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure whose Lie algebra is given by (3.30). Where by a Ricci soliton we mean a triple (V, λ, g) defined on a Riemannian manifold (M, g) such that $\frac{1}{2}\mathcal{L}_V g + S = \lambda g$. According to this result and our Theorem 1.1, we have the following.

Remark 3.1. If the Ricci operator of an almost Kenmotsu 3-h-manifold M is transversely Killing, then M is a Ricci soliton.

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