CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 23, Pages 105–116 (June 21, 2019)
https://doi.org/10.1090/ecgd/335

ALMOST KENMOTSU METRIC AS A CONFORMAL RICCI SOLITON

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Abstract. In the present paper, we characterize $(k,\mu)'$ and generalized $(k,\mu)'$-almost Kenmotsu manifolds admitting the conformal Ricci soliton. It is also shown that a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} does not admit conformal gradient Ricci soliton (g,V,λ) with V collinear with the characteristic vector field ξ. Finally an illustrative example is presented.

1. Introduction

Hamilton [9] introduced the concept of Ricci flow in 1982 and proved its existence. The Ricci flow is an evolution equation for metrics on a Riemannian manifold given by

$$\frac{\partial g}{\partial t} = -2S,$$

where g is the Riemannian metric and S denotes the Ricci tensor.

A self-similar solution to the Ricci flow [9], [14] is called a Ricci soliton [10] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\pounds_V g + 2S = 2\lambda g,$$

where $\pounds_X$ is the Lie derivative, S is the Ricci tensor, g is the Riemannian metric, V is a vector field, and λ is a scalar. The Ricci soliton is denoted by (g,V,λ) and said to be shrinking, steady, and expanding according to whether λ is positive, zero, and negative, respectively.

In [8], Fischer developed the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M where M is considered as a smooth, closed, connected, oriented n-manifold is defined by the equation [8]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg \quad \text{and} \quad r = -1,$$

where p is a non-dynamical scalar field which is time dependent, r is the scalar curvature of the manifold, and n is the dimension of the manifold.

Received by the editors July 19, 2018.

2010 *Mathematics Subject Classification.* Primary 53D15; Secondary 53A30, 53C25.

Key words and phrases. Conformal Ricci soliton, conformal gradient Ricci soliton, almost Kenmotsu manifolds, nullity distributions.

The first author was supported by the Council of Scientific and Industrial Research, India (File no: 09/028(1010)/2017-EMR-1) in the form of Junior Research Fellowship.

In 2015, Basu and Bhattacharyya [1] introduced the notion of the conformal Ricci soliton equation on Kenmotsu manifold M^{2n+1} as

$$\pounds_V g + 2S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.1}$$

where λ is constant.

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation. It was later studied by Dutta et al. [7] in Lorentzian α-Sasakian manifolds and Nagaraja and Venu [12] in f-Kenmotsu manifolds.

A conformal Ricci soliton is said to be a conformal gradient Ricci soliton if the vector field V is a gradient of some smooth function on a manifold M. In this case, the conformal gradient Ricci soliton is given by

$$\nabla\nabla f + S = [2\lambda - (p + \frac{2}{2n+1})]g, \tag{1.2}$$

where f is the gradient of the potential vector field V.

The paper is organized as follows.

After preliminaries, in Section 2, we consider a conformal Ricci soliton on $(k,\mu)'$ and generalized $(k,\mu)'$-almost Kenmotsu manifolds. Section 3 deals with a conformal gradient Ricci soliton on $(k,\mu)'$-almost Kenmotsu manifolds. Finally, in Section 4, an example is presented which verifies our theorem.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to have a (ϕ,ξ,η)-structure or an almost contact structure if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ, and a 1-form η satisfying ([2], [3]),

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1) easily.

If a manifold M with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any X, Y on M. The condition for an almost contact metric manifold being normal is equivalent to the vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi,\phi] + 2d\eta\otimes\xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ [2]. Recently in [4], [5], [6], [13], almost contact metric manifolds such that η is closed and $d\Phi = 2\eta\wedge\Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X$ for any vector fields X,Y. It is well known [11] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I\times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t, and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = \mathrm{Ker}(\eta) = \mathrm{Im}(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [13]:

$$h\xi = 0,\ l\xi = 0,\ \mathrm{tr}(h) = 0,\ \mathrm{tr}(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X\xi = X - \eta(X)\xi - \phi hX (\Rightarrow \nabla_\xi\xi = 0), \tag{2.3}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y\phi h)X - (\nabla_X\phi h)Y \tag{2.5}$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([4], [16])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.6}$$

In [4], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' = \{Z \in T_p(M) : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &+\mu[g(Y,Z)h'X - g(X,Z)h'Y]\}. \end{aligned} \tag{2.7}$$

The above notion is called generalized nullity distributions when one allows k, μ to be smooth functions.

Let $X \in \mathcal{D}$ be the eigenvector of h' corresponding to the eigenvalue α. Then from (2.5) it is clear that $\alpha^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\alpha = \pm\sqrt{-k-1}$. We denote by $[\alpha]'$ and $[-\alpha]'$ the corresponding eigenspaces related to the non-zero eigenvalue α and $-\alpha$ of h', respectively. In [4], it is proved that in a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, $k < -1$, $\mu = -2$, and $\mathrm{Spec}(h') = \{0, \alpha, -\alpha\}$ with 0 as a simple eigenvalue and $\alpha = \sqrt{-k-1}$. Also

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{2.8}$$

In [15], Wang and Liu proved that for a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, the Ricci operator Q of M^{2n+1} is given by

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{2.9}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$. From (2.7), we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{2.10}$$

where $k, \mu \in \mathbb{R}$. Also we get from (2.10)

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{2.11}$$

Contracting X in (2.10), we have

$$S(Y,\xi) = 2nk\eta(Y). \tag{2.12}$$

Using (2.3), we have

$$(\nabla_X\eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y). \tag{2.13}$$

For further details on almost Kenmotsu manifolds, we refer the reader to go through the references ([15]-[18]).

3. Conformal Ricci soliton

In this section, we study the conformal Ricci soliton on $(k,\mu)'$ and generalized $(k,\mu)'$-almost Kenmotsu manifolds. Before proving our main theorems, we first prove the following lemmas.

Lemma 3.1. *In a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h' \neq 0$, the following relation holds:*

$$(\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = -4n(k+2)g(h'X,Y)\eta(Z).$$

Proof. From (2.9), we have

$$S(X,Y) = -2ng(X,Y) + 2n(k+1)\eta(X)\eta(Y) - 2ng(h'X,Y) \tag{3.1}$$

for any vector fields X, Y on M^{2n+1}.

Taking a covariant derivative of the foregoing equation along any vector field Z we have

$$\begin{aligned}\nabla_Z S(X,Y) &= -2n\nabla_Z g(X,Y) + 2n(k+1)(\nabla_Z \eta(X))\eta(Y)\\ &\quad +2n(k+1)\eta(X)(\nabla_Z \eta(Y)) - 2n\nabla_Z g(h'X,Y).\end{aligned} \tag{3.2}$$

Now, we have

$$(\nabla_Z S)(X,Y) = \nabla_Z S(X,Y) - S(\nabla_Z X,Y) - S(X,\nabla_Z Y).$$

Using (3.1) and (3.2) in the foregoing equation, we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2n(k+1)(\nabla_Z \eta)X)\eta(Y) + 2n(k+1)\eta(X)(\nabla_Z \eta)Y\\ &\quad -2ng((\nabla_Z h')X,Y).\end{aligned} \tag{3.3}$$

Now, using (2.8) and (2.13) in (3.3) we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= 2n(k+1)\eta(Y)(g(X,Z) - \eta(X)\eta(Z) + g(h'X,Z))\\ &\quad +2n(k+1)\eta(X)(g(Y,Z) - \eta(Y)\eta(Z) + g(h'Y,Z))\\ &\quad +2ng(h'Z + h'^2 Z, X)\eta(Y) + 2n\eta(X)g(h'Z + h'^2 Z, Y).\end{aligned} \tag{3.4}$$

Similarly, we obtain the following:

$$\begin{aligned}(\nabla_X S)(Y,Z) &= 2n(k+1)\eta(Z)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y))\\ &\quad +2n(k+1)\eta(Y)(g(X,Z) - \eta(X)\eta(Z) + g(h'X,Z))\\ &\quad +2ng(h'X + h'^2 X, Y)\eta(Z) + 2n\eta(Y)g(h'X + h'^2 X, Z)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}(\nabla_Y S)(X,Z) &= 2n(k+1)\eta(Z)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y))\\ &\quad +2n(k+1)\eta(X)(g(Y,Z) - \eta(Y)\eta(Z) + g(h'Y,Z))\\ &\quad +2ng(h'Y + h'^2 Y, X)\eta(Z) + 2n\eta(X)g(h'Y + h'^2 Y, Z).\end{aligned} \tag{3.6}$$

Using (3.4)-(3.6), we infer that

$$\begin{aligned}&(\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\\ = &-4n(k+1)\eta(Z)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y))\\ &-4ng(h'X + h'^2 X, Y)\eta(Z),\end{aligned} \tag{3.7}$$

where the symmetry of h' is used. Now, using (2.6) and then (2.1) in (3.7) we complete the proof. □

Lemma 3.2. *In a* $(k,\mu)'$*-almost Kenmotsu manifold* M^{2n+1}, $(\pounds_X h')Y = 0$ *for any* $X,\ Y \in [\alpha]'$ *or* $X,\ Y \in [-\alpha]'$, *where* $\mathrm{Spec}(h') = \{0, \alpha, -\alpha\}$.

Proof. We consider a local orthonormal basis $\{\xi, e_i, \phi e_i\}$, $i = 1, 2, \ldots, n$ with $e_i \in [\alpha]'$ for M^{2n+1} and for any $X,\ Y \in [\alpha]'$, we have

$$\begin{aligned}\nabla_X Y &= \sum_i g(\nabla_X Y, e_i)e_i + g(\nabla_X Y, \xi)\xi \\ &= \sum_i g(\nabla_X Y, e_i)e_i - (1+\alpha)g(X,Y)\xi.\end{aligned}$$

For details of the above equation, see Proposition 4.1 of [4]. Now,

$$\begin{aligned}(\pounds_X h')Y &= \pounds_X h'Y - h'(\pounds_X Y) \\ &= \alpha \pounds_X Y - h'(\pounds_X Y) \\ &= \alpha(\nabla_X Y - \nabla_Y X) - h'(\nabla_X Y - \nabla_Y X) \\ &= \alpha(\nabla_X Y - \sum_i g(\nabla_X Y, e_i)e_i) - \alpha(\nabla_Y X - \sum_i g(\nabla_Y X, e_i)e_i) \\ &= -\alpha(1+\alpha)g(X,Y)\xi + \alpha(1+\alpha)g(X,Y)\xi \\ &= 0.\end{aligned}$$

Similarly, one can prove the same when $X,\ Y \in [-\alpha]'$. Hence, the proof is complete. □

Theorem 3.3. *A* $(k,\mu)'$*-almost Kenmotsu manifold* M^{2n+1} *with* $h' \neq 0$ *admitting conformal Ricci soliton* (g, V, λ) *is locally isometric to* $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ *or the conformal Ricci soliton* (i) *expanding,* (ii) *steady, or* (iii) *shrinking according to whether the non-dynamical scalar field* p *is*

(i) $p < -4nk - \frac{2}{2n+1}$,
(ii) $p = -4nk - \frac{2}{2n+1}$,
(iii) $p > \frac{8n^2+4n-2}{2n+1}$.

Proof. From (1.1) we have

$$(\pounds_V g)(X,Y) + 2S(X,Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X,Y). \tag{3.8}$$

Differentiating the above equation covariantly along any vector field Z we get

$$(\nabla_Z \pounds_V g)(X,Y) = -2(\nabla_Z S)(X,Y). \tag{3.9}$$

It is well known that ([19, p. 23])

$$(\pounds_V \nabla_X g - \pounds_X \nabla_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y).$$

Since g is parallel with respect to the Levi-Civita connection ∇, then the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{3.10}$$

Since $\pounds_V \nabla$ is symmetric, then it follows from (3.10) that

$$\begin{aligned}g((\pounds_V \nabla)(X,Y),Z) &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) \\ &\quad -\frac{1}{2}(\nabla_Z \pounds_V g)(X,Y).\end{aligned} \tag{3.11}$$

Using (3.9) in (3.11) we have

$$g((\pounds_V\nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{3.12}$$

Now using Lemma 3.1 in (3.12) we have

$$g((\pounds_V\nabla)(X,Y),Z) = -4n(k+2)g(h'X,Y)\eta(Z),$$

which implies

$$(\pounds_V\nabla)(X,Y) = -4n(k+2)g(h'X,Y)\xi. \tag{3.13}$$

Substituting $Y = \xi$ in (3.13) we get $(\pounds_V\nabla)(X,\xi) = 0$. From this we obtain $\nabla_Y(\pounds_V\nabla)(X,\xi) = 0$. This gives

$$(\nabla_Y\pounds_V\nabla)(X,\xi) + (\pounds_V\nabla)(\nabla_Y X,\xi) + (\pounds_V\nabla)(X,\nabla_Y\xi) = 0. \tag{3.14}$$

Using $(\pounds_V\nabla)(X,\xi) = 0$, (3.12), and (2.3) in (3.14) we infer that

$$(\nabla_Y\pounds_V\nabla)(X,\xi) = 4n(k+2)(g(h'X,Y) + g(h'^2X,Y))\xi. \tag{3.15}$$

Using the foregoing equation in the following formula ([19, p. 23])

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z),$$

we obtain

$$(\pounds_V R)(X,\xi)\xi = (\nabla_X\pounds_V\nabla)(\xi,\xi) - (\nabla_\xi\pounds_V\nabla)(X,\xi) = 0. \tag{3.16}$$

Now, substituting $Y = \xi$ in (3.8) and using (2.12) we have

$$(\pounds_V g)(X,\xi) = [2\lambda - (p + \frac{2}{2n+1}) - 4nk]\eta(X). \tag{3.17}$$

Lie-differentiating $g(X,\xi) = \eta(X)$ along V and using (3.17) we obtain

$$(\pounds_V\eta)X - g(X,\pounds_V\xi) - [2\lambda - (p + \frac{2}{2n+1}) - 4nk]\eta(X) = 0. \tag{3.18}$$

From (3.18), after putting $X = \xi$ we can easily obtain that

$$\eta(\pounds_V\xi) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - 2nk]. \tag{3.19}$$

From (2.10), we have

$$R(X,\xi)\xi = k(X - \eta(X)\xi) - 2h'X. \tag{3.20}$$

Now, using (3.18)-(3.20) and (2.10)-(2.11) we obtain

$$\begin{aligned} (\pounds_V R)(X,\xi)\xi &= \pounds_V R(X,\xi)\xi - R(\pounds_V X,\xi)\xi - R(X,\pounds_V\xi)\xi - R(X,\xi)\pounds_V\xi \\ &= k[2\lambda - (p + \frac{2}{2n+1}) - 4nk](X - \eta(X)\xi) - 2(\pounds_V h')X \\ &\quad -2[2\lambda - (p + \frac{2}{2n+1}) - 4nk]h'X - 2n\eta(X)h'(\pounds_V\xi) \\ &\quad -2g(h'X,\pounds_V\xi)\xi. \end{aligned} \tag{3.21}$$

Equating (3.16) and (3.21) and then taking an inner product with Y yields

$$\begin{aligned} & k[2\lambda - (p + \frac{2}{2n+1}) - 4nk](g(X,Y) - \eta(X)\eta(Y)) \\ & -2g((\pounds_V h')X,Y) - 2[2\lambda - (p + \frac{2}{2n+1}) - 4nk]g(h'X,Y) \\ & -2n\eta(X)g(h'(\pounds_V\xi),Y) - 2g(h'X,\pounds_V\xi)\eta(Y) = 0. \end{aligned}$$

Replacing X by ϕX in the above equation, we infer that

$$k[2\lambda - (p + \frac{2}{2n+1}) - 4nk]g(\phi X, Y) - 2g((\pounds_V h')\phi X, Y)$$
$$-2[2\lambda - (p + \frac{2}{2n+1}) - 4nk]g(h'\phi X, Y) = 0. \tag{3.22}$$

Letting $X \in [-\alpha]'$ and $V \in [\alpha]'$, then $\phi X \in [\alpha]'$. Then from (3.22), we have

$$(k - 2\alpha)[2\lambda - (p + \frac{2}{2n+1}) - 4nk]g(\phi X, Y) - 2g((\pounds_V h')\phi X, Y) = 0. \tag{3.23}$$

Since, $V,\ \phi X \in [\alpha]'$, using Lemma 3.2 we have $(\pounds_V h')\phi X = 0$. Therefore, equation (3.23) reduces to

$$(k - 2\alpha)[2\lambda - (p + \frac{2}{2n+1}) - 4nk]g(\phi X, Y) = 0,$$

which implies either $k = 2\alpha$ or $2\lambda = (p + \frac{2}{2n+1}) + 4nk$.

Case 1. If $k = 2\alpha$, then from $\alpha^2 = -(k+1)$ we get $\alpha = -1$ and hence $k = -2$. Then from Proposition 4.2 of [4], we have

$$R(X_\alpha, Y_\alpha)Z_\alpha = 0$$

and

$$R(X_{-\alpha}, Y_{-\alpha})Z_{-\alpha} = -4[g(Y_{-\alpha}, Z_{-\alpha})X_{-\alpha} - g(X_{-\alpha}, Z_{-\alpha})Y_{-\alpha}]$$

for any $X_\alpha, Y_\alpha, Z_\alpha \in [\alpha]'$ and $X_{-\alpha}, Y_{-\alpha}, Z_{-\alpha} \in [-\alpha]'$. Also noticing $\mu = -2$ it follows from Proposition 4.3 of [4] that $K(X, \xi) = -4$ for any $X \in [-\alpha]'$ and $K(X, \xi) = 0$ for any $X \in [\alpha]'$. Again from Proposition 4.3 of [4] we see that $K(X, Y) = -4$ for any $X, Y \in [-\alpha]'$ and $K(X, Y) = 0$ for any $X, Y \in [\alpha]'$. As is shown in [4] the distribution $[\xi] \oplus [\alpha]'$ is integrable with totally geodesic leaves and the distribution $[-\alpha]'$ is integrable with totally umbilical leaves by $H = -(1-\alpha)\xi$, where H is the mean curvature tensor field for the leaves of $[-\alpha]'$ immersed in M^{2n+1}. Here $\alpha = -1$; then the two orthogonal distributions $[\xi] \oplus [\alpha]'$ and $[-\alpha]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Case 2. Let $2\lambda = (p + \frac{2}{2n+1}) + 4nk$. Now, the conformal Ricci soliton is expanding, steady, or shrinking according to whether $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively. Therefore, the conformal Ricci soliton is expanding when $p < -4nk - \frac{2}{2n+1}$, steady when $p = -4nk - \frac{2}{2n+1}$,and shrinking when $p > \frac{8n^2+4n-2}{2n+1}$, where the fact $k \leq -1$ is used in the case of shrinking. This completes the proof. □

Theorem 3.4. *If (g, ξ, λ) is a conformal Ricci soliton in a generalized $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1}, then M^{2n+1} is η-Einstein and $\lambda = \frac{p}{2} + \frac{1}{2n+1} + 2nk$.*

Proof. Since (g, ξ, λ) is a conformal Ricci soliton in M^{2n+1}, we have from (1.1)

$$(\pounds_\xi g)(X, Y) + 2S(X, Y) = [2\lambda - (p + \frac{2}{2n+1})]g(X, Y). \tag{3.24}$$

Now, using (2.3) we obtain

$$\begin{aligned} (\pounds_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\ &= 2g(X, Y) - 2\eta(X)\eta(Y) - 2g(\phi h X, Y). \end{aligned} \tag{3.25}$$

Substituting (3.25) in (3.24) we get

$$\begin{aligned} & 2g(X,Y)-2\eta(X)\eta(Y)-2g(\phi hX,Y)+2S(X,Y) \\ (3.26) \qquad = \; & [2\lambda-(p+\frac{2}{2n+1})]g(X,Y). \end{aligned}$$

From (2.9), we get

$$g(\phi hX,Y)=\frac{1}{2n}S(X,Y)+g(X,Y)-(k+1)\eta(X)\eta(Y). \tag{3.27}$$

Now, substituting (3.27) in (3.26) we get

$$S(X,Y)=\frac{n[2\lambda-(p+\frac{2}{2n+1})]}{2n-1}g(X,Y)-\frac{2nk}{2n-1}\eta(X)\eta(Y),$$

which shows that the manifold is η-Einstein.

Putting $X=Y=\xi$ in the foregoing equation, we obtain

$$2nk=\frac{n[2\lambda-(p+\frac{2}{2n+1})]}{2n-1}-\frac{2nk}{2n-1}.$$

From above, it follows that $\lambda=\frac{p}{2}+\frac{1}{2n+1}+2nk$. □

4. Conformal gradient Ricci soliton

In this section we consider a conformal gradient Ricci soliton in the framework of $(k,\mu)'$-almost Kenmotsu manifolds. If V is any vector field collinear with ξ, then there is a smooth function b on M such that $V=b\xi$. In this case, $h'V=0$. Here we prove the following theorem.

Theorem 4.1. *A $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} does not admit conformal gradient Ricci soliton (g,V,λ) with V collinear with the characteristic vector field ξ.*

The proof of the above theorem relies on the following lemma.

Lemma 4.2. *In a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} admitting conformal gradient Ricci soliton (g,V,λ), the following relation holds:*

$$R(X,Y)Df=2n(k+2)(\eta(X)h'Y-\eta(Y)h'X), \tag{4.1}$$

where $f: M^{2n+1}\to\mathbb{R}$ is a smooth function such that $V=Df$, D is the gradient operator.

Proof. From (1.2) we can write

$$\nabla_X Df=[\lambda-(\frac{p}{2}+\frac{1}{2n+1})]X-QX. \tag{4.2}$$

Taking the covariant derivative of the above equation along Y we get

$$\nabla_Y\nabla_X Df=[\lambda-(\frac{p}{2}+\frac{1}{2n+1})]\nabla_Y X-\nabla_Y QX. \tag{4.3}$$

Interchanging X and Y in (4.3) we get

$$\nabla_X\nabla_Y Df=[\lambda-(\frac{p}{2}+\frac{1}{2n+1})]\nabla_X Y-\nabla_X QY. \tag{4.4}$$

Again, from (4.2) we obtain

$$\nabla_{[X,Y]}Df=[\lambda-(\frac{p}{2}+\frac{1}{2n+1})](\nabla_X Y-\nabla_Y X)-Q(\nabla_X Y-\nabla_Y X). \tag{4.5}$$

Using (4.3)-(4.5) in the following:

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df$$

we obtain

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{4.6}$$

Now, using (2.8), (2.9), and (2.13) we obtain

$$\begin{aligned} (\nabla_Y Q)X &= \nabla_Y QX - Q(\nabla_Y X) \\ &= 2n(k+1)(g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y))\xi \\ &\quad +2n(k+1)\eta(X)(Y - \eta(Y)\xi - \phi hY) + 2ng(h'Y + h'^2Y, X)\xi \\ &\quad +2n\eta(X)(h'Y + h'^2Y). \end{aligned} \tag{4.7}$$

Interchanging X and Y in the above equation we obtain $(\nabla_X Q)Y$. Now, substituting $(\nabla_Y Q)X$ and $(\nabla_X Q)Y$ in (4.6) and using (2.6) we complete the proof. □

Proof of Theorem 4.1. Putting $X = \xi$ in (4.1) we have

$$R(\xi, Y)Df = 2n(k+2)h'Y,$$

which implies

$$g(R(\xi,Y)Df, X) = 2n(k+2)g(h'Y, X). \tag{4.8}$$

Again, using (2.11) we have

$$\begin{aligned} g(R(\xi,Y)Df, X) &= -g(R(\xi,Y)X, Df) \\ &= -kg(X,Y)(\xi f) + k\eta(X)(Yf) \\ &\quad +2g(h'X,Y)(\xi f) - 2\eta(X)((h'Y)f). \end{aligned} \tag{4.9}$$

From (4.8) and (4.9) we get

$$\begin{aligned} & -kg(X,Y)(\xi f) + k\eta(X)(Yf) + 2g(h'X,Y)(\xi f) - 2\eta(X)((h'Y)f) \\ = \ & 2n(k+2)g(h'Y, X). \end{aligned}$$

Antisymmetrizing the foregoing equation we obtain

$$k\eta(X)(Yf) - k\eta(Y)(Xf) - 2\eta(X)((h'Y)f) + 2\eta(Y)((h'X)f) = 0. \tag{4.10}$$

Now, $(h'X)f = g(h'X, Df) = g(X, h'(Df)) = 0$ for any vector field X as $h'V = h'(Df) = 0$ by hypothesis. Hence, from (4.10) we get

$$\eta(X)(Yf) - \eta(Y)(Xf) = 0,$$

as $k \leq -1$. Putting $X = \xi$ in the above equation we obtain

$$Df = (\xi f)\xi. \tag{4.11}$$

Differentiating (4.11) covariantly along X, we obtain

$$\nabla_X Df = (X(\xi f))\xi + (\xi f)(X - \eta(X)\xi - \phi hX). \tag{4.12}$$

Equating (4.2) and (4.12) we obtain

$$QX = ([\lambda - (\frac{p}{2} + \frac{1}{2n+1})] + (\xi f))X + ((\xi f)\eta(X) - X(\xi f))\xi + (\xi f)\phi hX. \tag{4.13}$$

Comparing (2.9) and (4.13) we have the following:

$$[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] + (\xi f) = -2n, \tag{4.14}$$

$$(\xi f)\eta(X) - X(\xi f) = 2n(k+1)\eta(X), \tag{4.15}$$

$$(\xi f) = 2. \tag{4.16}$$

Using (4.16) in (4.14) we get $2\lambda - (p + \frac{2}{2n+1}) = -4n - 4$ which implies $\lambda = \frac{p}{2} + \frac{1}{2n+1} - 2n - 2$. Again using (4.16) in (4.15) we get $2\eta(X) = 2n(k+1)\eta(X)$ for any vector field X which implies $k = -1 + \frac{1}{n} > -1$ which is a contradiction as $k \leq -1$. Hence, a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} does not admit conformal gradient Ricci soliton (g, V, λ) such that the potential vector field V is collinear with the characteristic vector field ξ.

5. Example of a 5-dimensional almost Kenmotsu manifold

We consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Let ξ, e_2, e_3, e_4, e_5 be five vector fields in $\mathbb{R}^5$ which satisfies [4]

$$[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [\xi, e_4] = 0,\ [\xi, e_5] = 0,$$

$[e_i, e_j] = 0$, where $i, j = 2, 3, 4, 5$.

Let g be the Riemannian metric defined by

$$g(\xi, \xi) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1$$

and $g(\xi, e_i) = g(e_i, e_j) = 0$ for $i \neq j$; $i, j = 2, 3, 4, 5$.

Let η be the 1-form defined by $\eta(Z) = g(Z, \xi)$, for any $Z \in T(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0,\ \phi(e_2) = e_4,\ \phi(e_3) = e_5,\ \phi(e_4) = -e_2,\ \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g, we have

$$\eta(\xi) = 1,\ \phi^2(Z) = -Z + \eta(Z)\xi,\ g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$$

for any $Z, U \in T(M)$.

Moreover, $h'\xi = 0,\ h'e_2 = e_2,\ h'e_3 = e_3,\ h'e_4 = -e_4,\ h'e_5 = -e_5$.

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\nabla_\xi \xi = 0,\ \nabla_\xi e_2 = 0,\ \nabla_\xi e_3 = 0,\ \nabla_\xi e_4 = 0,\ \nabla_\xi e_5 = \xi,$$

$$\nabla_{e_2} \xi = 2e_2,\ \nabla_{e_2} e_2 = -2\xi,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,$$

$$\nabla_{e_3} \xi = 2e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -2\xi,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,$$

$$\nabla_{e_4} \xi = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = 0,\ \nabla_{e_4} e_5 = 0,$$

$$\nabla_{e_5} \xi = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h'X$$

for any $X \in T(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$R(\xi, e_2)\xi = 4e_2,\ R(\xi, e_2)e_2 = -4\xi,\ R(\xi, e_3)\xi = 4e_3,\ R(\xi, e_3)e_3 = -4\xi,$$

$$R(\xi, e_4)\xi = R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0,$$

$$R(e_2, e_3)e_2 = 4e_3,\ R(e_2, e_3)e_3 = -4e_2,\ R(e_2, e_4)e_2 = R(e_2, e_4)e_4 = 0,$$

$$R(e_2, e_5)e_2 = R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0,$$

$$R(e_3, e_5)e_3 = R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0.$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution with $k = -2$ and $\mu = -2$. Therefore, from $\alpha^2 = -(k + 1)$, we get $\alpha = \pm 1$. Without lose of generality we consider $\alpha = -1$. Then by the same argument as in Theorem 3.3 we can say that the manifold is locally isometric to $\mathbb{H}^3(-4) \times \mathbb{R}^2$.

Using the expressions of the curvature tensor R we have

$$R(X, Y)Z = -4[g(Y, Z)X - g(X, Z)Y].$$

From the above equation we obtain

$$S(Y, Z) = -16g(Y, Z), \text{ which implies } r = -80.$$

Now, it is easy to see that

$$(\pounds_\xi g)(\xi, \xi) = (\pounds_\xi g)(e_4, e_4) = (\pounds_\xi g)(e_5, e_5) = 0,$$

$$(\pounds_\xi g)(e_2, e_2) = (\pounds_\xi g)(e_3, e_3) = 4.$$

Consider $V = \xi$ and then tracing (1.1) we obtain $\lambda = \frac{p}{2} + \frac{1}{5} + \frac{76}{5}$. Hence, (g, ξ, λ) is a conformal Ricci soliton on M. Thus Theorem 3.3 is verified.

Acknowledgment

The authors would like to thank the anonymous referee for his or her careful reading and valuable suggestions that have improved the paper.

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