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ALMOST LINEARITY OF ϵ-BI-LIPSCHITZ MAPS BETWEEN REAL BANACH SPACES

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ABSTRACT. Let X and Y be real Banach spaces. A map f between X and Y is called an ϵ-bi-Lipschitz map if $(1-\epsilon)\|x-y\| \le \|f(x)-f(y)\| \le (1+\epsilon)\|x-y\|$ for all $x, y \in X$. In this note we show that if f is an ϵ-bi-Lipschitz map with $f(0)=0$ from X onto Y, then f is almost linear. We also show that if $f : X \longrightarrow Y$ is a surjective ϵ-bi-Lipschitz map with $f(0)=0$, then there exists a linear isomorphism $I : X \to Y$ such that

$$\|I(x)-f(x)\| \le E(\epsilon,\alpha)(\|x\|^{\alpha}+\|x\|^{2-\alpha})$$

where $E(\epsilon,\alpha) \to 0$ as $\epsilon \to 0$ and $0<\alpha<1$.

1. INTRODUCTION

It is a well-known classical result of Mazur and Ulam [4] that an isometry f from a real Banach space X onto a real Banach space Y with $f(0)=0$ is automatically linear. A map f between Banach spaces X and Y is called an (m, M)-rigid map if

$$m\|x-y\| \le \|f(x)-f(y)\| \le M\|x-y\|$$

for all $x, y \in X$. We denote a $(1-\epsilon, 1+\epsilon)$-rigid map by ϵ-bi-Lipschitz map. The following theorem follows from [1, Proposition 2].

Theorem. *Let f be an ϵ-bi-Lipschitz map from a real Banach space X onto a real Banach space Y, with $f(0)=0$ and $0 \le \epsilon < \frac{1}{3}$. Then*

$$\|f(x+y)-f(x)-f(y)\| \le c_1(3\epsilon)^{c_2}(\|x\|+\|y\|), \quad \text{for } x, y \in X$$

where

$$c_1 = (8/3)((\alpha/8)(\alpha^2+4\alpha-1)/(\alpha-1)\,+\,1)$$

and

$$c_2 = (\log(2/\alpha))(\log((\alpha+7)/(\alpha-1))+\log(2/\alpha))^{-1}$$

for $1<\alpha<2$.

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Note that c_2 attains its maximum for $\alpha \in (1,2)$ at $\alpha = 1.1572\ldots$ and for this value of α, c_1 and c_2 have values $5.5704\ldots$ and $0.1216\ldots$, respectively.

Jarosz [2] conjectured that if f is an ϵ-bi-Lipschitz map from a real Banach space X onto a real Banach space Y, then X and Y are linearly isomorphic for sufficiently small ϵ. Throughout, R represents the real number field. There is an ϵ-bi-Lipschitz map f on R which is not a linear map on R.

Example. We define an ϵ-bi-Lipschitz map $f : R \to R$ such that

$$f(x) = \begin{cases} (1+\epsilon)x, & \text{if } x \leq 2, \\ (1-\epsilon)(x-2) + 2(1+\epsilon), & \text{if } x > 2. \end{cases}$$

Throughout this paper, X and Y shall denote real Banach spaces. The r-neighborhood of a set A is denoted by $B(A,r)$ and we abbreviate $B(\{x\},r)$ and $B(\{0\},r)$ by $B(x,r)$ and $B(r)$, respectively. The corresponding closed balls are denoted by $\overline{B}(x,r)$ and $\overline{B}(r)$. As usual, $[x,y]$ denotes the closed segment determined by x and y. A bound subset A of X is said to be symmetric with respect to a point a if x in A implies that $2a - x$ is also in A. If A is bounded and symmetric with respect to a, we define $radA$ to be $\inf\{r > 0 : A \subset B(a,r)\}$. We denote the Blaschke distance function on sets by D; that is, for $A_1, A_2 \subset X$, $D(A_1, A_2) = \inf\{r > 0 : A_1 \subset B(A_2, r) \text{ and } A_2 \subset B(A_1, r)\}$. For $A \subset X$ and $\alpha > 0$ we define $T(A,\alpha) = \{x \in X : A \subset \overline{B}(x,\alpha)\} = \bigcap\{\overline{B}(y,\alpha) : y \in A\}$. For $x, y \in X$ we define $S(x,y,\alpha) = T([x,y],\alpha) = T(\{x,y\},\alpha) = \overline{B}(x,\alpha) \cap \overline{B}(y,\alpha)$. We also define $C(A,\alpha) = A \cap T(A,\alpha)$. The following lemmas are due to Gevirtz [1] and John [3].

Lemma 1 [1, Lemma 2]. *Let $x, y \in X$ and let $\beta, \gamma > |x-y|/2$. Then*

$$D(S(x,y,\beta), S(x,y,\gamma)) \leq \frac{|\beta - \gamma|}{1 - |x-y|/2\beta}.$$

Lemma 2 [1, Lemma 3]. *Let $\delta > 0$. For every bounded convex symmetric subset $A \neq \varnothing$ of X with center a and all β, γ for which $B(a,\delta) \subset C(A,\beta)$ and $\gamma > radA$ there holds*

$$D(C(A,\beta), C(A,\gamma)) \leq (1 + \frac{rad\, A}{\delta})|\beta - \gamma|.$$

Lemma 3 [1, Lemma 4]. *Let $\delta > 0$. For every bounded convex symmetric subset $A \neq \varnothing$ of X with center a and $F \subset X$ and β for which $C(F,\beta) \neq \varnothing$ and $B(a,\delta) \subset C(A,\beta)$ there holds*

$$D(C(A,\beta), C(F,\beta)) \leq (1 + \frac{4\, rad\, A}{\delta}) D(A,F).$$

Lemma 4 [1, Lemma 5]. *Let $A \subset U \subset X$ and let $f : U \longrightarrow Y$ be (m,M)-rigid. Then for all $\alpha > 0$*

$$f(U \cap T(A,\alpha)) \subset f(U) \cap T(f(A), M\alpha) \subset f(U \cap T(A, \frac{M}{m}\alpha)).$$

Lemma 5 [3, Theorem II]. *Let $f : X \to Y$ be (m,M)-rigid. Then $f(B(a,r)) \supset B(f(a), mr)$.*

2. The results

Definition 6. A map f from a real Banach space X into a real Banach space Y is an almost linear map if it satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq D(\epsilon)(\|x\| + \|y\|), \text{ for } x, y \in X$$

and

$$\|f(\lambda x) - \lambda f(x)\| \leq D(\epsilon)E(\lambda)\|x\|, \text{ for } \lambda \in R, x \in X$$

where $D(\epsilon) \to 0$ as $\epsilon \to 0$ and $E(\lambda) \to 0$ as $\lambda \to 0$.

Let $x \neq y$ be points in X and let $d = \|x - y\|$. For $\alpha > 1$ we define $S_n = S_n(x, y, d, \alpha)$ recursively as follows : $S_1 = S(x, y, \frac{\alpha d}{2})$ and, for $n \geq 1$, $S_{n+1} = C(S_n, \frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)d)$.

Lemma 7. *Let $x \neq y$ be points in X, $a = \frac{x+y}{2}$ and $\alpha > 1$. Then for $n \geq 1, S_n$ is convex and symmetric with respect to a and*

$$B(a, \frac{\alpha - 1}{2}d) \subset S_n \subset B(a, \frac{\alpha d}{2^n} + \sum_{i=2}^{n} \frac{1}{2^i}(\alpha - 1)d).$$

Proof. It is easy to verify that the assertion is true for $n = 1$. Assume inductively that it is true for a given $n \geq 1$. The inductive hypothesis implies that S_{n+1} is convex and symmetric with respect to a. If $z \in B(a, \frac{\alpha-1}{2}d)$ and $u \in S_n$, then

$$\begin{aligned}\|z - u\| &\leq \|z - a\| + \|a - u\| \\ &< \frac{\alpha - 1}{2}d + \frac{\alpha}{2^n}d + \sum_{i=2}^{n} \frac{1}{2^i}(\alpha - 1)d \\ &= \frac{\alpha}{2^n}d + \sum_{i=1}^{n} \frac{1}{2^i}(\alpha - 1)d.\end{aligned}$$

This implies that $z \in S_{n+1}$. If $u \in S_{n+1}$, then $\|u - z\| \leq \frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha - 1)d$ for all $z \in S_n$. Since S_{n+1} is symmetric with respect to a, $2a - u \in S_{n+1} \subset S_n$. Putting $2a - u$ instead of z in the above formula, we have $\|u - a\| \leq \frac{\alpha d}{2^{n+1}} + \sum_{i=2}^{n+1} \frac{1}{2^i}(\alpha - 1)d$. That is, $rad\, S_{n+1} \leq \frac{\alpha d}{2^{n+1}} + \sum_{i=2}^{n+1} \frac{1}{2^i}(\alpha - 1)d$.

Proposition 8. *Let $x, y \in X$, $\|x - y\| = d, \alpha > 1$ and $0 < m \leq M$. Let $f : X \longrightarrow Y$ be an (m, M)-rigid map. Writing S_n, S'_n and μ for $S_n(x, y, d, \alpha)$, $S_n(f(x), f(y), Md, \alpha)$ and $\frac{M}{m}$, respectively, there holds*

$$D(f(S_n), S'_n) \leq K(m, M, \alpha)\|x - y\|$$

where $K(m, M, \alpha) = M(\mu - 1)(\frac{7\alpha-1}{\alpha-1})^{n-1}(\frac{31\alpha^3-18\alpha^2+10\alpha-2}{(\alpha-1)(6\alpha-4)})$.

Proof. Replacing U, A by $X, \{x, y\}$, respectively, in Lemma 4, we obtain

$$f(S_1) \subset S'_1 \subset f(S(x, y, \alpha\mu d/2)). \tag{1}$$

Since $S_{n+1} = C(S_n, \frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)d)$, Lemma 4 with $A = U = S_n$ implies that

$$f(S_{n+1}) \subset C(f(S_n), Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)]) \tag{2}$$
$$\subset f(C(S_n, \mu d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)])).$$

Formula (1) gives

$$D(f(S_1), S_1') \leq D(f(S(x,y,\frac{\alpha d}{2})), f(S(x,y,\frac{\alpha\mu d}{2})))$$
$$\leq MD(S(x,y,\frac{\alpha d}{2}), S(x,y,\frac{\alpha\mu d}{2})).$$

Lemma 1 with $\beta = \frac{\alpha d}{2}$ and $\gamma = \frac{\alpha\mu d}{2}$ implies that we have $D(f(S_1), S_1') \leq M\alpha^2 d\frac{(\mu-1)}{2(\alpha-1)}$. Put $D_n = D(f(S_n), S_n')$. Let $n \geq 1$ and

$$W = C(f(S_n), Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)]).$$

Then (2) implies $W \neq \varnothing$, and so

$$D_{n+1} \leq D(f(S_{n+1}), W) + D(S_{n+1}', W). \tag{3}$$

Formula (2) gives

$$D(f(S_{n+1}), W) \leq D(f(S_{n+1}), f(C(S_n, \mu d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)])))$$
$$= D(f(C(S_n, d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)])),$$
$$f(C(S_n, \mu d[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)]))).$$

By Lemma 2 and Lemma 7,

$$D(f(S_{n+1}), W) \leq d(\mu-1)M\left(1 + \frac{\alpha d/2^n + \sum_{i=2}^n(\alpha-1)d/2^i}{(\alpha-1)d/2}\right) \tag{4}$$
$$\times [\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)]$$
$$\leq d(\mu-1)M\left(1 + \frac{\alpha}{\alpha-1}\right)\left(\frac{\alpha}{2} + \frac{\alpha-1}{2}\right)$$
$$\leq d(\mu-1)M\frac{(2\alpha-1)^2}{2(\alpha-1)}.$$

Similarly, we may apply Lemma 3 with $a = \frac{f(x)+f(y)}{2}$, $A = S'_n$, $F = f(S_n)$, $\beta = M[\frac{\alpha d}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)d]$ and $\delta = M\frac{\alpha-1}{2}d$.

$$\begin{aligned} D(S'_{n+1}, W) &= D(C(S'_n, Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)]), \\ &\qquad C(f(S_n), Md[\frac{\alpha}{2^n} + \sum_{i=1}^n \frac{1}{2^i}(\alpha-1)])) \\ &\le \left(1 + \frac{4(\alpha d/2^n + \sum_{i=2}^n (\alpha-1)d/2^i)}{(\alpha-1)d/2}\right) D_n \\ &\le \left(\frac{5\alpha-1}{\alpha-1}\right) D_n. \end{aligned} \tag{5}$$

From (4) and (5) we get

$$D_{n+1} \le d(\mu-1)M\frac{(2\alpha-1)^2}{2(\alpha-1)} + \left(\frac{5\alpha-1}{\alpha-1}\right) D_n.$$

Let $G = d(\mu-1)M\frac{(2\alpha-1)^2}{2(\alpha-1)}$, $H = \frac{5\alpha-1}{\alpha-1}$. By induction we then get

$$\begin{aligned} D_n &\le G(1 + H + \cdots + H^{n-2}) + H^{n-1}D_1 \\ &= G\left(\frac{H^{n-1}-1}{H-1}\right) + H^{n-1}D_1 \\ &\le (\mu-1)M\left(\frac{5\alpha-1}{\alpha-1}\right)^{n-1}\left(\frac{8\alpha^3-8\alpha^2+5\alpha-1}{8\alpha(\alpha-1)}\right)\|x-y\|. \end{aligned}$$

Proposition 9. *Let $x, y \in X$, $0 < m \le M$, $\mu = \frac{M}{m} < 2$. Let $f : X \longrightarrow Y$ be (m, M)-rigid. Then*

$$\|\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\| \le c_1(\mu-1)^{c_2}M\|x-y\|$$

where $c_1 = \frac{3\alpha-1}{2} + \frac{(9\alpha-5)(17\alpha^3-25\alpha^2+16\alpha-4)}{8(2\alpha-1)(\alpha-1)^2}$ and $c_2 = \frac{\log 2}{\log\frac{2(9\alpha-5)}{\alpha-1}}$ for $\alpha > 1$.

Proof. Put $a = \frac{x+y}{2}$ and $p = \frac{f(x)+f(y)}{2}$. By Lemma 5 and Lemma 7, we obtain $B(f(a), m\frac{\alpha-1}{2}d) \subset f(S_n)$ and $S'_n \subset B(p, M\frac{\alpha d}{2^n} + M\frac{\alpha-1}{2}d)$. Since $D_n = D(f(S_n), S'_n)$, $f(S_n) \subset B(S'_n, D_n)$. Thus we have

$$\begin{aligned} B(f(a), m\frac{\alpha-1}{2}d) &\subset B(B(p, M\frac{\alpha d}{2^n} + M\frac{\alpha-1}{2}d), D_n) \\ &= B(p, M\frac{\alpha d}{2^n} + M\frac{\alpha-1}{2}d + D_n). \end{aligned}$$

Hence Proposition 8 implies

$$\begin{aligned} \|f(a) - p\| \le &(M-m)\frac{\alpha-1}{2}d + M\frac{\alpha d}{2^n} \\ &+ d(\mu-1)M\left(\frac{5\alpha-1}{\alpha-1}\right)^{n-1}\frac{8\alpha^3-8\alpha^2+5\alpha-1}{8\alpha(\alpha-1)}. \end{aligned} \tag{6}$$

Let $E = \frac{1}{2}$ and $F = \frac{5\alpha-1}{\alpha-1}$. For given α, μ we use an integer n which is chosen in such a way that the last expression takes the form $c_1(\mu-1)^{c_2}M\|x-y\|$, apart from negligible differences. Explicitly, we write n in the form

$$n = -\beta(\log(\mu-1))/\log F + \xi + 1$$

where $0 < \beta < 1$ and $0 \leq \xi < 1$. Since $\mu < 2$, we have that $n \geq 1$, $(\mu-1)F^{n-1} = (\mu-1)^{1-\beta}F^{\xi} \leq (\mu-1)^{1-\beta}F$ and $E^n = (\mu-1)^{-\beta\log\frac{1}{2}/\log F}E^{\xi} \leq (\mu-1)^{\beta\log 2/\log F}$. If β is determined so that $1-\beta = \beta\log 2/\log F$, then $\beta = \frac{\log F}{\log 2+\log F}$. Put $c_2 = 1-\beta$. Then

$$\begin{aligned} dM[\alpha\left(\frac{1}{2}\right)^n &+ (\mu-1)\left(\frac{5\alpha-1}{\alpha-1}\right)^{n-1}\frac{8\alpha^3-8\alpha^2+5\alpha-1}{8\alpha(\alpha-1)}] \\ &\leq dM(\mu-1)^{c_2}\left(\alpha + \frac{(5\alpha-1)(8\alpha^3-8\alpha^2+5\alpha-1)}{8\alpha(\alpha-1)^2}\right). \end{aligned}$$

Since $0 < c_2 < 1$ and $\mu \leq 2$

$$(M-m)\frac{\alpha-1}{2}d \leq M(\mu-1)^{c_2}\frac{\alpha-1}{2}d.$$

Thus

$$\|f(a)-p\| \leq M(\mu-1)^{c_2}\left(\frac{3\alpha-1}{2} + \frac{(5\alpha-1)(8\alpha^3-8\alpha^2+5\alpha-1)}{8\alpha(\alpha-1)^2}\right)\|x-y\|.$$

Remark. It is easy to show that c_2 converges to $\log 2$ as α converges to infinity. If we choose $\alpha = 10$, then $c_1 = 6.9314\ldots$ and $c_2 = 0.2902\ldots$. We denote these values by K_1 and K_2, respectively. The later value is greater than $0.1216\ldots$ which is the maximum value of c_2 in [1, Proposition 2]. If $\mu \geq 2$, there exists $\delta > 0$ such that $\|\frac{1}{2}(f(x)+f(y)) - f(\frac{x+y}{2})\| \leq \delta\|x-y\|$, for $x, y \in X$.

Lemma 10. *Let $f : X \longrightarrow Y$ be a continuous map such that $\|\frac{1}{2}(f(x)+f(y)) - f(\frac{x+y}{2})\| \leq K\|x-y\|$, for some $K > 0$ and for all $x, y \in X$, with $f(0) = 0$. Then*

$$\|f(\lambda x) - \lambda f(x)\| \leq 2K\|x\|, \qquad 0 \leq \lambda \leq 1, x \in X.$$

Proof. Let $Q_n = \{\frac{p}{2^n} | p = 1, 2, \ldots, 2^n\}$ and $Q = \bigcup_{n=1}^{\infty} Q_n$. Then Q is dense in $\{\lambda | \, 0 \leq \lambda \leq 1\}$. Since f is continuous, it is sufficient to show that $\|\lambda f(x) - f(\lambda x)\| \leq \sum_{i=0}^{n-1}\frac{1}{2^i}K\|x\|$ for all $x \in X, \lambda \in Q_n$. We prove the above formula by induction. It is clearly true for $n = 1$, and we assume it is true if we have Q_{n-1}, for $n \geq 2$. We take $\lambda \in Q_n$, $\lambda = \frac{p}{2^n}$. If p is divided by 2, then $\lambda \in Q_{n-1}$. Otherwise there exists an integer r such that $p = 2r+1$ and so $r \leq 2^{n-1}-1$. Thus $\frac{r}{2^{n-1}}, \frac{r+1}{2^{n-1}} \in Q_{n-1}$. Hence we obtain

$$\begin{aligned} \|f(\frac{p}{2^n}x) - \frac{p}{2^n}f(x)\| &\leq \frac{K}{2^{n-1}}\|x\| + \frac{1}{2}\|f(\frac{r}{2^{n-1}}x) - \frac{r}{2^{n-1}}f(x) \\ &\qquad + f(\frac{r+1}{2^{n-1}}x) - \frac{r+1}{2^{n-1}}f(x)\| \\ &\leq \frac{K}{2^{n-1}}\|x\| + K\sum_{i=0}^{n-2}\frac{1}{2^i}\|x\| \\ &= K\sum_{i=0}^{n-1}\frac{1}{2^i}\|x\|. \end{aligned}$$

That is, $\|f(\lambda x) - \lambda f(x)\| \leq 2K\|x\|$ for $0 \leq \lambda \leq 1, x \in X$.

Theorem 11. *Let $f : X \longrightarrow Y$ be an ϵ-bi-Lipschitz map with $f(0) = 0$, and with $\epsilon < \frac{1}{3}$. Then f is an almost linear map.*

Proof. Let $x, y \in X$. By Proposition 9 and the above Remark

$$\|\frac{f(x)+f(y)}{2} - f(\frac{x+y}{2})\| \le \frac{4}{3}K_1(3\epsilon)^{K_2}\|x-y\|. \tag{7}$$

We put $C(\epsilon) = \frac{4}{3}K_1(3\epsilon)^{K_2}$. Then

$$\begin{aligned}\|\frac{1}{2}(f(x)+f(y)) - \frac{1}{2}f(x+y)\| &\le \|\frac{1}{2}(f(x)+f(y)) - f(\frac{x+y}{2})\| \\ &\quad + \|f(\frac{x+y}{2}) - \frac{1}{2}f(x+y)\| \\ &\le 2C(\epsilon)(\|x\|+\|y\|).\end{aligned}$$

That is,

$$\|f(x)+f(y)-f(x+y)\| \le 4C(\epsilon)(\|x\|+\|y\|). \tag{8}$$

(7) and Lemma 10 imply

$$\|\lambda f(x) - f(\lambda x)\| \le 2C(\epsilon)\|x\|, \text{ for } 0 \le \lambda \le 1,\ x \in X. \tag{9}$$

For each $x \in X$, we define $g_x(\lambda) = \|\lambda f(x) - f(\lambda x)\|$ on R. Then for $\lambda_1, \lambda_2 \in R$

$$\begin{aligned}|g_x(\lambda_1) - g_x(\lambda_2)| &= |\|\lambda_1 f(x) - f(\lambda_1 x)\| - \|\lambda_2 f(x) - f(\lambda_2 x)\|| \\ &\le \|\lambda_1 f(x) - \lambda_2 f(x)\| + \|f(\lambda_1 x) - f(\lambda_2 x)\| \\ &\le 2|\lambda_1 - \lambda_2|(1+\epsilon)\|x\|.\end{aligned}$$

Putting $\lambda_2 = 0$, we obtain

$$\|\lambda f(x) - f(\lambda x)\| \le 2|\lambda|(1+\epsilon)\|x\|. \tag{10}$$

We define $g_1(\lambda) = 2\lambda(1+\epsilon)\|x\|$ and $g_2(\lambda) = 2C(\epsilon)\|x\|$, for $\lambda \ge 0$. Then g_1 and g_2 have the common value at $\lambda = \frac{C(\epsilon)}{1+\epsilon}$. By simple calculation for all $0 < \alpha < 1$,

$$\min\{g_1(\lambda), g_2(\lambda)\} \le \frac{8}{3}C(\epsilon)^{1-\alpha}\lambda^{\alpha}\|x\|.$$

Then (9) and (10) imply

$$\|\lambda f(x) - f(\lambda x)\| \le \frac{8}{3}C(\epsilon)^{1-\alpha}\lambda^{\alpha}\|x\|, \text{ for } 0 \le \lambda \le 1.$$

We replace x by $\frac{1}{\lambda}x$ and multiply $\frac{1}{\lambda}$ $(0 < \lambda \le 1)$ in the above formula and we get

$$\|\frac{1}{\lambda}f(x) - f(\frac{1}{\lambda}x)\| \le \frac{8}{3}C(\epsilon)^{1-\alpha}(\frac{1}{\lambda})^{2-\alpha}\|x\|.$$

Thus

$$\|\lambda f(x) - f(\lambda x)\| \leq \frac{8}{3} C(\epsilon)^{1-\alpha}(\lambda^{\alpha} + \lambda^{2-\alpha})\|x\| \quad \text{for } \lambda \geq 0.$$

Using (8),

$$\|f(x) + f(-x)\| \leq 4C(\epsilon)\|x\|.$$

For $\lambda < 0$, put $-\lambda = \beta$. Then

$$\begin{aligned}\|\lambda f(x) - f(\lambda x)\| &= \|f(-\beta x) + \beta f(x)\| \\ &\leq 4C(\epsilon)\beta\|x\| + \frac{8}{3} C(\epsilon)^{1-\alpha}(\beta^{\alpha} + \beta^{2-\alpha})\|x\| \\ &\leq \frac{40}{3}(C(\epsilon) + C(\epsilon)^{1-\alpha})(|\lambda|^{\alpha} + |\lambda|^{2-\alpha})\|x\|.\end{aligned}$$

That is,

$$\|f(\lambda x) - \lambda f(x)\| \leq D(\epsilon, \alpha)(|\lambda|^{\alpha} + |\lambda|^{2-\alpha})\|x\|$$

where $D(\epsilon, \alpha) = \frac{40}{3}(C(\epsilon) + C(\epsilon)^{1-\alpha})$, $0 < \alpha < 1$. This completes the proof of the theorem.

Let f be an ϵ-bi-Lipschitz map from a finite dimensional Banach space X onto a finite dimensional real Banach space Y with $f(0) = 0$. Then there exists a linear map I near f.

Theorem 12. *Let X and Y be finite dimensional real Banach spaces. If $f : X \longrightarrow Y$ is a surjective ϵ-bi-Lipschitz map with $f(0) = 0$ and $\epsilon < \frac{1}{3}$, then there exists a linear isomorphism $I : X \longrightarrow Y$ such that*

$$\|I(x) - f(x)\| \leq E(\epsilon, \alpha)(\|x\|^{\alpha} + \|x\|^{2-\alpha})$$

where $E(\epsilon, \alpha) \to 0$ as $\epsilon \to 0$ and $0 < \alpha < 1$.

Proof. Since f is a homeomorphism from X onto Y, there exists a basis

$$\{e_1, e_2, \ldots, e_n\}$$

for X such that $\{f(e_1), f(e_2), \ldots, f(e_n)\}$ are linearly independent in Y and $\|e_i\| = 1$, $i = 1, 2, \ldots, n$. We define a surjective linear map $I : X \to Y$ by

$$I(\sum_{i=1}^{n} \alpha_i e_i) = \sum_{i=1}^{n} \alpha_i f(e_i), \qquad \alpha_i \in R,\ i = 1, 2, \ldots, n.$$

If $x \in X$, there exist $\alpha_1, \alpha_2, ..., \alpha_n$ in R such that $x = \sum_{i=1}^{n} \alpha_i e_i$. By Theorem 11,

$$\begin{aligned}
\|f(x) - I(x)\| &= \|f(\sum_{i=1}^{n} \alpha_i e_i) - \sum_{i=1}^{n} \alpha_i f(e_i)\| \\
&\le \|f(\sum_{i=1}^{n} \alpha_i e_i) - f(\sum_{i=1}^{n-1} \alpha_i e_i) - f(\alpha_n e_n)\| \\
&+ \|f(\sum_{i=1}^{n-1} \alpha_i e_i) - f(\sum_{i=1}^{n-2} \alpha_i e_i) - f(\alpha_{n-1} e_{n-1})\| \\
&\cdots \\
&+ \|f(\alpha_1 e_1 + \alpha_2 e_2) - f(\alpha_1 e_1) - f(\alpha_2 e_2)\| \\
&+ \|f(\alpha_1 e_1) - \alpha_1 f(e_1)\| + \|f(\alpha_2 e_2) - \alpha_2 f(e_2)\| + \cdots \\
&+ \|f(\alpha_n e_n) - \alpha_n f(e_n)\| \\
&\le (n-1) D(\epsilon, \alpha)(\sum_{i=1}^{n} |\alpha_i|) \\
&+ D(\epsilon, \alpha)(\sum_{i=1}^{n} (|\alpha_i|^{\alpha} + |\alpha_i|^{2-\alpha})) \\
&\le n^2 D(\epsilon, \alpha) K(\|x\|^{\alpha} + \|x\|^{2-\alpha}), \ \text{ for some } K > 0.
\end{aligned}$$

Put $E(\epsilon, \alpha) = n^2 D(\epsilon, \alpha) K$. This completes the proof of the theorem.

References

1. J. Gevirtz, *Injectivity in Banach spaces and the Mazur-Ulam Theorem on isometries*, Trans. Amer. Math. Soc. **274** (1982), 307-318. MR **84h:**46024
2. K. Jarosz, *Ultraproducts and small bound perturbations*, Pacific J. Math. **148** (1991), 81-88. MR **91m:**46022
3. F. John, *On quasi-isometric mappings,* I, Comm. Pure Appl. Math. **21** (1968), 77-110. MR **36:**5716
4. S. Mazur and S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C.R. Acad. Sci. Paris Sér. **194** (1932), 946-948.

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