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Almost Periodic Schrödinger Operators

III. The Absolutely Continuous Spectrum in One Dimension

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Abstract. We discuss the absolutely continuous spectrum of $H=-d^2/dx^2+V(x)$ with V almost periodic and its discrete analog $(hu)(n)=u(n+1)+u(n-1)+V(n)u(n)$. Especial attention is paid to the set, A, of energies where the Lyaponov exponent vanishes. This set is known to be the essential support of the a.c. part of the spectral measure. We prove for a.e. V in the hull and a.e. E in A, H and h have continuum eigenfunctions, u, with $|u|$ almost periodic. In the discrete case, we prove that $|A|\leqq 4$ with equality only if $V=$const. If k is the integrated density of states, we prove that *on* A, $2k\,dk/dE \geqq \pi^{-2}$ in the continuum case and that $2\pi \sin \pi k\, dk/dE \geqq 1$ in the discrete case. We also provide a new proof of the Pastur-Ishii theorem and that the multiplicity of the absolutely continuous spectrum is 2.

1. Introduction

This paper discusses the theory of one dimensional stochastic Schrödinger operators and Jacobi matrices, that is $H=-d^2/d^2x+V_\omega(x)$ on $L^2(-\infty,\infty)$ and $u \mapsto (hu)(n)=u(n+1)+u(n-1)+V_\omega(n)u(n)$ on $\ell^2(Z)$, where V_ω is a stationary ergodic process on R or Z. This set includes the highly random case and also the almost periodic (a.p.) case. As we will explain, our theorems are vacuous in the highly random case and are only of interest in cases close to the almost periodic case. A major role will be played by the integrated density of states, $k(E)$, and the Lyaponov exponent, $\gamma(E)$, defined, e.g. in [2] or in [8] [in the latter, the rotation number $\alpha(E)=\pi k(E)$ is discussed].

In this paper, our primary goal will be to study the absolutely continuous (a.c.) spectrum. Much of what we do should be viewed as a development of themes of Moser [12], Johnson and Moser [8], and most especially Kotani [10] (see Simon [18] for Kotani theory in the Jacobi matrix case). Virtually all the theorems we

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prove are true under the sole assumption that $V_\omega(\cdot)$ is ergodic. However, one of Kotani's results is that if V_ω is non-deterministic, the a.c. spectrum is empty so our theorems are vacuous unless V_ω is deterministic, i.e. close to almost periodic.

One of our original motivations was to extend a remarkable inequality of Moser [12], who proved that for $E \in \mathrm{spec}(H)$

$$2\alpha \frac{d\alpha}{dE} \geqq 1 \tag{1.1}$$

for periodic and then, by a limiting argument, suitable limit periodic potentials. Inequality (1.1) cannot be true for the general stochastic case. For suppose $b(t)$ is two sided Brownian motion on a compact Riemannian manifold and $V(x)=f(b(x))$. Take f to be a continuous function whose minimum value is -1 but so that the measure of the set f, where $f(m)<0$ is very small in the normalized measure on the manifold. Then $\alpha(0)$ will be very small. But $[-1,0] \subset \mathrm{spec}(H)$, so if (1.1) holds, it would imply that $\alpha(0) \geqq 1$. There are also strongly coupled a.p. cases where (1.1) can be seen to fail. Our first realization is that (1.1) shouldn't be required to hold on all of $\mathrm{spec}(H)$ but only on a smaller set which equals $\mathrm{spec}(H)$ in the periodic case.

Given any absolutely continuous measure, $d\mu_{\mathrm{a.c.}}$ it is mutually a.c. with respect to a measure of the form $\chi_A dx$, where A is uniquely determined up to sets of measure zero. A is called the *essential* or *minimal support* of $d\mu_{\mathrm{a.c.}}$. Given any measure, $d\mu$, the essential support, A, of its absolutely continuous part is determined by the following pair of properties:

(1) There is a set of Lebesgue measure zero, B, so that $\mu(R \backslash (A \cup B))=0$.

(2) If $\mu(C)=0$, then $A \cap C$ has Lebesgue measure zero.

In the context of multidimensional stochastic Schrödinger operators, it is a theorem of Kunz and Souillard [11] (see also Kirsch and Martinelli [9]) that the a.c.-spectrum is a.e. constant (a.e. in ω). We expect that in that generality, it is even true that the essential support of the a.c. part of the spectral measure is constant, but we don't know how to prove it. In one dimension, however, one has the following beautiful theorem of Kotani [10] which uses the Lyaponov exponent, $\gamma(E)$.

Theorem 1.1 ([10]. *For* a.e. ω, *the support of the* a.c. *part of the spectral measure is equal to* $\{E \mid \gamma(E)=0\}$.

Remarks. 1. This is only implicit in Kotani [10]. It follows from Theorem 4.1 of his paper.

2. Kotani deals with the Schrödinger case. The extension of his ideas to the Jacobi case can be found in Simon [18].

In interpreting (1.1), we begin by noting that since α is a monotone function, by a well-known theorem in measure theory (see e.g. Saks [15]), the symmetrized derivative $\lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1}[\alpha(E+\varepsilon)-\alpha(E-\varepsilon)]$ exists for a.e. E. We denote it by $d\alpha/dE$. The extension of (1.1) that we will prove in this paper is

Theorem 1.2. *In the Schrödinger case, for* a.e. E *in the set where* $\gamma(E)=0$, *we have that* $d\alpha^2(E)/dE \geqq 1$.

As we have already noted, another theorem of Kotani [10] says that if $\{E|\gamma(E)=0\}$ has positive measure, then V is deterministic, so our Theorem 1.2 is only interesting in case V is deterministic, e.g. in the almost periodic case. An immediate consequence of Theorem 1.2 and the *inequality* $g(b)-g(a)\geqq\int_a^b \frac{dg}{dx}dx$ for g monotone is:

Corollary 1.3. *Let* $A=\{E|\gamma(E)=0\}$. *Then for* $a<b$:

$$\alpha^2(b)-\alpha^2(a)\geqq|A\cap a,b)|,$$

where $|\cdot|$ *is Lebesgue measure.*

We warn the reader that, in principle, $|\sigma_{\text{a.c.}}(H)\cap(a,b)|$ can be much larger than $|A\cap(a,b)|$.

If Theorem 1.2 and Corollary 1.3 were true in the Jacobi case (they are, as we will see), they are especially interesting since in that case we can take $b\to\infty$ and $a\to-\infty$ and obtain an absolute bound $|A|\leqq\pi^2$. However, in the Jacobi case, $2\alpha d\alpha/dE\geqq1$ cannot be optimal. For, if we replace V by $-V$, the resulting h is unitarily equivalent to the negative of the original h [under the unitary map $u(n)\to(-1)^n u(n)$, which flips the sine of h_0]. Thus, the new α, call it $\tilde{\alpha}$, is related to the old α by $\tilde{\alpha}(E)=\pi-\alpha(-E)$, so if $2\alpha d\alpha/dE\geqq1$ and the same for $\tilde{\alpha}$, we see that $F(\alpha)d\alpha/dE\geqq1$, where

$$\begin{aligned} F(\alpha)&=2\alpha &&\text{if} \quad \alpha\leqq\pi/2\\ &=2\pi-2\alpha &&\text{if} \quad \alpha\geqq\pi/2. \end{aligned}$$

It is unreasonable that this F should be optimal. A hint of what is the correct F is that in the Schrödinger case $2\alpha d\alpha/dE\geqq1$ has equality when $V=0$. This suggests the correct function should be the one that gives equality in the free case. This led us to find the following:

Theorem 1.4. *In the Jacobi case, for* a.e. *E in the set where* $\gamma(E)=0$, *we have that* $2\sin\alpha d\alpha/dE\geqq1$.

As before, this implies

Corollary 1.5. *Let* $A=\{E|\gamma(E)=0\}$. *Then for* $a<b$:

$$2\cos\alpha(a)-2\cos\alpha(b)\geqq|A\cap(a,b)|. \tag{1.2}$$

In particular, $|A|\leqq4$.

This inequality is new and, we feel, striking, even in the case where V is periodic (although in that case, one can use perturbation theory to check it for small and large coupling). In general, it says the size of the set of energies where there are extended states shrinks. It fits in well with the idea that in the strong coupling almost periodic case, one wants to have some spectrum that isn't a.c. (but doesn't prove it).

We will prove Theorems 1.2 and 1.4 as "boundary values" of inequalities in the upper half plane. It is a basic fact [8] (essentially a version of the Thouless formula [6, 19] – see [2]) that $\beta(E)\equiv-\gamma(E)+i\alpha(E)$ is the boundary value of an analytic function [which we also call $\beta(E)$] in the upper half plane; $\gamma \equiv -\mathrm{Re}\beta$ is the Lyaponov exponent in that half plane. Moreover, in $\mathrm{Im}\,E>0$:

$$\gamma(E) \geqq 0; \quad \alpha(E) \geqq 0; \quad \alpha(E) \leqq \pi \quad \text{(Jacobi case)}. \tag{1.3}$$

The relevant inequalities are:

Theorem 1.6 (Kotani [10]). *In the Schrödinger case,*

$$2\alpha(E)\gamma(E) \geqq \mathrm{Im}\,E. \tag{1.4}$$

Theorem 1.7. *In the Jacobi case*

$$2\sin\alpha(E)\sinh\gamma(E) \geqq \mathrm{Im}\,E. \tag{1.5}$$

Intuitively, the idea is that if $\gamma(E_0)=0$, then these inequalities are non-trivial for $E_0+i\varepsilon$ when ε is small and yield an inequality involving $\partial\gamma(E_0+iy)/\partial y$. By Cauchy-Riemann equations, this derivative should be $\partial\alpha/\partial E$.

In Sect. 2, we show that Theorems 1.6 and 1.7 imply Theorems 1.2 and 1.4. In Sect. 3, we give a simple proof of Theorem 1.6 which, like Kotani's original proof [10], uses Jensen's inequality (in the form of the Schwarz inequality), albeit in a different way. We don't see how to use Jensen's inequality to get Theorem 1.7. We give a completely different proof of Theorem 1.7 in Sect. 4 which uses nothing but the Thouless formula. The extension of this proof to yield an alternate proof of Theorem 1.6 requires a new result on the asymtotics of $k(E)$ at high energy. This result, of interest even in the random case, appears in an appendix. In Sect. 5 we show that in many cases, equality in the various inequalities implies that V is a constant. For example, in Corollary 1.5, $|A|=4$ implies that V is constant.

After presenting this set of ideas, we turn to studying eigenfunctions of $-u''+Vu=Eu$ for E on thhe real axis. In [10], Kotani only proved that the essential support of the a.c. part of the spectral measure isn't any smaller than $\{E|\gamma(E)=0\}$. That it isn't any larger is an older result of Pastur [13] and Ishii [7]. In Sect. 6 we study eigenfunctions for those E real with $\gamma(E)>0$ and prove the Pastur-Ishii theorem by ideas close to those Kotani used for the other half of the theorem. Unlike Pastur, we make no use of the existence of eigenfunction expansions. The main tool is to study the boundary values as $\varepsilon\downarrow 0$ of the eigenfunctions for $E+i\varepsilon$ which are L^2 at $+\infty$ or at $-\infty$. Using these same boundary values, we study eigenfunctions on the set where $\gamma(E)=0$ in Sect. 7. We prove the important result that if case V is almost periodic, these eigenfunctions at least have an absolute value that is almost periodic. We have learned that Kotani found this result some months before us, and plans to have it appear in the final version of [10]. In Sect. 8, we study the generalized eigenfunctions for energies in the spectrum where $\gamma(E)>0$. In Sect. 9, we prove the a.c. spectrum has multiplicity 2.

Throughout, we have an underlying probability measure space (Ω,μ) and a one parameter family T_y ($y\in R$ or $y\in Z$ depending on whether we are in the Schrödinger or Jacobi case) of measure preserving transformations which are ergodic and so that $V_\omega(y)=f(T_y\omega)$ for some function f on ω.

Unless we specify otherwise, the statement a.e. when applied to subsets of R denotes "with respect to Lebesgue measure"; "a.e." when applied to Ω means with respect to μ. We would like to thank Tom Wolff for supplying us with the proof in Appendix B.

2. Reduction of Theorems 1.2 and 1.4 to Inequalities for Complex E

Our goal in this section is to prove Theorems 1.2 and 1.4, assuming Theorems 1.6 and 1.7. We will use the following, which is a consequence of the Thouless formula [2], or alternatively a direct result of Johnson and Moser [8]:

Theorem 2.1. *$d\beta/dE$ also has a positive imaginary part in the region* $\operatorname{Im} E > 0$. *In fact, in that region*

$$\frac{d\beta}{dE} = \int \frac{dk(E')}{E' - E}. \tag{2.1}$$

As a consequence, we have

Proposition 2.2. *For almost all E*

$$\lim_{\varepsilon \downarrow 0} \frac{\partial \gamma}{\partial \varepsilon}(E_0 + i\varepsilon) = \frac{d\alpha}{dE}(E_0). \tag{2.2}$$

For almost all E_0 with $\gamma(E_0) = 0$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \gamma(E_0 + i\varepsilon) = \frac{d\alpha}{dE}(E_0). \tag{2.3}$$

Proof. Equation (2.3) follows from (2.2) and the mean value theorem, so we only need (2.2). By the Cauchy-Riemann equations, for $\beta = -\gamma + i\alpha$, $\frac{\partial \gamma}{\partial \varepsilon}(E_0 + i\varepsilon) = \frac{\partial \alpha}{\partial E}(E_0 + i\varepsilon)$, so (2.2) is equivalent to

$$\lim_{\varepsilon \downarrow 0} \operatorname{Im} \frac{d\beta}{dE}(E_0 + i\varepsilon) = \frac{d\alpha}{dE}. \tag{2.4}$$

We need some standard facts in measure theory [15]: if $d\mu$ is any measure with $\int (1 + |x|)^{-1} d\mu(x) < \infty$, then

(a) $F(x) = \lim_{\varepsilon \downarrow 0} \operatorname{Im} \int \frac{d\mu(y)}{y - x - i\varepsilon}$ exists for a.e. x.

(b) $d\mu_{\text{a.c.}} = \pi^{-1} F(x) dx$, where $d\mu_{\text{a.c.}}$ is the absolutely continuous part of μ.

(c) $G(x) = \int_{-\infty}^{x} d\mu(x)$ has a classical derivative at a.e. x.

(d) $d\mu_{\text{a.c.}} = G'(x) dx$.

Looking at (2.1) and recalling that $\alpha = \pi k$, we see that (2.4) follows from (a)–(d) above. □

Proof of Theorems 1.2 and 1.4 (assuming Theorems 1.6 and 1.7). $2\alpha(E_0 + i\varepsilon)\gamma(E_0 + i\varepsilon) \geqq \varepsilon$ and $\gamma(E_0) = 0$ implies $2\alpha d\alpha/dE(E_0) \geqq 1$ for a.e. E_0 by

Proposition 2.2. Similarly, since $\lim_{\varepsilon\downarrow 0}\sinh\gamma(E_0+i\varepsilon)/\varepsilon=\lim_{\varepsilon\downarrow 0}\gamma(E_0+i\varepsilon)/\varepsilon$ if $\gamma(E_0)=0$, we see that $2\sin\alpha(E_0+i\varepsilon)\sinh\gamma(E_0+i\varepsilon)\geqq\varepsilon$ implies that $2\sin\alpha d\alpha/dE(E_0)\geqq 1$. □

3. Complex Energy Inequality: The Schrödinger Case

Kotani proved (1.4) in [10] using Jensen's inequality [in the form $E(e^f)\geqq\exp(E(f))$] on a suitable integral. Here is an alternate version: In the Schrödinger case Kotani introduces a function $h_+(\omega,E)$ defined to be $u'_+(0)/u_+(0)$, where u_+ solves $-u''_+ + V_\omega u_+ = Eu_+$ with $\int_0^\infty |u_+|^2dx<\infty$ and h_+ is Herglotz. For $\operatorname{Im}E>0$, he proves:

Theorem 3.1.

$$E(1/\operatorname{Im}h_+)=2\gamma(E)/\operatorname{Im}E. \tag{3.1}$$

Johnson and Moser [8] prove for $\operatorname{Im}E>0$.

Theorem 3.2.

$$E(\operatorname{Im}h_+)=\alpha(E). \tag{3.2}$$

Given these results, we have

Proof of Theorem 1.6. By the Schwarz inequality

$$1\leqq E(1/\operatorname{Im}h_+)E(\operatorname{Im}h_+).$$

Now use (3.1) and (3.2). □

For the Jacobi case, the analogs of (3.1) and (3.2) are written in terms of the function $m_+(\omega,E)=-u_+(1)/u_+(0)$, where $u_+(n)$ solves $u_+(n+1)+u_+(n-1)+V_\omega(n)u_+(n)=Eu_+(n)$ and $\sum_1^\infty |u_+(n)|^2<\infty$. They say [18]:

$$\operatorname{Exp}(\ln(1+[\operatorname{Im}e/\operatorname{Im}m_+]))=2\gamma(E), \tag{3.3}$$

$$\operatorname{Im}[E(\ln(m_+))]=\alpha(E). \tag{3.4}$$

We do not see to obtain Theorem 1.4 from these relations and Jensen's inequality, so we turn to a different proof in the next section.

4. Harmonic Function Proof

In this section we will prove Theorem 1.7 for bounded potentials. The idea will be to obtain (1.5) by noting that both sides of the inequality are harmonic functions, so we need only prove the inequality near infinity and on the real axis. Near the real axis, it is trivial and near infinity we will have asymptotic equality. To be more explicit, let

$$F(E)=-2\cosh(\beta(E))\equiv-2\cosh(-\gamma(E)+i\alpha(E)).$$

Then (1.5) is equivalent to:

$$\operatorname{Im} F(E) \geqq \operatorname{Im} E. \tag{4.1}$$

Theorem 4.1. *Consider the Jacobi case. Suppose*

$$\int E dk(E) = \operatorname{Exp}(V(0)) = 0. \tag{4.2}$$

Then

$$F(E) = E + O(|E|^{-1}) \tag{4.3}$$

uniformly as $|E| \to \infty$ in the upper half-plane.

Proof. The Thouless formula says that

$$\beta(E) = -\int \ln(E' - E) dk(E'), \tag{4.4}$$

where the branch of ln is taken with $\ln(-1) = -i\pi$ and $\ln(z)$ continuous in the region $\operatorname{Im} z \leqq 0$. By (4.2) and (4.4)

$$\beta(E) = -\ln(-E) + O(1/|E|^2).$$

Since $|e^x - e^y| \leqq |x-y|[|e^x| + |e^y|] \leqq |e^x|(e+1)|x-y|$ if $|x-y| \leqq 1$, we see that

$$|e^{-\beta(E)} + E| \leqq |E| O(|E|^{-2}) = O(|E|^{-1}),$$

and so

$$e^{-\beta(E)} = -E + O(|E|^{-1}),$$
$$e^{\beta(E)} = O(|E|^{-1}),$$

and thus (4.2) follows. □

Proof of Theorem 1.7. Fix $\varepsilon > 0$ and let $H_\varepsilon(E) = [\operatorname{Im} E - \varepsilon]$. Let $D_{R,\varepsilon} = \{E \mid |E| < R, \operatorname{Im} E > \varepsilon\}$. Then we claim that for all sufficiently large R

$$\operatorname{Im} F(E) \geqq H_\varepsilon(E) \quad \text{if} \quad E \in \partial D_{R,\varepsilon}. \tag{4.5}$$

For, letting $E_0 = \operatorname{Exp}(V(0))$, (4.3) becomes

$$F(E) = (E - E_0) + O(|E|^{-1}),$$

so $\operatorname{Im} F(E) = \operatorname{Im} E + O(|E|^{-1})$. This yields (4.5) on the segment of $\partial D_{R,\varepsilon}$ with $|E| = R$ so long as R is large. On the segment with $\operatorname{Im} E = \varepsilon$, $H_\varepsilon(E) = 0 \leqq \operatorname{Im} F(E) = 2 \sin\alpha \sinh\gamma$, since $\alpha \in [0, \pi]$, $\gamma \geqq 0$.

This verifies (4.5). Since both sides are harmonic, the inequality holds inside $D_{R,\varepsilon}$ and thus on $\bigcup_{R \text{ large}} D_{R,\varepsilon} = \{E \mid \operatorname{Im} E > \varepsilon\}$. Now take $\varepsilon \to 0$. One obtains (4.1) and so (1.5). □

One can ask about whether Theorem 1.6 also has a harmonic function proof. We only see how to do this with a mild regularity condition on V. The analog of (4.4) is

$$\beta(E) = \sqrt{-E} + \int \frac{[k(E') - k_0(E')]}{(E' - E)} dE', \tag{4.6}$$

where k_0 is the free density of states, i.e. $k_0(E')=\pi^{-1}\sqrt{\max(0,E')}$. To see (4.6), we start with the following formula from [2]:

$$\beta(E)=\sqrt{-E}-\lim_{R\to\infty}\int_{-\infty}^{R}\ln(E'-E)d(k-k_0)(E'),$$

and integrate by parts using the fact [2] that $(k-k_0)(E')=O((E')^{-1/2})$. Alternatively (4.6) is proven in Kotani [10]. The analog of Theorem 4.1 which is required is

Theorem 4.2. *Consider the Schrödinger case. Suppose that* $\operatorname{Exp}(V(0))=0$, $\operatorname{Exp}(|V(0)|^2)<\infty$ *and* $\operatorname{Exp}(|V(x)-V(0)|)=o(1/|\ln x|)$ *as* $x\to 0$, $G(E)=-\beta(E)^2$. *Then as* $|E|\to\infty$

$$G(E)=E+o(1)$$

uniformly in each region $\operatorname{Im}E>\varepsilon$.

Proof. A new bound on the high energy behavior of k which we prove in Appendix A says that under the above hypotheses $k(E')-k_0(E')=o((E')^{-1/2}(\ln|E'|)^{-1})$, so that by an elementary estimate in the region $\operatorname{Im}E>\varepsilon$:

$$\beta(E)=\sqrt{-E'}+o(|E|^{-1/2}),$$

and thus

$$-(\beta(E))^2=E+o(1). \quad \square$$

Given this, we obtain an alternate proof of Theorem 1.6, but only with the weak regularity condition $\operatorname{Exp}(|V(x)-V(0)|)=o(1/|\ln x|)$. Of course, the trivial proof in Sect. 3 doesn't require this.

5. Conditions for $V=\text{const}$

In this section, we will prove that unless V equals a constant, most inequalities in Sect. 1 are strict, e.g. (1.4) and (1.5) are strict at all E with $\operatorname{Im}E>0$ and at *almost* every E with $\gamma(E)=0$, we have that $d\alpha^2/dE>1$ in the Schrödinger case, and $2\sin\alpha d\alpha/dE>1$ in the Jacobi case. In particular, in the Jacobi case $|\{E|\gamma(E)=0\}|<4$ if $V\neq\text{const}$. In his paper, Kotani [10] proves related theorems which imply that $V=\text{const}$, and his work motivated this section.

We begin by proving that if $\beta(E)$ is the free one, then $V=0$. We note that since k is the boundary value of $\operatorname{Im}\beta$, k is the free one if β is the free one. The converse of this is also true; it follows from (4.4) and (4.6).

Proposition 5.1. *Consider the Jacobi case. If k is the free one, then* $V=0$.

Proof. We have that

$$\operatorname{Exp}[(\delta_0,H^2\delta_0)]=\int E^2dk(E)=\int E^2dk_0(E)=(\delta_0,H_0^2\delta_0).$$

But $(\delta_0,H^2\delta_0)=\|H\delta_0\|^2=(\delta_0,H_0^2\delta_0)+|V(0)|^2$, so

$$\operatorname{Exp}(|V(0)|^2)=0.$$

Thus $V(0)=0$ a.e. and so by stationary $V\equiv 0$. □

There is a similar argument in the Schrödinger case [using $(H+a)^{-1}$ as $a\to\infty$] but it requires some regularity on V. Here is an argument that requires nothing:

Proposition 5.2. *Consider the Schrödinger case. Suppose that* $\beta(E)=\sqrt{-E}$. *Then* $V=0$.

Proof. Fix E_0 in the upper half-plane. Then $2\gamma(E_0)\alpha(E_0)=-\operatorname{Im}\beta(E_0)^2=\operatorname{Im}E_0$, so by the proof in Sect. 3 (equality in Schwarz), $\operatorname{Im}h_+$ is a.e. constant and that constant must be $E(\operatorname{Im}h_+)=\operatorname{Im}(\sqrt{-E_0})$. Thus, for a.e. V, the function $r(x)=u'_+(x)/u_+(x)$, where u_+ solves $-u''_+ + Vu_+ = E_0 u$ with $\int_0^\infty |u_+|^2 dx$, obeys $\operatorname{Im}r(x)=\operatorname{Im}(\sqrt{-E_0})$ for a.e. x and then by continuity for all x. Now r obeys the Ricatti equation

$$r'=(V(x)-E_0)-r^2\,.$$

Since $\operatorname{Im}r'=0$, we see that

$$\operatorname{Im}(r^2)=-E_0\,.$$

It follows that $\operatorname{Re}r(x)=\pm\operatorname{Re}(-\sqrt{-E_0})$, so, by continuity, the same sign works at all x. Thus $r'=0$ so $V_\omega(x)$ is constant. By ergodicity, the constant is ω independent. By the value of $\operatorname{Re}\beta(E_0)$, the constant is 0. □

Actually, the above only used $\beta(E_0)=\sqrt{-E_0}$ at a single E_0 in the upper half-plane. This is not surprising since the next result says that a weaker fact implies that $\beta(E)\equiv\sqrt{-E}$.

Theorem 5.3. *If equality holds in* (1.4) *(respectively* (1.5)*) at a single point in the upper half-plane, then* $V=\text{const}$.

Proof. Both sides of the inequality are harmonic functions, indeed $\operatorname{Im}(-\beta^2)\geqq\operatorname{Im}E$ [respectively $\operatorname{Im}(-2\cosh\beta)\geqq\operatorname{Im}E$]. Hence equality at one point implies equality at all points, and then by analyticity we see that $\beta^2=-(E-E_0)$ [respectively $\cosh\beta=-(E-E_0)$] for a real E_0. Hence, replacing V by $V-E_0$, we see that the $\beta(E+E_0)$ is the free β and thus, by the last two propositions, $V=E_0$ a.e. □

Our final circle of results concerns when equality holds on a set of positive measure on the real axis. We will need the following result which Tom Wolff proved for us; we give his proof in Appendix B.

Theorem 5.4. *Let G be a function analytic in the upper half-plane with a derivative dG/dz which is Herglotz. Then $G(z)$ has boundary values on the real axis $G(x+i0)$ for all x and for almost all x_0 in the set where* $\operatorname{Re}G(x+i0)=0$, *we have that*

$$\lim_{\varepsilon\downarrow 0}\operatorname{Re}\frac{dG}{dz}(x_0+i\varepsilon)=0\,.$$

Of course, this theorem is only interesting if $\operatorname{Re}G(E+i0)=0$ on a set of positive Lebesgue measure.

Theorem 5.5. *Consider the Schrödinger case. If there is a subset, S in R, of positive Lebesgue measure on which $\gamma(E)=0$ and $d\alpha^2/dE=1$, then V is constant.*

Proof. By the last theorem and the fact that $d\beta/dE$ is Herglotz, we see that $\lim_{\varepsilon\downarrow 0}\frac{\partial\gamma}{\partial E}(E_0+i\varepsilon)=0$ for a.e. E_0 in S. By the proof of Proposition 2.2, $\lim_{\varepsilon\downarrow 0}\frac{\partial\alpha}{\partial E}(E_0+i\varepsilon)=\frac{d\alpha}{dE}(E_0)$ for a.e. E_0 in S. Thus on S, $d\beta/dE(E_0+i0)$ is a.e. $id\alpha/dE(E_0)$. Obviously $\beta(E_0+i0)$ is $i\alpha(E_0)$. Thus $G(E)\equiv 2\beta d\beta/dE$ has a boundary value which is -1 by hypothesis. It follows that the Herglotz function $\sqrt{G(E)}$ has a boundary value i on almost all of S and so on a set of positive measure. Since boundary values of Herglotz functions on sets of positive measure uniquely determine the function $\sqrt{G}=i$ and so $d\beta^2/dE=-1$ which yields $\beta=\sqrt{-(E-E_0)}$. As above, V is a constant. □

Theorem 5.6. *Consider the Jacobi case. If there is a subset, S in R, of positive Lebesgue measure on which $\gamma(E)=0$ and $2\sin\alpha d\alpha/dE=1$, then V is constant.*

Proof. As in the last theorem, $d\beta/dE(E_0+i0)$ is $id\alpha/dE(E_0)$ for a.e. $E_0\in S$. Similarly the Herglotz function $\sinh(\beta(E))$ has boundary values $i\sin\alpha$. The above argument applied to $G(E)\equiv 2\sinh\beta d\beta/dE$ implies that $d/dE(2\cosh\beta)=-1$. This implies that V is a constant. □

With this last theorem, we can improve Corollary 1.5:

Theorem 5.7. *In the Jacobi case, $|\{E|\gamma(E)=0\}|\leqq 4$ with equality if and only if V is a constant.*

6. A New Proof of the Pastur-Ishii Theorem

Pastur [13] and Ishii [7] proved that on the set where $\gamma(E)>0$ there is no absolutely continuous spectrum. Kotani [10] used their result for one half of his theorem that $\{E|\gamma(E)=0\}$ is the essential support of the absolutely continuous spectrum. In this section we will give a proof of the Pastur-Ishii theorem using the same philosophy (and even the same equality) that Kotani used for his half of his theorem. Our proof is related to that of Ishii.

Let $h_+(\omega,E)$ be the function defined in Sect. 3 for $\operatorname{Im}E>0$. Then [10]

$$h_+(\omega,E)=\int\frac{d\mu_+(E',\omega)}{E'-E}$$

for a positive measure $d\mu_+$. Fix E_0 real. From this representation, we see that

$$\varepsilon^{-1}\operatorname{Im}h_+(\omega,E_0+i\varepsilon)=\int\frac{d\mu_+(E',\omega)}{\varepsilon^2+(E'-E_0)^2} \tag{6.1}$$

is monotone decreasing in ε, so

$$S_+(\omega,E_0)=\lim_{\varepsilon\downarrow 0}\operatorname{Im}h_+(\omega,E_0+i\varepsilon)/\varepsilon$$

exists in $[0, \infty]$. Applying the monotone convergence theorem to Kotani's relation (3.1), we see that

$$E(1/S_+(\omega, E_0)) = 2\gamma(E_0). \tag{6.2}$$

This implies that S_+ is a.e. infinite when $\gamma(E_0)$ is 0. What we will show is that S_+ is a.e. finite when $\gamma(E_0) > 0$. Parenthetically, we note that if $\alpha(E_0) > 0$, then

$$\begin{aligned} E(S_+(\omega, E_0)) &= \lim E(\operatorname{Im} h_+(\omega, E_0 + i\varepsilon)/\varepsilon) \\ &= \lim \alpha(E_0 + i\varepsilon)/\varepsilon = \infty \end{aligned}$$

so, even if S_+ is a.e. finite it is unbounded and non-L^1.

Below we will prove that

Theorem 6.1 (Continuous Case). *If* $\gamma(E) > 0$, *then* S_+ *is* a.e. *finite.*

Assuming this, we have

Theorem 6.2 (Pastur-Ishii Theorem). *Let* $P_\omega^{\text{a.c.}}$ *be the spectral measure class of* H_ω. *Then for* a.e. ω

$$P_\omega^{\text{a.c.}}(\{E \mid \gamma(E) > 0\}) = 0.$$

Proof (Continuous Case). Fix E_0 with $\gamma(E_0) > 0$. By Theorem 6.1, for a.e. ω, $S < \infty$ so $\operatorname{Im} h_+(\omega, E_0 + i0) = 0$. Similarly, $\operatorname{Im} h_-(\omega, E_0 + i0) = 0$. For a.e. pair (ω, E_0), $h_+ + h_-$ has a non-zero limit. If $h_+ + h_-$ has a non-zero limit and $\operatorname{Im} h_+$ and $\operatorname{Im} h_-$ have a zero limit, then $\operatorname{Im}(-1/h_+ + h_-)(\omega, E_0 + i0) = 0$. Thus for a.e. (ω, E) with $\gamma(E) > 0$, we have that $\operatorname{Im}((-1/(h_+ + h_-))(\omega, E + i0)) = 0$. Since this is the Green's function, we conclude that $P_\omega^{\text{a.c.}}(\{E \mid \gamma(E) > 0\}) = 0$. □

To prove Theorem 6.1, we define, following Kotani [10], for $\operatorname{Im} E > 0$, the function $f_+(\omega, E)$ to be the solution of $-f'' + (V - E)f = 0$ with $f_+(0) = 1$ and $f'_+(0) = h_+(\omega, E)$ so f is L^2 at ∞.

Lemma 6.3 [≡ Eq. (1.7) of Kotani]. $\int_0^\infty |f_+(x, \omega, E)|^2 dx = \operatorname{Im} h_+(\omega, E)/\operatorname{Im} E$.

Proof. Let

$$w(x) = \overline{f'_+(x)} f_+(x) - f'_+(x) \overline{f_+}(x).$$

Then $dw/dx = 2i \operatorname{Im} E |f_+(x)|^2$ and $w(x) \to 0$ ar ∞ with $w(0) = -2i \operatorname{Im} h_+(\omega, E)$ so the equality follows. □

Since f_+ is the unique solution L^2 at $+\infty$ with value 1 at $x = 0$, we see that

$$f_+(x, T_y\omega, E) = f_+(x + y, \omega, E)/f_+(y, \omega, E), \tag{6.3}$$

from which we see by the lemma that for $y > 0$:

$$\frac{\operatorname{Im} h_+(T_y\omega, E)}{\operatorname{Im} E} = f_+(y, \omega, E)^{-2} \left[\frac{\operatorname{Im} h_+(\omega, E)}{\operatorname{Im} E} - \int_0^y |f_+(x, \omega, E)|^2 dx \right]. \tag{6.4}$$

Next we want to note:

Proposition 6.4. *Suppose that* $S_+(\omega, E_0) < \infty$. *Then* $h_+(\omega, E_0 + i\varepsilon)$ *has a finite limit.*

Proof. By (6.1), if $S_+(\omega, E_0) < \infty$, then

$$\int \frac{d\mu_+(E', \omega)}{(E' - E_0)^2} < \infty .$$

Since $\int(|E'|+1)d\mu_+(E',\omega) < \infty$ also, by the dominated convergence theorem, h_+ has a finite limit. □

Proof of Theorem 6.1. By (6.2), $S_+(\omega, E_0) < \infty$ is finite on a set of positive measure, so by the ergodic theorem there exist for a.e. ω, y_0's with $S_+(T_{y_0}\omega, E_0) < \infty$ (indeed a set of y_0's with positive density). We will show that if $S_+(T_{y_0}\omega, E_0) < \infty$, then $S_+(T_y\omega, E_0) < \infty$ for a.e. y, in which case by the ergodic theorem again $S_+ < \infty$ a.e. By changing the meaning of ω, we can suppose $y_0 = 0$. By Proposition 6.4, $h_+(\omega, E_0 + i\varepsilon)$ has a finite limit, so $f_+(x, \omega, E_0 + i\varepsilon)$ has a finite limit $f_+(x, \omega, E_0 + i0)$ for all x. This f_+ solves the Schrödinger equation, so its set of zeros has measure zero. Equation (6.4) shows that if $f_+(y, \omega, E_0 + i0) \neq 0$ and $S_+(\omega, E_0) < \infty$, then $S_+(T_y\omega, E_0) < \infty$; so we have that $S_+(T_y\omega, E_0) < \infty$ for a.e. y. □

The above proof has to be slightly modified in the discrete case. The zeros of f_+ no longer have measure zero since the measure on Z is discrete. Define S_+ now as the limit of $\mathrm{Im}\, m_+/\mathrm{Im}\, E$. We replace Theorem 6.1 with

Theorem 6.5 (Discrete Case). *Let* $\gamma(E_0) > 0$. *Then for* a.e. ω *one of the following is true:*

(a) $S_+(\omega, E_0) < \infty$, $\mathrm{Im}\, m_+(\omega, E_0 + i0) = 0$ *and* $m_+(\omega, E_0 + i0)$ *has a finite limit.*

(b) $S_+(\omega, E_0) = \infty$, $\mathrm{Im}\, m_+(\omega, E_0 + i0) = \infty$ *and there is a non-zero solution, u, of the Schrödinger equation which is* ℓ_2 *at* $+\infty$ *and with* $u(0) = 0$.

Proof. (6.2) has to be replaced by

$$E(\log(1 + [S_+(\omega, E_0)]^{-1})) = 2\gamma(E_0),$$

which follows from Simon's formula (3.3). As above, for a.e. ω, there exist some $n < 0$, with $S_+(T^{-n}\omega, E_0) < \infty$. Using this n, we find a solution $f_+(m+n, T^{-n}\omega, E_0) \equiv u(m)$ of the Schrödinger equation for V_ω which is ℓ^2 at $+\infty$. If $u(0) \neq 0$, then using the analog of (6.4), we see that $S_+(\omega, E_0) < \infty$ from which alternative (a) follows by Proposition 6.4. If $u(0) = 0$, then the half-line operator has a Dirichlet eigenvalue, so $\mathrm{Im}\, m_+(\omega, E_0 + i0) = \infty$ so, *a fortiori* $S_+(\omega, E_0) = \infty$. This verifies alternative (b). □

With Theorem 6.5 replacing Theorem 6.1, the proof of the Pastur-Ishii theorem goes through in the discrete case. If $\mathrm{Im}\, m_+ = \infty$, then $m_+ + m_- \to \infty$, so $\mathrm{Im}(-1/m_+ + m_-)(\omega, E_0 + i0) = 0$. Similarly this is true if $\mathrm{Im}\, m_- = \infty$. Theorem 6.5 says that for a.e. ω, either $\mathrm{Im}\, m_+ = \infty$ or $\mathrm{Im}\, m_- = \infty$ or $\mathrm{Im}(m_+ + m_-) = 0$. In the latter case, the argument in Theorem 6.2 shows that for a.e. ω, E, $\mathrm{Im}(-1/m_+ + m_-)(\omega, E_0 + i0) = 0$. This proves Theorem 6.2 in the discrete case.

We suspect that alternative (b) of Theorem 6.5 always occurs on a set of measure zero, but we don't need to know that.

7. Continuum Eigenfunctions on the A.C. Spectrum

In this section, we will prove a basic result which we learned Kotani proved some months before us.

Theorem 71. *For* a.e. *pairs* $(\omega, E_0) \in \Omega \times \{E|\gamma(E)=0\}$, *there exist linearly independent eigenfunctions* $u\pm(x,\omega,E_0)$ *of* $-u''+(V_\omega-E_0)u=0$ *and for* a.e. E_0 *in* $\{E|\gamma(E)=0\}$ *a function* $H(\omega,E_0)$ *on* Ω,

(i) $u_+=\bar{u}_-$,

(ii) $\overline{\lim}_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} |u_\pm(x,\omega,E_0)|^2 dx \in (0,\infty)$,

(iii) $\int_\Omega |H(\omega,E_0)|^2 d\mu(\omega) < \infty$,

(iv) $|u_\pm(x,\omega,E_0)| = H(T_x\omega,E_0)$.

We have not been able to control the phase of $u_\pm$ but we *conjecture* that $u_\pm(x,\omega,E_0)e^{\mp i\alpha x} = U_\pm(T_x\omega,E_0)$ for a complex valued function on Ω.

Conditions (iii) and (iv) say that if case V is almost periodic, $|u_\pm|$ are L^2-almost periodic with the same frequency module as V. Our conjecture would imply that on the set where $\gamma(E)=0$, there are Bloch waves with quasi-momentum exactly equal to α. For this reason, we regard the proof or disproof of our conjecture as a significant open problem. It would have interesting consequences; e.g. by Aubry duality and Gordon's theorem [5, 17] one would obtain that in the almost Mathieu equation with Liouville frequency, there is only singular continuous spectrum also for coupling $\lambda \leqq 2$ (this is known of $\lambda > 2$ [2]).

We remark that if the solution u_+ is written $u_+(x)=r(x)e^{i\theta(x)}$, then θ will obey $\dot{\theta}=1/r^2$ and we will see that $\text{Exp}(1/r^2)=\alpha$. Thus our conjecture that $\theta-\alpha x$ is almost periodic with the same frequency module is seen to be related to a small divisor problem. In the periodic case, there is no problem and we obtain a rather involved proof of the existence of Bloch waves for the periodic case. We recall that for certain almost periodic potentials, Dinaburg and Sinai [4] have constructed Bloch waves: If our conjecture were proven, the essence of their result would be that $\{E|\gamma(E)=0\}$ had large measure for small coupling or large energy.

As the proof below shows, $u_\pm$ will be the $f_\pm$ of Kotani [10] but normalized with the normalization preferred by Moser [12].

Proof of Theorem 7.1. For a.e. pairs (ω, E_0), $h_\pm(\omega, E_0+i\varepsilon)$ have finite limits. Moreover, by Kotani's argument [10], $\text{Im}\, h_\pm(\omega,E_0)>0$ for a.e. pairs. If $h_\pm(\omega,E_0)$ are finite, we can form the limits $f_\pm(x,\omega,E_0)$ and by the limit of (6.3)

$$f_\pm(x, T_y\omega, E_0) = c_\pm f_\pm(x+y,\omega,E_0), \tag{7.1}$$

where $c_\pm$ are functions of y,ω and E_0 but not of x. If also $\text{Im}\, h_\pm \neq 0$, define

$$u_\pm(x,\omega,E_0) = f_\pm(x,\omega,E_0)/\sqrt{\text{Im}\, h_\pm(\omega,E_0)}. \tag{7.2}$$

Since the Wronskian of f_+ and $\bar{f}_+$ at 0 is $\mp 2i\, \text{Im}\, h_\pm$, we see that

$$\bar{u}'_\pm u_\pm - u'_\pm \bar{u}_\pm = \mp 2i. \tag{7.3}$$

While (7.3) is proven initially at 0, it holds at all x since $\bar{u}_+$ also solves the Schrödinger equation (this is where E_0 real enters). Since (7.1) holds and (7.3) holds at any point for both $u_+(x, \omega, E_0)$ and $u_+(\cdot, T_y\omega, E_0)$, we conclude that the constant relating $u_+(\cdot, \omega, E_0)$ and $u_+(\cdot, T_y\omega, E)$ must have magnitude 1, i.e.

$$|u_+(x, T_y\omega, E_0)| = |u_+(x+y, \omega, E_0)|.$$

Since $f_+(0, \omega, E_0) \equiv 1$, we conclude that

$$|u_+(x, \omega, E_0)| = [\operatorname{Im} h_+(T_x\omega, E_0)]^{-1/2}, \tag{7.4a}$$

proving (iv) with

$$H(\omega, E) = [\operatorname{Im} h_+(\omega, E_0)]^{-1/2}. \tag{7.4b}$$

(iii) then follows from Kotani's relation

$$E(1/\operatorname{Im} h_+(\omega, E_0)) \leqq \lim_{\varepsilon \downarrow 0} 2\gamma(E_0 + i\varepsilon)/\varepsilon. \tag{7.5}$$

(ii) is then a consequence of the ergodic theorem, indeed

$$\lim_{R \to \infty} \frac{1}{2R} \int_{-R}^{R} |u_\pm(x, \omega, E_0)|^2 = \operatorname{Exp}(1/\operatorname{Im} h_+(\omega, E_0)). \tag{7.6}$$

Finally, (i) follows if we note that

$$u_\pm(0, \omega, E_0) = 1/[\operatorname{Im} h_\pm(\omega, E_0)]^{1/2},$$

$$u'_\pm(0, \omega, E_0) = \pm h_\pm(\omega, E_0)/[\operatorname{Im} h_\pm(\omega, E_0)]^{1/2},$$

and then use Kotani's relation [10], that for a.e. (ω, E_0) in $\Omega \times \{E \mid \gamma(E) = 0\}$, we have that

$$h_-(\omega, E_0) = -\overline{h_+(\omega, E_0)}. \quad \square$$

While we give the above proof in the Schrödinger case, it extends without essential change to the Jacobi case if we replace Kotani's work by Simon [18]. Since $m_\pm$ is defined to be $-u_\pm(\pm 1)/u_\pm(0)$ in the complex plane, we have this relation in the limit. Then (Eq. (2.6) in [18])

$$\frac{u_-(1)}{u_-(0)} = m_- + E - V(0)$$

and $u_+ = \bar{u}_-$ requires

$$m_- + E - V(0) = -\bar{m}_+,$$

which is exactly (3.6) and (3.7) of [18]. The above proof shows the significance of Kotani's relation $h_- = -\bar{h}_+$. It is an expression of the fact that the solutions $u_\pm$ are complex conjugates of each other, which as we will explain, we believe is an expression of the fact that almost periodic potentials are reflectionless.

Inequality (7.5) and our argument in Sect. 2 imply that

$$\lim_{R \to \infty} \frac{1}{R} \int_0^R |u_\pm(x, \omega, E_0)|^2 dx \leqq 2 \frac{d\alpha}{dE}(E_0). \tag{7.7}$$

Moreover, since $\bar{u}_+$ and u_+ have Wronskian $-2i$, we see that if $u_+(x,\omega,E) = r(x,\omega,E)e^{i\theta(x,\omega,E)}$, then $r^2 d\theta/dx = 1$, so

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R |u_\pm(x,\omega,dE_0)|^{-2} dx = \alpha(E_0). \tag{7.8}$$

These two relations and the Schwarz inequality yield a proof that $2\alpha \frac{d\alpha}{dE} \geqq 1$; indeed, this is just the analog of Moser's proof in the periodic case [12]. The astute reader will see that this proof is not really any different from the one in Sects. 2 and 3.

In the discrete case, one has the analog of (7.7), viz

$$\lim_{N\to\infty} \frac{1}{N} \sum_0^{N-1} |u_\pm(j,\omega,E_0)|^2 \leqq 2\frac{d\alpha}{dE}(E_0). \tag{7.7'}$$

It is an interesting *open* question to see if the proper analog of (7.8) holds, viz whether

$$\lim_{N\to\infty} \frac{1}{N} \sum_0^{N-1} |u_\pm(j,\omega,E_0)|^{-2} = E(\operatorname{Im} h_\pm(\omega,E_0))$$

is smaller than $\sin\alpha$.

We note a rather striking formula implicit in the above construction:

$$\int_0^x \operatorname{Re} h_+(T_y\omega,E_0)dy = \tfrac{1}{2}\ln[\operatorname{Im} h_+(\omega)] - \tfrac{1}{2}\ln[\operatorname{Im} h_+(T_x\omega)]. \tag{7.8'}$$

This formula just says

$$\int_0^x \operatorname{Re}(u'_+(y)/u_+(y))dy = \ln|u_+(x)| - \ln|u_+(0)|.$$

We believe that (except perhaps for sets of measure zero with respect to *both* Lebesgue and spectral measures) there are only eigenfunctions with $\lim \frac{1}{R}\int_0^R |u_+|^2 dx > 0$ if $\gamma(E) = 0$. Specifically, we expect (but cannot prove) that for a.e. E *(with respect to the spectral measure)* in the singular spectrum there are solutions with $\lim_{R\to\infty} \frac{1}{2R}\int_{-R}^R |u(x)|^2 dx = 0$. Such a result would imply that the singular spectrum has multiplicity 1.

Using the differential equation and assuming the boundedness of V, it is easy to see that

$$\overline{\lim_{R\to\infty}} \frac{1}{R}\int_0^R [|u_+|^2 + |u'_+(x)|^2]dx < \infty,$$

and from this one sees that

$$\overline{\lim_{x\to\infty}} |x|^{-1/2} |u_+(x,\omega,E_0)| < \infty,$$

and in the almost periodic case

$$\lim |x|^{-1/2} |u_+(x, \omega, E_0)| = 0.$$

This mildly improves the $|x|^{1/2+\varepsilon}$ bound which is automatic a.e. from eigenfunction expansions (see e.g. [16]).

The existence of two linearly independent polynomially bounded eigenfunctions suggests that the a.c. spectrum has multiplicity 2 and this could probably be proven directly from these solutions. However, since one easy direct proof exists which we give in the next section, we don't pursue this here.

In [3], Davies and Simon show that for any bounded potential, one can find four spaces $\mathscr{H}^{\pm}_{\ell,r}$ so that

$$\mathscr{H}_\ell^- \oplus \mathscr{H}_r^- = \mathscr{H}_\ell^+ \oplus \mathscr{H}_r^+ = \mathscr{H}_{\text{a.c.}},$$

the absolutely continuous space for H, and so that

$$\mathscr{H}^{\pm}_{\ell,r} = \left\{ \varphi \in \mathscr{H}_{\text{a.c.}} \,\middle|\, \lim_{t \to \pm\infty} \|\chi_{r,\ell} e^{-itH} \varphi\|^2 = 0 \right\},$$

where $\chi_{r(\ell)}$ is the characteristic function of the right (left) half line. They call H reflectionless if $\mathscr{H}_\ell^- = \mathscr{H}_r^+$, i.e. if states which are on the left in the distant past are on the right in the distant future. We believe that almost periodic potentials are reflectionless and that this is connected with Kotani's relation $h_+ = -\bar{h}_-$. For the fact that u_+ is a boundary value from $\operatorname{Im} E > 0$ of functions decaying at $+\infty$ suggests that $\mathscr{H}_\ell^-$ is the "span" of the functions u_+, and similarly $\mathscr{H}_r^-$ is the span of the functions u_-. But time reversal implies that (see [3]) $\mathscr{H}_r^+ = \overline{\mathscr{H}_r^-}$, so $u_+ = \bar{u}_-$ would be an expression of the reflectionless nature of almost periodic Hamiltonians. Of course, this is only a vision without any proofs yet. In the periodic case, where one can easily identify $\mathscr{H}^{\pm}_{\ell,r}$ in terms of eigenfunctions [3], one can check that this vision is correct. The periodic case will differ in one way from the almost periodic case: Using stationary phase, one finds a dense set in $\mathscr{H}_\ell^-$ in the periodic case for which φ decays rapidly in x and for which $\|\chi_r e^{-itH} \varphi\|$ decays faster than any power of t. The occurrence of only recurrent spectrum in the a.p. case [1] will not allow that.

8. The Singular Support

We have seen that $\{E \mid \gamma(E) = 0\}$ is the a.e. ω common essential support of the a.c. part of the spectral measure. Here for each ω, we want to define a natural set, $\mathfrak{S}_\omega$, in R which supports the singular part of the spectral measure and which is intrinsic to it. Basic to our definition is the theorem of de Vallee Poussin (see Saks [15]):

Theorem 8.1. *The singular part of any measure μ is supported on the set of E where* $\lim_{\varepsilon \downarrow 0} \operatorname{Im} \int \frac{d\mu(E')}{E' - E - i\varepsilon} = \infty$.

A fortiori, it is supported on the set where $\overline{\lim} \ldots = \infty$ (and it is quite a bit easier to prove this weaker result). This motivates

Definition (Continuous Case). Let $G_\omega(x, y; E)$ be the integral kernel of $(H_\omega - E)^{-1}$. Then $\mathfrak{S}_\omega$ is the set of E_0 in R for which

$$\overline{\lim_{\varepsilon\downarrow 0}} \operatorname{Im}[G_\omega(0,0;E_0+i\varepsilon)] + \operatorname{Im}\left[\frac{\partial^2}{\partial x \partial_y} G_\omega(0,0;E_0+i\varepsilon)\right] = \infty .$$

(Discrete Case). Let $G_\omega(n, m; E)$ be the integral kernel of $(H_\omega - E)^{-1}$. Then $\mathfrak{S}_\omega$ is the set of E_0 in R for which

$$\overline{\lim_{\varepsilon\downarrow 0}} \{\operatorname{Im}[G_\omega(0,0;E_0+i\varepsilon)] + \operatorname{Im}[G_\omega(1,1;E_0+i\varepsilon)]\} = \infty .$$

Our goal in this section is to prove the following pair of theorems, and to relate $\mathfrak{S}_\omega$ to $h_\pm$ and to solutions like $f_\pm$.

Theorem 8.2. *$\mathfrak{S}_\omega$ is translation invariant, i.e.* $\mathfrak{S}_{T_y\omega} = \mathfrak{S}_\omega$.

Theorem 8.3. *For every* $E_0 \in R$, $\{\omega | E_0 \in \mathfrak{S}_\omega\}$ *has measure zero.*

Theorem 8.3 extends the result that $\{\omega | E_0$ is an eigenvalue of $H_\omega\}$ has measure zero (see e.g. [2]).

Our final result supplements the fact that a.e. on $\{E | \gamma(E) = 0\}$, we have that $h_+(\omega, E_0 + i0) \equiv -\overline{h_-(\omega, E_0 + i0)}$ (continuous case) or $m_+(\omega, E_0 + i0)$ $= -\overline{m_-(\omega, E_0 + i0)} - E_0 - V(0)$ (discrete case).

Theorem 8.4. (a) *(Continuous Case.)* $E_0 \in \mathfrak{S}_\omega$ *if and only if either* $\underline{\lim} |h_+(\omega, E_0 + i\varepsilon) + h_-(\omega, E_0 + i\varepsilon)| = 0$ *or* $\underline{\lim} |h_+(\omega, E_0 + i\varepsilon)^{-1} + h_-(\omega, E_0 + i\varepsilon)^{-1}| = 0$.

(b) *(Discrete Case.)* $E_0 \in \mathfrak{S}_\omega$ *if and only if either* $\underline{\lim} |m_+(\omega, E_0 + i\varepsilon) + m_-(\omega, E_0 + i\varepsilon) + E_0 + i\varepsilon + V(0)| = 0$ *or* $\underline{\lim} |m_+(\omega, E_0 + i\varepsilon)^{-1} + [m_-(\omega, E_0 + i\varepsilon) + E_0 + i\varepsilon + V(0)]^{-1} = 0$.

Remark. Thus (a) can be paraphrased by saying $h_+ = -h_-$, although the possibility that both limits are infinite must be allowed as must the possibility that we go through a subsequence.

Proof. We prove (a). (b) is similar. $\operatorname{Im} G$ has an infinite limit only if $(h_+ + h_-)^{-1} \to \infty$ and $\operatorname{Im} \partial^2 G/\partial x dy$ has an infinite limit only if $h_+ h_-(h_+ + h_-)^{-1} = (h_+^{-1} + h_-^{-1})$ $\to \infty$. □

To link $\mathfrak{S}_\omega$ to solutions of the Schrödinger equation, we need to deal with the fact that $f_+(T_y\omega, x, E_0 + i\varepsilon)$ can be singular because $f_+(\omega, y, E_0 + i\varepsilon)$ has a zero as $\varepsilon \downarrow 0$. We thus define

$$\eta_\pm(\omega, x, E) = f_\pm(\omega, x, E)/(1 + |h_\pm(\omega, E)|^2)^{1/2} ,$$

so η is normalized by $\eta_\pm(\omega, 0, E) > 0$, $|\eta_\pm(\omega, 0, E)|^2 + |\eta'_\pm(\omega, 0, E)|^2 = 1$.

Theorem 8.5. $E_0 \in \mathfrak{S}_\omega$ *if and only if, for some sequence* $\varepsilon_n \downarrow 0$, $\lim \eta_\pm(\omega, x, E_0 + i\varepsilon_n) \equiv \eta_\pm(\omega, x, E_0 + i0)$ *exists and for a constant* c:

$$\eta_+(\omega, x, E_0 + i0) = c\eta_-(\omega, x, E_0 - i0) . \tag{8.1}$$

Proof. Define two dimensional vectors $a_\pm = (1, \pm h_\pm)/\sqrt{1 + |h_\pm|^2}$. Using compactness of the unit vectors, we see that $E_0 \in \mathfrak{S}_\omega$ if and only if there exists $\varepsilon_n \downarrow 0$ so that

$a_\pm(E_0+i\varepsilon_n)$ have limits which are equal up to a factor (the factor is only necessary if $\lim|h_\pm|=\infty$). From this, (8.1) follows (and c is only necessary of $\lim|h_\pm|=\infty$). □

This last theorem illustrates the difference between the a.c. spectrum, where $u_+=\bar{u}_-\neq u_-$ and the singular spectrum and supports our belief that the singular spectrum has multiplicity 1.

Proof of Theorem 8.2. If $\eta_+(\omega,x,E_0+i\varepsilon_n)$ has a limit, so does $\eta_+(T_y\omega,x,E_0+i\varepsilon_n)$ for all y and it equals $\text{const}\,\eta_+(\omega,x+y,E_0+i\varepsilon_n)$, so if (8.1) holds for ω, it holds for $T_y\omega$. □

Proof of Theorem 8.3. By Lemma 6.3 and Fatou's lemma, $\int\limits_0^\infty |\eta_\pm(x,\omega,E_0)|^2dx \leqq S_\pm(\omega,E_0)$. Thus if $S_+<\infty$ and $S_-<\infty$ and $E_0\in\mathfrak{S}_\omega$, we see that E_0 is an eigenvalue of H_ω. Thus, in the continuous case, where $S_+<\infty$ and $S_-<\infty$ a.e. have that

$$\{\omega|E_0\in\mathfrak{S}_\omega\}\subset\{\omega|E_0 \text{ is an eigenvalue of } H_\omega\}\,(\text{mod. measure zero}). \quad (8.2)$$

The set on the right has measure zero [2]. In the discrete case, we use Theorem 6.5 and note that if $\operatorname{Im}m_+=\infty$ and $E_0\in\mathfrak{S}_\omega$, then $\operatorname{Im}m_-=\infty$. Thus if $E_0\in\mathfrak{S}_\omega$, either $S_-<\infty$, and $S_+<\infty$ or else there exist ℓ_2 solutions at $+\infty$ and $-\infty$ vanishing at zero, or we are in a set of measure zero; so again (8.2) holds. □

9. Multiplicity of the Absolutely Continuous Spectrum

We want to note the following, which is connected to ideas of Davies and Simon [3] and the stationarity which says V looks the same near $+\infty$ and $-\infty$.

Theorem 9.1. *The absolutely continuous spectrum of a stochastic Schrödinger operator or Jacobi matrix is uniformly of multiplicity two.*

Proof. We give the details in the Schrödinger case. The Jacobi case is similar. If H_ω^+ is the operator on $[0,\infty)$ with Dirichlet boundary conditions, then $\lim\limits_{\substack{y>x>0\\ y\downarrow 0}}\frac{\partial^2}{\partial x\partial y}G(x,y)=h_+(\omega,E)$, where $G(x,y)$ is the kernel of $(H_\omega-E)^{-1}$ and h_+ is the function discussed in Sect. 3. By the arguments in Kotani [10], for a.e. ω, $\lim h_+(\omega,E+i0)$ is non-zero and finite precisely for a.e. E in $\{E|\gamma(E)=0\}$. Thus H_ω^+ has as its essential spectral support this set. By general principles [16], the spectral multiplicity of H_ω^+ is exactly 1. Thus $H_\omega^+\oplus H_\omega^-$ has a.c. spectrum of uniform multiplicity 2. But by the Kato-Birman theory (see e.g. [14]), the absolutely continuous spectrum of $H_0+V_\omega\equiv H_\omega$ is unitarily equivalent to $H_\omega^+\oplus H_\omega^-$ [since $(H_\omega+i)^{-1}-(H_\omega^++i)^{-1}\oplus(H_\omega^-+i)^{-1}$ is finite rank]. □

The point spectrum clearly has multiplicity 1. We conjecture that the singular spectrum also has multiplicity 1. This conjecture would follow from one that says that for the continuum eigenfunctions associated to the singular spectrum, we have

$$\lim_{R\to\infty}\frac{1}{R}\int\limits_{-R}^{R}|u'(x)|^2+|u(x)|^2dx=0.$$

Appendix A. High Energy Behavior of the Integrated Density of States

Let $k(E)$ be the integrated density of states for $-d^2/dx^2+V(x)$, where $V(x)$ is a stationary ergodic stochastic process. When V is uniformly bounded, Avron and Simon [2, remark following Theorem 3.2] showed that $|k(E)-\pi^{-1}\sqrt{E}| = O(E^{-1/2})$. In Sect. 4, we need to obtain the $O(E^{-1/2})$ term. We will prove this using the "rotation number" point of view for k [8], rather than the direct density of states arguments in [2]. Let $E=\kappa^2$, and let u solve $-u''+Vu=Eu$ with boundary conditions $du/dx(0)=0$. Make an energy dependent Prüfer transformation to

$$u'(x) = -\kappa r(x)\sin\theta(x),$$
$$u(x) = r(x)\cos\theta(x).$$

Straightforward calculations show that θ obeys:

$$\frac{d\theta(x)}{dx} = \kappa - \kappa^{-1}V(x)\cos^2\theta(x). \tag{A.1}$$

Moreover, it is a basic fact [8, 2] that

$$k(E) = \lim_{x\to\infty}(\pi x)^{-1}\theta(x). \tag{A.2}$$

To illustrate the power of (A.1), we remove the boundedness hypothesis of the above quoted result of [2]:

Theorem A.1. *If* $\mathrm{Exp}(|V(0)|)<\infty$, *then*

$$|k(E)-\pi^{-1}\sqrt{E}| \leqq \pi^{-1}E^{-1/2}\,\mathrm{Exp}(|V(0)|).$$

Proof. By (A.1)

$$|\theta(x)-\kappa x| \leqq \kappa^{-1}\int_0^x |V(y)|\,dy.$$

If we divide by x, use (A.2) and the ergodic theorem, the theorem results. □

Our main result in this appendix is:

Theorem A.2. *If* $\mathrm{Exp}(|V(0)|^2)<\infty$, *then*

$$k(E) = \pi^{-1}\sqrt{E} - \tfrac{1}{2}\pi^{-1}E^{-1/2}\,\mathrm{Exp}(V(0)) + o(E^{-1/2}). \tag{A.3}$$

Remarks. 1. We emphasize that by stationary stochastic process, we mean a *separable* probability measure space (Ω, Σ, μ) and one parameter *measurable* family of measure preserving transformations T_x with $V_\omega(x)=f(T_x\omega)$ for some function f. Since $U_x f = f\circ T_x$ is continuous on L^2, $f\in L^2$ [i.e. $\mathrm{Exp}(|V(0)|^2)<\infty$] implies also that $\mathrm{Exp}(|V(x)-V(0)|) \leqq \mathrm{Exp}(|V(x)-V(0)|^2)^{1/2}$ goes to zero as $x\downarrow 0$. We use this below.

2. After the proof, we discuss improved estimates on the $o(E^{-1/2})$ term in special cases.

Proof. We rewrite (A.1) as

$$\frac{d\theta}{dx} = \kappa - \frac{1}{2}\kappa^{-1}V(x) - \frac{1}{2}\kappa^{-1}V(x)\cos 2\theta\,. \tag{A.4}$$

The idea will be that the last term wants to average to zero, since the cos oscillates faster and faster and at a more uniform rate as $\kappa \to \infty$. Explicitly, set $x_n = \pi n/\kappa$, $\Delta x = \pi/\kappa$, and $\theta_n = \theta(x_n)$. Note first that by (A.1)

$$|\theta(x) - \theta_n - \kappa(x - x_n)| \leqq \kappa^{-1}\int_{x_n}^{x} |V(y)|\,dy\,. \tag{A.5}$$

Set

$$A_n \equiv -\left[\frac{\theta(x_{n+1}) - \theta(x_n)}{\Delta x} - \left(\kappa - \frac{1}{2\kappa}\frac{1}{\Delta x}\int_{x_n}^{x_{n+1}} V(x)\,dx\right)\right].$$

By (A.2), we must show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} |A_j| = o(E^{-1/2})\,.$$

In fact, since k is a.e. independent of ω, we only need prove that

$$\operatorname{Exp}\left(\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} |A_j|\right) = o(E^{-1/2})\,. \tag{A.6}$$

By (A.4)

$$A_n = (2\kappa)^{-1}(\Delta x)^{-1}\int_{x_n}^{x_{n+1}} V(x)\cos(2\theta(x))\,dx \equiv B_n + C_n\,,$$

$$B_n \equiv (2\kappa\Delta x)^{-1}\int_{x_n}^{x_{n+1}} V(x)\cos[2\kappa(x - x_n) + 2\theta_n]\,dx$$

$$= (2\kappa\Delta x)^{-1}\int_{x_n}^{x_{n+1}} [V(x) - V(x_n)]\cos[2\kappa(x - x_n) + 2\theta_n]\,dx\,,$$

since cos integrated over one period integrates to zero. Thus

$$\operatorname{Exp}\left(\overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{j=0}^{n-1} |B_j|\right) \leqq (2\kappa\Delta x)^{-1}\int_{0}^{\Delta x} E(|V(x) - V(0)|)\,dx\,.$$

By the first remark above, this is $o(\kappa^{-1})$ as $\kappa \to \infty$.

$$C_n \equiv (2\kappa\Delta x)^{-1}\int_{x_n}^{x_{n+1}} V(x)\,[\cos 2\theta(x) - \cos(2\kappa(x - x_n) + 2\theta_n)]\,dx\,,$$

so by (A.5) and Schwarz

$$|C_n| \leqq (2\kappa^2)^{-1}(\Delta x)^{-1}\left(\int_{x_n}^{x_{n+1}} |V(x)|\,dx\right)^2$$

$$\leqq (2\kappa^2)^{-1}\int_{x_n}^{x_{n+1}} |V(x)|^2\,dx\,.$$

Thus

$$\mathrm{Exp}\left(\overline{\lim_{n\to\infty}}\ \frac{1}{n}\sum_{j=0}^{n-1}|C_j|\right)\leqq(\pi/2\kappa^3)\,\mathrm{Exp}(|V(0)|^2).$$

This is $O(\kappa^{-3})$, and so (A.6) is proven. □

The above proof shows that

$$|k(E)-\pi^{-1}\sqrt{E}-\tfrac{1}{2}\pi^{-1}E^{-1/2}\,\mathrm{Exp}(V(0))|\leqq b+c$$

with (recall $\Delta x\equiv\pi E^{-1/2}$)

$$b=\tfrac{1}{2}E^{-1/2}\left[(\Delta x)^{-1}\int_0^{\Delta x}E(|V(x)-V(0)|)dx\right],$$

$$c=\tfrac{1}{2}\pi E^{-3/2}\,\mathrm{Exp}(|V(0)|^2).$$

In explicit cases, one can show b is better than $o(E^{-1/2})$. For example, if V is a smooth function of Brownian motion on a manifold, $E(|V(x)-V(0)|)=O(x^{1/2})$, and thus since $(\Delta x)^{1/2}=O(E^{-1/4})$, we see that the error is $O(E^{-3/4})$. If V is itself smooth so V' is in L^∞, then by the above proof the error is $O(E^{-1})$. We believe that with more effort, one could get $O(E^{-3/2})$ with sufficient smoothness on V. If, as we require in Sect. 4, $E(|V(x)-V(0)|)=o(1/|\ln x|)$, then $b=E^{-1/2}o(1/|\ln\Delta x|)$ $=o(E^{-1/2}(\ln|E|)^{-1})$.

Appendix B. A Theorem from Hard Analysis

In this appendix we want to give Wolff's proof of Theorem 5.4. Since dG/dE has boundary values which are finite a.e., so does $G(E_0)=-\int_0^1\frac{dG}{dE}(E_0+iy)dy$ $+G(E_0+i)$. Let $\gamma(x,y)=\mathrm{Re}\,G(x+iy)$. The real point is to control $\lim_{y\downarrow 0}\frac{\partial\gamma}{\partial x}(x,y)$. We introduce the non-tangential maximal function:

$$W^*(x_0)=\sup\{|dG/dz(x+iy)|\,|\,0<y\leqq 1,|x-x_0|\leqq y\}.$$

It is easy to see that W^* is lower semicontinuous and in particular, $\{W^*(x_0)>\lambda\}$ is open for any λ. Since $\ln(dG/dz)$ is locally in H^2, we can apply results on the non-tangential maximal function for H^2 (which is controlled using the Hardy-Littlewood maximal function) to see that

Proposition B.1. *For each interval* (c,d)

$$|\{x\,|\,W^*(x)>\lambda;c<x<d\}|\leqq D/\ln(|\lambda|+2),$$

and in particular, the measure goes to zero as $\lambda\to\infty$.

By this proposition, it suffices to prove for each λ,c,d, that $\lim_{y\downarrow 0}\frac{\partial\gamma}{\partial x}(x,y)=0$ for a.e. x in the set where $W^*(x)\leqq\lambda$, $c<x<d$ and $\gamma(x)=0$. Henceforth, fix λ,c,d. Define $S=\{x\,|\,W^*(x)\leqq\lambda;\ c<x<d\}$ and for $x\in(c,d)$,

$$y(x)\equiv\mathrm{dist}(x,S)\,;$$

y trivially obeys $|y(x)-y(x')|\leqq|x-x'|$, so it has a derivative y' a.e. and it is the integral of its derivative. Indeed, since S is closed, $(c,d)\backslash S=\bigcup(a_i,b_i)$ is a union of disjoint open intervals and $y'=0$ on $S\backslash\{a_i\}\cup\{b_i\}$, $y'=+1$ on each $(a_i,\frac{1}{2}(a_i+b_i))$ and $=-1$ on each $((\frac{1}{2}a_i+b_i),b_i)$. For each $\varepsilon>0$, $f_\varepsilon(x)\equiv\gamma(x,y(x)+\varepsilon)$ is Lipschitz and thus

$$f_\varepsilon(x_1)-f_\varepsilon(x_0)=\int_{x_0}^{x_1}\left[\frac{\partial\gamma}{\partial x}(x,y(x)+\varepsilon)+\frac{\partial\gamma}{\partial y}(x,y(x)+\varepsilon)y'(x)\right]dx.$$

Now suppose that $x_0,x_1\in S$ and $|x_1-x_0|<2$. Then for all $x\in(x_0,x_1)$ and all ε small, we have that $(x,y(x)+\varepsilon)$ lies in $\bigcup_{x_2\in S}\{(x_3y_3)|y_3\leqq1;|x_3-x_2|\leqq y_3\}$, and so in the last integral the integrand is uniformly bounded (by 2λ). Thus, by dominated convergence, we can take ε to zero and find that if $x_0,x_1\in S$ and $|x_1-x_0|<2$, then

$$\gamma(x_1)-\gamma(x_0)=\int_{x_0}^{x_1}g(x)dx,$$

where

$$\begin{aligned}g(x)&=\lim_{y\downarrow0}\frac{\partial\gamma}{\partial x}(x,y) &&\text{if}\quad x\in S\\ &=\frac{\partial\gamma}{\partial x}(x,y(x))+\frac{\partial\gamma}{\partial y}(x,y(x))y'(x) &&\text{if}\quad x\notin S.\end{aligned}$$

Now we need only use Lebesgue's theorem on differentiation of integrals twice. First, applying that theorem to the characteristic function of $T=\{x\in S|\gamma(x)=0\}$, we see that a.e. x_0 in T is a limit of other points x_1 in T. Secondly, applying that theorem to $g\in L^\infty$, we see that for a.e. $x_0\in T, g(x_0)\equiv\lim_{y\downarrow0}\frac{\partial\gamma}{\partial \mathrm{x}}(\mathrm{x},\mathrm{y})=\lim_{x_1\to x_0}$ $(x_1-x_0)^{-1}\int_{x_0}^{x_1}g(x)dx$. Taking the limit through a subsequence in T, we see that for a.e. x_0 in T, we have $g(x_0)=0$, which is the desired result. □

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