Open Mathematics

Research Article

Yalong Xue, Xiangdong Xie*, and Qifa Lin

Almost periodic solutions of a commensalism system with Michaelis-Menten type harvesting on time scales

https://doi.org/10.1515/math-2019-0134
Received April 3, 2019; accepted November 2, 2019

Abstract: In this paper, we consider an almost periodic commensal symbiosis model with nonlinear harvesting on time scales. We establish a criterion for the existence and uniformly asymptotic stability of unique positive almost periodic solution of the system. Our results show that the continuous system and discrete system can be unify well. Examples and their numerical simulations are carried out to illustrate the feasibility of our main results.

Keywords: almost periodic solutions; commensal symbiosis model; Lyapunov functional; Michaelis-Menten type harvesting; time scales

MSC 2010: 34C25, 92D25, 34D20, 34D40

1 Introduction

Many results of differential equations can be easily generalized to difference equations, while other results seem to be completely different from their continuous counterparts. A major task of mathematics today is to harmonize continuous and discrete analysis. The theory of time scale, which was first introduced by Stefan Hilger in his PhD thesis [1], can handle this problem well. For example, it can model insect populations that are continuous while in season (and many follows a difference scheme with variable), die out in (say) winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population [2]. More generally, time scales calculus can be applied to the system whose time domains are more complex. A good example can be found in economics: a consumer receives income at one point in time, asset holdings are adjusted at a different point in time, and consumption takes place at yet another point in time [3]. The time scales calculus has a tremendous potential for applications (see [4-10]).

Many scholars have recently studied the influence of the harvesting to predator-prey or competition system. Some of them (e.g., [11-14]) argued that nonlinear harvesting is more feasible. Also consider that the almost periodic phenomenon and non-autonomous model are more accurate to describe the actual situation (e.g., [15,16]). Therefore, we investigate the following commensalism system incorporating Michaelis-Menten

Yalong Xue: College of Mathematics and Physics, Ningde Normal University, Ningde, Fujian, 352100, P.R. China; E-mail: xueyalong@ndnu.edu.cn
***Corresponding Author: Xiangdong Xie:** College of Mathematics and Physics, Ningde Normal University, Ningde, Fujian, 352100, P.R. China; E-mail: xueyalong@ndnu.edu.cn
Qifa Lin: College of Mathematics and Physics, Ningde Normal University, Ningde, Fujian, 352100, P.R. China; E-mail: T9101@ndnu.edu.cn

type harvesting:

$$\begin{cases} x^{\Delta}(t) = a(t) - b(t)\exp\{x(t)\} + c(t)\exp\{y(t)\}, \\ y^{\Delta}(t) = d(t) - e(t)\exp\{y(t)\} - \dfrac{q(t)E(t)\exp\{y(t)\}}{E(t) + m(t)\exp\{y(t)\}}, \end{cases} \tag{1.1}$$

where $x(t), y(t)$ are the density of species x, y at time $t \in T$ (T is a time scale). x^{Δ}, y^{Δ} express the delta derivative of the functions $x(t), y(t)$. $E(t)$ denotes the harvesting effort and $q(t)$ is the catch ability coefficient. The coefficients are bounded positive almost periodic functions and we use the notations $g^l = \inf_{t\in T^+} g(t), g^u = \sup_{t\in T^+} g(t)$.

Obviously, let $x(t) = \ln x_1(t), y(t) = \ln y_1(t)$, if $T = R^+$, then system (1.1) is reduced to a continuous version:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(t)\Big(a(t) - b(t)x_1(t) + c(t)y_1(t)\Big), \\ \dfrac{dy_1}{dt} = y_1(t)\Big(d(t) - e(t)y_1(t)\Big) - \dfrac{q(t)E(t)y_1^2(t)}{E(t) + m(t)y_1(t)}, \end{cases} \tag{1.2}$$

if $T = Z^+$, then system (1.1) can be simplified as the following discrete system:

$$\begin{cases} x_1(t+1) = x_1(t)\exp\Big(a(t) - b(t)x_1(t) + c(t)y_1(t)\Big), \\ y_1(t+1) = y_1(t)\exp\Big(d(t) - e(t)y_1(t) - \dfrac{q(t)E(t)y_1(t)}{E(t) + m(t)y_1(t)}\Big). \end{cases} \tag{1.3}$$

The rest of this paper is arranged as follows. The next part we present some notations. After that, sufficient conditions for the uniformly asymptotic stability of unique almost periodic solution are established. We end this paper with two examples to verify the validity of our criteria.

2 Preliminaries

A time scale T is an arbitrary nonempty closed subset of the real numbers. A point $t \in T$ is called left-dense if $t > \inf T$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup T$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If T has a left-scattered maximum m, then $T^k = T\backslash\{m\}$; otherwise $T^k = T$. If T has a right-scattered minimum m, then $T_k = T\backslash\{m\}$; otherwise $T_k = T$.

A function $p : T \to R$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in T^k$. The set of all regressive and rd-continuous functions $p : T \to R$ will be denoted by $\mathcal{R} = \mathcal{R}(T) = \mathcal{R}(T, R)$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(T, R) = \{p \in \mathcal{R}: 1 + \mu(t)p(t) > 0 \text{ for all } t \in T\}$.

If p is a regressive function, then the generalized exponential function e_p is defined by

$$e_p(t, s) = \exp\Big\{\int_t^s \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big\}$$

for all $s, t \in T$, with the cylinder transformation $\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h} & if\ h \neq 0, \\ z & if\ h = 0. \end{cases}$ For further reading we refer to the book by Bohner and Peterson [2].

Definition 2.1 (see [2]). Let T be a time scale. For $t \in T$ we define the forward and backward jump operators $\sigma, \rho : T \to T$ and the graininess function $\mu : T \to R^+$ by

$$\sigma(t) = \inf\{s \in T : s > t\}, \quad \rho(t) = \sup\{s \in T : s < t\}, \quad \mu(t) = \sigma(t) - t,$$

and $\mu_1 = \inf_{t\in T^+} \mu(t),\ \mu_2 = \sup_{t\in T^+} \mu(t)$.

Definition 2.2 (see [6]). A time scale T is called an almost periodic time scale if

$$\prod := \{\tau \in R : T + \tau \in T, \forall t \in T\} \neq \{0\}.$$

Definition 2.3 (see [6]). Let T be an almost periodic time scale. A function $x \in C(T, R^n)$ is called an almost periodic function if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in \prod : |x(t+\tau) - x(t)| < \varepsilon, \ \forall t \in T\}$$

is a relatively dense set in T for all $\varepsilon > 0$, that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon) \in E\{\varepsilon, x\}$ such that

$$|x(t+\tau) - x(t)| < \varepsilon, \ \forall t \in T.$$

Definition 2.4 (see [6]). Let T be an almost periodic time scale and D denotes an open set in R^n. A function $f \in C(T \times D, R^n)$ is called an almost periodic function in $t \in T$ uniformly for $x \in D$ if the ε-translation set of f

$$E\{\varepsilon, f, S\} = \{\tau \in \prod : |f(t+\tau, x) - f(t, x)| < \varepsilon, \ \forall (t, x) \in T \times S\}$$

is a relatively dense set in T for all $\varepsilon > 0$ and for each compact subset S of D, that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \ \forall (t, x) \in T \times S.$$

Lemma 2.5 (see [7]). Assume that $a > 0$, $b > 0$ and $-a \in \mathcal{R}^+$. Then

$$y^{\Delta}(t) \geq (\leq) b - ay(t), \quad y(t) > 0, \quad t \in [t_0, \infty)_T \tag{2.1}$$

implies

$$y(t) \geq (\leq) \frac{b}{a}\left[1 + \left(\frac{ay(t_0)}{b} - 1\right) e_{(-a)}(t, t_0)\right], \quad t \in [t_0, \infty)_T. \tag{2.2}$$

Consider the following system

$$x^{\Delta}(t) = f(t, x) \tag{2.3}$$

and its associate product system

$$x^{\Delta}(t) = f(t, x), \quad z^{\Delta}(t) = f(t, z) \tag{2.4}$$

where $f : T^+ \times S_H \to R^n$, $S_H = \{x \in R^n : \|x\| < H\}$, $f(t, x)$ is almost periodic in t uniformly for $x \in S_H$ and is continuous in x.

Lemma 2.6 (see [8]). Suppose that there exists a Lyapnov function $V(t, x, z)$ defined on $T^+ \times S_H \times S_H$ satisfying the following conditions

(i) $a(\|x - z\|) \leq V(t, x, z) \leq b(\|x - z\|)$, where $a, b \in K$, $K = \{\alpha \in C(R^+, R^+) : \alpha(0) = 0$ and α is increasing$\}$;

(ii) $|V(t, x, z) - V(t, x_1, z_1)| \leq L(\|x - x_1\| + \|z - z_1\|)$, where $L > 0$ is a constant;

(iii) $D^+ V^{\Delta}_{(2.4)}(t, x, z) \leq -cV(t, x, z)$, where $c > 0$, $-c \in \Re^+$.

Moreover, if there exists a solution $x(t) \in S$ of system (2.3) for $t \in T^+$, where $S \subset S_H$ is a compact set, then there exists a unique almost periodic solution $q(t) \in S$ of system (2.3), which is uniformly asymptotically stable.

Lemma 2.7 If

$$d^u > e^l, \quad a^u + c^u \exp\{M_y\} > b^l, \tag{2.5}$$

then any positive solution $(x(t), y(t))$ of system (1.1) satisfies

$$\begin{aligned} \limsup_{t \to +\infty} y(t) &\leq M_y \stackrel{def}{=} (d^u - e^l)/e^l, \\ \limsup_{t \to +\infty} x(t) &\leq M_x \stackrel{def}{=} (a^u + c^u \exp\{M_y\} - b^l)/b^l. \end{aligned} \tag{2.6}$$

Proof. From the second equation of system (1.1) it follows

$$\begin{aligned} y^{\Delta}(t) &\leq d(t) - e(t)\exp\{y(t)\} \leq d(t) - e(t)(y(t) + 1) \\ &\leq (d^u - e^l) - e^l y(t). \end{aligned} \tag{2.7}$$

By using Lemma 2.5 we get

$$\limsup_{t\to+\infty} y(t) \le (d^u - e^l)/e^l \overset{def}{=} M_y. \tag{2.8}$$

For a sufficiently small $\varepsilon > 0$, from (2.5) and (2.8), there exists a $t_1 \in T^+$ such that

$$\begin{aligned} &y(t) \le M_y + \varepsilon, \quad \forall t > t_1. \\ &a^u + c^u \exp\{M_y + \varepsilon\} - b^l > 0, \end{aligned} \tag{2.9}$$

From (2.9) and the first equation of system (1.1), we have

$$x^\Delta(t) \le a(t) + c(t)\exp\{M_y + \varepsilon\} - b(t)\exp\{x(t)\} \le (a^u + c^u \exp\{M_y + \varepsilon\} - b^l) - b^l x(t). \tag{2.10}$$

By using Lemma 2.5 again, we have

$$\limsup_{t\to+\infty} x(t) \le (a^u + c^u \exp\{M_y + \varepsilon\} - b^l)/b^l. \tag{2.11}$$

Setting $\varepsilon \to 0$, one has

$$\limsup_{t\to+\infty} x(t) \le (a^u + c^u \exp\{M_y\} - b^l)/b^l \overset{def}{=} M_x. \tag{2.12}$$

Lemma 2.8 Under the hypothesis (2.5) and

$$\frac{E^l(d^l - e^u)}{(e^u - d^l)m^l + q^u E^u} > \exp\{M_y\} > \exp\{N_y\} > \frac{b^u - a^l}{c^l}, \tag{2.13}$$

then any positive solution $(x(t), y(t))$ of system (1.1) satisfies

$$\begin{aligned} &\liminf_{t\to+\infty} y(t) \ge N_y \overset{def}{=} \ln \frac{d^l(E^l + m^l \exp\{M_y\}) - q^u E^u \exp\{M_y\}}{e^u(E^l + m^l \exp\{M_y\})}, \\ &\liminf_{t\to+\infty} x(t) \ge N_x \overset{def}{=} \ln \frac{a^l + c^l \exp\{N_y\}}{b^u}. \end{aligned} \tag{2.14}$$

Proof. Lemma 2.7 means that for any $\varepsilon > 0$, there exists a $t_2 > t_1$ (the definition of t_1 in Lemma 2.7) such that

$$y(t) \le M_y + \varepsilon, \quad x(t) \le M_x + \varepsilon, \quad \forall t > t_2. \tag{2.15}$$

It follows from the second equation of system (1.1) that

$$y^\Delta(t) \ge d^l - e^u \exp\{y(t)\} - \frac{q^u E^u \exp\{M_y + \varepsilon\}}{E^l + m^l \exp\{M_y + \varepsilon\}}, \quad \forall t > t_2. \tag{2.16}$$

We claim that for $t \ge t_2$,

$$d^l - e^u \exp\{y(t)\} - \frac{q^u E^u \exp\{M_y + \varepsilon\}}{E^l + m^l \exp\{M_y + \varepsilon\}} \le 0. \tag{2.17}$$

Suppose that there exists a $\hat{t} \ge t_2$ such that

$$d^l - e^u \exp\{y(\hat{t})\} - \frac{q^u E^u \exp\{M_y + \varepsilon\}}{E^l + m^l \exp\{M_y + \varepsilon\}} > 0 \tag{2.18}$$

and for any $t \in [t_2, \hat{t})_{T^+}$

$$d^l - e^u \exp\{y(t)\} - \frac{q^u E^u \exp\{M_y + \varepsilon\}}{E^l + m^l \exp\{M_y + \varepsilon\}} \le 0. \tag{2.19}$$

Then

$$y(\hat{t}) < \ln \frac{d^l(E^l + m^l \exp\{M_y + \varepsilon\}) - q^u E^u \exp\{M_y + \varepsilon\}}{e^u(E^l + m^l \exp\{M_y + \varepsilon\})} \tag{2.20}$$

and for any $t \in [t_2, \hat{t})_{T^+}$,

$$y(t) \ge \ln \frac{d^l(E^l + m^l \exp\{M_y + \varepsilon\}) - q^u E^u \exp\{M_y + \varepsilon\}}{e^u(E^l + m^l \exp\{M_y + \varepsilon\})}, \tag{2.21}$$

which implies $y^{\Delta}(t) < 0$. It is a contradiction, so that (2.17) holds, i.e.,

$$y(t) \geq \ln \frac{d^l(E^l + m^l \exp\{M_y + \varepsilon\}) - q^u E^u \exp\{M_y + \varepsilon\}}{e^u(E^l + m^l \exp\{M_y + \varepsilon\})}, \tag{2.22}$$

thereby

$$\liminf_{t \to +\infty} y(t) \geq \ln \frac{d^l(E^l + m^l \exp\{M_y + \varepsilon\}) - q^u E^u \exp\{M_y + \varepsilon\}}{e^u(E^l + m^l \exp\{M_y + \varepsilon\})}. \tag{2.23}$$

Setting $\varepsilon \to 0$,

$$\liminf_{t \to +\infty} y(t) \geq \ln \frac{d^l(E^l + m^l \exp\{M_y\}) - q^u E^u \exp\{M_y\}}{e^u(E^l + m^l \exp\{M_y\})} \overset{def}{=} N_y. \tag{2.24}$$

For above ε and (2.24), there exists a $t_3 > t_2$ such that

$$y(t) \geq N_y - \varepsilon, \quad \forall t > t_3. \tag{2.25}$$

It follows from the first equation of system (1.1) and above inequation that

$$x^{\Delta}(t) \geq a^l - b^u \exp\{x(t)\} + c^l \exp\{N_y - \varepsilon\}. \tag{2.26}$$

By analyzing (2.26) similar to (2.17)-(2.24), one has

$$\liminf_{t \to +\infty} x(t) \geq \ln \frac{a^l + c^l \exp\{N_y\}}{b^u} \overset{def}{=} N_x. \tag{2.27}$$

3 Positive almost periodic solution

From Lemma 2.7 and Lemma 2.8, let $\Omega = \{(x(t), y(t)) : (x(t), y(t))$ is a solution of (1.1) and $0 < N_x \leq x(t) \leq M_x, 0 < N_y \leq y(t) \leq M_y\}$. Obviously, Ω is an invariant set.

Theorem 3.1 Under the hypothesis (2.5) and (2.13), the $\Omega \neq \emptyset$.

Proof. Since the coefficients are almost periodic sequences, there exists a sequence $\{\tau_k\} \subseteq T^+$ with $\tau_k \to \infty$ as $k \to \infty$ such that

$$\begin{aligned} &a(t+\tau_k) \to a(t), \quad b(t+\tau_k) \to b(t), \quad c(t+\tau_k) \to c(t), \quad d(t+\tau_k) \to d(t), \\ &e(t+\tau_k) \to e(t), \;\; q(t+\tau_k) \to q(t), \;\; m(t+\tau_k) \to m(t), \;\; E(t+\tau_k) \to E(t). \end{aligned} \tag{3.1}$$

From Lemma 2.7 and Lemma 2.8, for sufficiently small $\varepsilon > 0$, there exists a $t_4 \in T^+$ such that

$$N_x - \varepsilon \leq x(t) \leq M_x + \varepsilon, \quad N_y - \varepsilon \leq y(t) \leq M_y + \varepsilon, \quad \forall t > t_4. \tag{3.2}$$

Write $x_k(t) = x(t+\tau_k)$ and $y_k(t) = y(t+\tau_k)$ for $t > t_4 - \tau_k$ and $k = 1, 2, \cdots$. For arbitrary $q \in N^+$, it is easy to see that there exists sequences $\{x_k(t) : k \geq q\}$ and $\{y_k(t) : k \geq q\}$ such that the sequence $\{x_k(t)\}$ and $\{y_k(t)\}$ has a subsequence, denoted by $\{x_k^\star(t)\}$ $(x_k^\star(t) = x(t+\tau_k^\star))$ and $\{y_k^\star(t)\}$ $(y_k^\star(t) = y(t+\tau_k^\star))$, respectively, converging on any finite interval of T^+ as $k \to \infty$. Therefore there exist two almost periodic sequences $\{z(t)\}$ and $\{w(t)\}$ such that for $t \in T^+$,

$$x_k^\star(t) \to z(t), \quad y_k^\star(t) \to w(t), \quad as \;\; k \to \infty. \tag{3.3}$$

Apparently the above sequence $\{\tau_k^\star\} \subseteq T^+$ with $\tau_k^\star \to \infty$ as $k \to \infty$ such that

$$\begin{aligned} &a(t+\tau_k^\star) \to a(t), \quad b(t+\tau_k^\star) \to b(t), \quad c(t+\tau_k^\star) \to c(t), \quad d(t+\tau_k^\star) \to d(t), \\ &e(t+\tau_k^\star) \to e(t), \;\; q(t+\tau_k^\star) \to q(t), \;\; m(t+\tau_k^\star) \to m(t), \;\; E(t+\tau_k^\star) \to E(t). \end{aligned} \tag{3.4}$$

which, together with (3.3) and

$$\begin{cases} x_k^{\star\Delta}(t) = a(t+\tau_k^\star) - b(t+\tau_k^\star)\exp\{x_k^\star(t)\} + c(t+\tau_k^\star)\exp\{y_k^\star(t)\}, \\ y_k^{\star\Delta}(t) = d(t+\tau_k^\star) - e(t+\tau_k^\star)\exp\{y_k^\star(t)\} - \dfrac{q(t+\tau_k^\star)E(t+\tau_k^\star)\exp\{y_k^\star(t)\}}{E(t+\tau_k^\star)+m(t+\tau_k^\star)\exp\{y_k^\star(t)\}}, \end{cases} \tag{3.5}$$

yields

$$\begin{cases} z^\Delta(t) = a(t) - b(t)\exp\{z(t)\} + c(t)\exp\{w(t)\}, \\ w^\Delta(t) = d(t) - e(t)\exp\{w(t)\} - \dfrac{q(t)E(t)\exp\{w(t)\}}{E(t)+m(t)\exp\{w(t)\}}. \end{cases} \tag{3.6}$$

Obviously, $(z(t), w(t))$ is a solution of system (1.1) and

$$N_x - \varepsilon \le z(t) \le M_x + \varepsilon, \quad N_y - \varepsilon \le w(t) \le M_y + \varepsilon, \quad \forall t \in T^+. \tag{3.7}$$

Since ε is arbitrary, thus

$$N_x \le z(t) \le M_x, \quad N_y \le w(t) \le M_y, \quad \forall t \in T^+. \tag{3.8}$$

Theorem 3.2 Assume that (2.5) and (2.13) are hold. If $\lambda = \min\{A, B\} > 0$ and $-\lambda \in \mathcal{R}^+$, then system (1.1) admits a unique almost periodic solution $(x(t), y(t))$, which is uniformly asymptotically stable and $(x(t), y(t)) \in \Omega$, where

$$\begin{aligned} A &= 2(\mu_1+1)b^l\exp\{N_x\} - (\mu_2^2+\mu_2)(b^u)^2\exp\{2M_x\} - 1, \\ B &= 2\left[e^l + \frac{q^lE^l}{(E^u+m^u\exp\{M_y\})^2}\right]\exp\{N_y\} \\ &\quad -\mu_2\left[e^u + \frac{q^uE^u}{(E^l+m^l\exp\{N_y\})^2}\right]^2\exp\{2M_y\} - (\mu_2+1)(c^u)^2\exp\{2M_y\}, \end{aligned} \tag{3.9}$$

Proof. By Theorem 3.1, there exists a solution $(x(t), y(t))$ such that

$$N_x \le x(t) \le M_x, \quad N_y \le y(t) \le M_y, \quad \forall t \in T^+. \tag{3.10}$$

Define

$$\|(x(t), y(t))\| = |x(t)| + |y(t)|. \tag{3.11}$$

Suppose that $U_1(t) = (x(t), y(t))$, $U_2(t) = (z(t), w(t))$ are arbitrary two positive solutions of system (1.1), then $\|U_1(t)\| \le M_x + M_y$, $\|U_2(t)\| \le M_x + M_y$. Consider the product system of (1.1)

$$\begin{cases} x^\Delta(t) = a(t) - b(t)\exp\{x(t)\} + c(t)\exp\{y(t)\}, \\ y^\Delta(t) = d(t) - e(t)\exp\{y(t)\} - \dfrac{q(t)E(t)\exp\{y(t)\}}{E(t)+m(t)\exp\{y(t)\}}, \\ z^\Delta(t) = a(t) - b(t)\exp\{z(t)\} + c(t)\exp\{w(t)\}, \\ w^\Delta(t) = d(t) - e(t)\exp\{w(t)\} - \dfrac{q(t)E(t)\exp\{w(t)\}}{E(t)+m(t)\exp\{w(t)\}}. \end{cases} \tag{3.12}$$

Construct the Lyapunov functional $V(t, U_1(t), U_2(t))$ on $T^+ \times \Omega \times \Omega$

$$V(t, U_1(t), U_2(t)) = (x(t) - z(t))^2 + (y(t) - w(t))^2. \tag{3.13}$$

The norm

$$\|U_1(t) - U_2(t)\| = |x(t) - z(t)| + |y(t) - w(t)| \tag{3.14}$$

is equivalent to

$$\|U_1(t) - U_2(t)\|_\star = [(x(t) - z(t))^2 + (y(t) - w(t))^2]^{\frac{1}{2}}, \tag{3.15}$$

i.e., there exist constants $B_1 > 0$ and $B_2 > 0$ such that

$$B_1\|U_1(t) - U_2(t)\| \le \|U_1(t) - U_2(t)\|_\star \le B_2\|U_1(t) - U_2(t)\|, \tag{3.16}$$

thus we have

$$(B_1\|U_1(t) - U_2(t))^2\| \le V(t, U_1(t), U_2(t)) \le (B_2\|U_1(t) - U_2(t)\|)^2. \tag{3.17}$$

Let $a, b \in C(R^+, R^+)$, $a(x) = B_1^2x^2$, $b(x) = B_2^2x^2$, then the assumption (i) of Lemma 2.6 is satisfied.

On the other side,

$$\begin{aligned}
|V(t, U_1(t), U_2(t)) - V(t, U_1^\star(t), U_2^\star(t))| &= |(x(t) - z(t))^2 + (y(t) - w(t))^2 - (x^\star(t) - z^\star(t))^2 - (y^\star(t) - w^\star(t))^2| \\
&\le |(x(t) - z(t)) - (x^\star(t) - z^\star(t))| \times |(x(t) - z(t)) + (x^\star(t) - z^\star(t))| \\
&\quad + |(y(t) - w(t)) - (y^\star(t) - w^\star(t))| \times |(y(t) - w(t)) + (y^\star(t) - w^\star(t))| \\
&\le |(x(t) - z(t)) - (x^\star(t) - z^\star(t))| \times (|x(t)| + |z(t)| + |x^\star(t)| + |z^\star(t)|) \\
&\quad + |(y(t) - w(t)) - (y^\star(t) - w^\star(t))| \times (|y(t)| + |w(t)| + |y^\star(t)| + |w^\star(t)|) \\
&\le \gamma(|x(t) - x^\star(t)| + |y(t) - y^\star(t)| + |z(t) - z^\star(t)| + |w(t) - w^\star(t)|) \\
&= \gamma(\|U_1(t) - U_1^\star(t)\| + \|U_2(t) - U_2^\star(t)\|),
\end{aligned} \tag{3.18}$$

where $U_1^\star(t) = (x^\star(t), y^\star(t))$, $U_2^\star(t) = (z^\star(t), w^\star(t))$, $\gamma = 4\max\{M_x, M_y\}$. Hence, the assumption (ii) of Lemma 2.6 is also satisfied.

Calculating the D^+V^Δ along the system (3.12),

$$\begin{aligned}
D^+V^\Delta_{(3.12)}(t, U_1(t), U_2(t)) &= (x(t) - z(t))^\Delta[(x(t) - z(t)) + (x(\sigma(t)) - z(\sigma(t)))] \\
&\quad + (y(t) - w(t))^\Delta[(y(t) - w(t)) + (y(\sigma(t)) - w(\sigma(t)))] \\
&= (x(t) - z(t))^\Delta[(x(t) - z(t)) + (\mu(t)x^\Delta(t) + x(t) - \mu(t)z^\Delta(t) + z(t))] \\
&\quad + (y(t) - w(t))^\Delta[(y(t) - w(t)) + (\mu(t)y^\Delta(t) + y(t) - \mu(t)w^\Delta(t) + w(t))] \\
&= (x(t) - z(t))^\Delta[2(x(t) - z(t)) + \mu(t)(x(t) - z(t))^\Delta] \\
&\quad + (y(t) - w(t))^\Delta[2(y(t) - w(t)) + \mu(t)(y(t) - w(t))^\Delta] \\
&= V_1 + V_2,
\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}
V_1 &= (x(t) - z(t))^\Delta[2(x(t) - z(t)) + \mu(t)(x(t) - z(t))^\Delta], \\
V_2 &= (y(t) - w(t))^\Delta[2(y(t) - w(t)) + \mu(t)(y(t) - w(t))^\Delta].
\end{aligned} \tag{3.20}$$

From (3.12), one has

$$\begin{cases}
(x(t) - z(t))^\Delta = -b(t)(\exp\{x(t)\} - \exp\{z(t)\}) + c(t)[\exp\{y(t)\} - \exp\{w(t)\}], \\
(y(t) - w(t))^\Delta = -e(t)(\exp\{y(t)\} - \exp\{w(t)\}) - q(t)E(t)\Big[\dfrac{\exp\{y(t)\}}{E(t) + m(t)\exp\{y(t)\}} - \dfrac{\exp\{w(t)\}}{E(t) + m(t)\exp\{w(t)\}}\Big].
\end{cases} \tag{3.21}$$

By the mean value theorem,

$$\begin{aligned}
&\exp\{x(t)\} - \exp\{z(t)\} = \exp\{\xi_1(t)\}(x(t) - z(t)), \\
&\exp\{y(t)\} - \exp\{w(t)\} = \exp\{\xi_2(t)\}(y(t) - w(t)), \\
&\frac{\exp\{y(t)\}}{E(t) + m(t)\exp\{y(t)\}} - \frac{\exp\{w(t)\}}{E(t) + m(t)\exp\{w(t)\}} = \frac{\exp\{\xi_3(t)\}(y(t) - w(t))}{(E(t) + m(t)\exp\{\xi_3(t)\})^2},
\end{aligned} \tag{3.22}$$

where $\xi_1(t)$ lies between $x(t)$ and $z(t)$ and $\xi_i(t)(i = 2, 3)$ lie between $y(t)$ and $w(t)$. Thus (3.21) can be expressed as follows

$$\begin{cases} (x(t) - z(t))^{\Delta} = -b(t)\exp\{\xi_1(t)\}(x(t) - z(t)) + c(t)\exp\{\xi_2(t)\}(y(t) - w(t)), \\ (y(t) - w(t))^{\Delta} = -e(t)\exp\{\xi_2(t)\}(y(t) - w(t)) - \left[\dfrac{q(t)E(t)\exp\{\xi_3(t)\}(y(t) - w(t))}{(E(t) + m(t)\exp\{\xi_3(t)\})^2}\right], \end{cases} \tag{3.23}$$

which together with (3.20) yield

$$\begin{aligned} V_1 &= \Big\{ - b(t)\exp\{\xi_1(t)\}(x(t) - z(t)) + c(t)\exp\{\xi_2(t)\}(y(t) - w(t))\Big\} \\ &\quad \Big[2(x(t) - z(t)) + \mu(t)\Big(- b(t)\exp\{\xi_1(t)\}(x(t) - z(t)) + c(t)\exp\{\xi_2(t)\}(y(t) - w(t))\Big)\Big] \\ &= \Big[- 2b(t)\exp\{\xi_1(t)\} + \mu(t)b^2(t)\exp\{2\xi_1(t)\}\Big]\Big(x(t) - z(t)\Big)^2 \\ &\quad +2c(t)\exp\{\xi_2(t)\}\Big[1 - \mu(t)b(t)\exp\{\xi_1(t)\}\Big](x(t) - z(t))(y(t) - w(t)) \\ &\quad +\mu(t)c^2(t)\exp\{2\xi_2(t)\}\Big(y(t) - w(t)\Big)^2 \\ &\le \Big[- 2(\mu(t) + 1)b(t)\exp\{\xi_1(t)\} + (\mu^2(t) + \mu(t))b^2(t)\exp\{2\xi_1(t)\} + 1\Big]\Big(x(t) - z(t)\Big)^2 \\ &\quad +(\mu(t) + 1)c^2(t)\exp\{2\xi_2(t)\}\Big(y(t) - w(t)\Big)^2 \\ &\le \Big[- 2(\mu_1 + 1)b^l\exp\{N_x\} + (\mu_2^2 + \mu_2)(b^u)^2\exp\{2M_x\} + 1\Big]\Big(x(t) - z(t)\Big)^2 \\ &\quad +(\mu_2 + 1)(c^u)^2\exp\{2M_y\}\Big(y(t) - w(t)\Big)^2, \end{aligned} \tag{3.24}$$

analogously

$$V_2 \le \Big\{ - 2\Big[e^l + \frac{q^lE^l}{(E^u + m^u\exp\{M_y\})^2}\Big]\exp\{N_y\} + \mu_2\Big[e^u + \frac{q^uE^u}{(E^l + m^l\exp\{N_y\})^2}\Big]^2\exp\{2M_y\}\Big\}\Big(y(t) - w(t)\Big)^2. \tag{3.25}$$

Therefore, one has

$$\begin{aligned} D^+V^{\Delta}_{(3.12)}(t, U_1(t), U_2(t)) &\le \Big\{ - 2(\mu_1 + 1)b^l\exp\{N_x\} + (\mu_2^2 + \mu_2)(b^u)^2\exp\{2M_x\} + 1\Big\}\Big(x(t) - z(t)\Big)^2 \\ &\quad +\Big\{(\mu_2 + 1)(c^u)^2\exp\{2M_y\} - 2\Big[e^l + \frac{q^lE^l}{(E^u + m^u\exp\{M_y\})^2}\Big]\exp\{N_y\} \\ &\quad +\mu_2\Big[e^u + \frac{q^uE^u}{(E^l + m^l\exp\{N_y\})^2}\Big]^2\exp\{2M_y\}\Big\}\Big(y(t) - w(t)\Big)^2 \\ &= -A\Big(x(t) - z(t)\Big)^2 - B\Big(y(t) - w(t)\Big)^2 \\ &\le -\lambda V(t, U_1(t), U_2(t)). \end{aligned} \tag{3.26}$$

Also, the assumption (iii) of Lemma 2.6 is satisfied.

By Lemma 2.6, there exists a unique uniformly asymptotically stable almost periodic solution $(x(t), y(t)) \in \Omega$ of system (1.1).

Now we consider the following single specie model with Michaelis-Menten type harvesting on time scales:

$$y^{\Delta}(t) = d(t) - e(t)\exp\{y(t)\} - \frac{q(t)E(t)\exp\{y(t)\}}{E(t) + m(t)\exp\{y(t)\}}, \tag{3.27}$$

For system (3.27), when we conduct the similar analysis of Lemma 2.7, Lemma 2.8, Theorem 3.1 and Theorem 3.2, one can easily obtain the following results and we omit the proof details here.

Lemma 3.3 If $d^u > e^l$, then positive solution $y(t)$ of system (3.27) satisfies

$$\limsup_{t\to+\infty} y(t) \le M_y' \overset{def}{=} (d^u - e^l)/e^l. \tag{3.28}$$

Lemma 3.4 If Lemma 3.3 and the following inequality

$$E^l(d^l - e^u) > \left((e^u - d^l)m^l + q^u E^u\right)\exp\{M_y'\} \tag{3.29}$$

hold, then any positive solution $y(t)$ of system (3.27) satisfies

$$\liminf_{t\to+\infty} y(t) \ge N_y' \overset{def}{=} \ln\frac{d^l(E^l + m^l\exp\{M_y'\}) - q^u E^u \exp\{M_y'\}}{e^u(E^l + m^l\exp\{M_y'\})}. \tag{3.30}$$

Let $\Omega' = \{y(t) : y(t)$ is a solution of (3.27) and $0 < N_y' \le y(t) \le M_y'\}$. It is obvious that Ω' is an invariant set.

Theorem 3.5 Assume that Lemma 3.3 and Lemma 3.4 hold, then $\Omega' \ne \emptyset$. Moreover, if $C > 0$, then system (3.27) admits a unique uniformly asymptotically stable almost periodic solution $y(t)$, and $y(t) \in \Omega'$, where

$$C = 2\left[e^l + \frac{q^l E^l}{(E^u + m^u\exp\{M_y\})^2}\right]\exp\{N_y\} - \mu_2\left[e^u + \frac{q^u E^u}{(E^l + m^l\exp\{N_y\})^2}\right]^2\exp\{2M_y\}. \tag{3.31}$$

4 Numerical Simulations

We give the following examples to illustrate the feasibility of our main results.

Example 4.1. Consider the continuous version:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(t)\Big(2 + 0.25\sin(\sqrt{2}t) - (1.5 - 0.5\cos(\sqrt{5}t))x_1(t) + (0.6 + 0.15\cos(\sqrt{5}t))y_1(t)\Big), \\ \dfrac{dy_1}{dt} = y_1(t)\Big(3.15 + 0.05\sin(\sqrt{3}t) - (2 + \sin(\sqrt{3}t))y_1(t)\Big) - \dfrac{(0.08 + 0.06\sin(\sqrt{6}t))y_1^2(t)}{1 + (1.2 + 0.2\cos(\sqrt{3}t))y_1(t)}, \end{cases} \tag{4.1}$$

By calculating, one has

$$\begin{aligned} &d^u - e^l = 1.2 > 0, \quad a^u + c^u\exp\{M_y\} - b^l = 2.6166 > 0, \\ &\frac{E^l(d^l - e^u)}{(e^u - d^l)m^l + q^u E^u} = 2.5 > \exp\{M_y\} = 1.8221 > \exp\{N_y\} = 1.0032 > \frac{b^u - a^l}{c^l} = 0.5556. \end{aligned} \tag{4.2}$$

Obviously, the assumption in (2.5) and (2.13) are satisfied. We obtain from Example 1.2 in [2] that $\mu(t) \equiv 0$, moreover, from (3.9) we have $\mu_1 \equiv 0$, $\mu_2 \equiv 0$. Thus

$$\begin{aligned} &A = 2b^l\exp\{N_x\} - 1 = 1.2014 > 0, \\ &B = 2\left[e^l + \frac{q^l E^l}{(E^u + m^u\exp\{M_y\})^2}\right]\exp\{N_y\} - (c^u)^2\exp\{2M_y\} = 2.1484 > 0, \end{aligned} \tag{4.3}$$

$\lambda = \min\{A, B\} > 0$ and $-\lambda \in \mathcal{R}^+$. From Figure 1, it is easy to see that for system (4.1) there exists a positive almost periodic solution denoted by $(x_1^\star(t), y_1^\star(t))$.

Example 4.2. Consider the discrete version:

$$\begin{cases} x_1(t+1) = x_1(t)\exp\Big(0.72 + 0.02\sin(\sqrt{2}t) - (0.72 + 0.02\cos(\sqrt{3}t))x_1(t) + (0.07 + 0.01\sin(\sqrt{5}t))y_1(t)\Big), \\ y_1(t+1) = y_1(t)\exp\Big(1 + 0.02\cos(\sqrt{2}t) - (0.25 - 0.01\sin(\sqrt{3}t))y_1(t) \\ \qquad - \dfrac{(0.02 + 0.001\cos(\sqrt{3}n))(1.5 + 0.5\sin(\sqrt{2}t))y_1(t)}{1.5 + 0.5\sin(\sqrt{2}t) + (1.3 + 0.1\cos(\sqrt{6}t))y_1(t)}\Big). \end{cases} \tag{4.4}$$

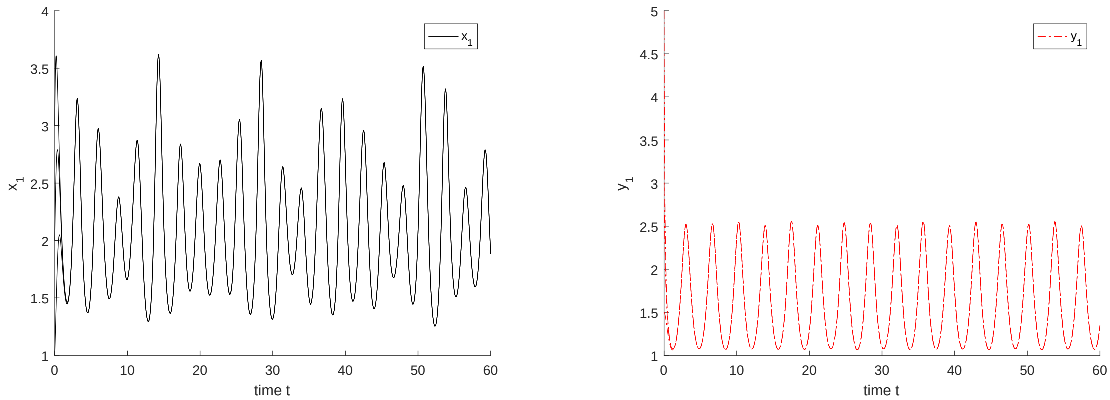

Figure 1: Dynamic behaviors of the solutions $(x_1^*(t), y_1^*(t))$ of system (4.1) with the initial conditions $(x_1^*(0), y_1^*(0)) = (1, 1.5)$, $(2, 3)$ and $(3, 5)$, respectively.

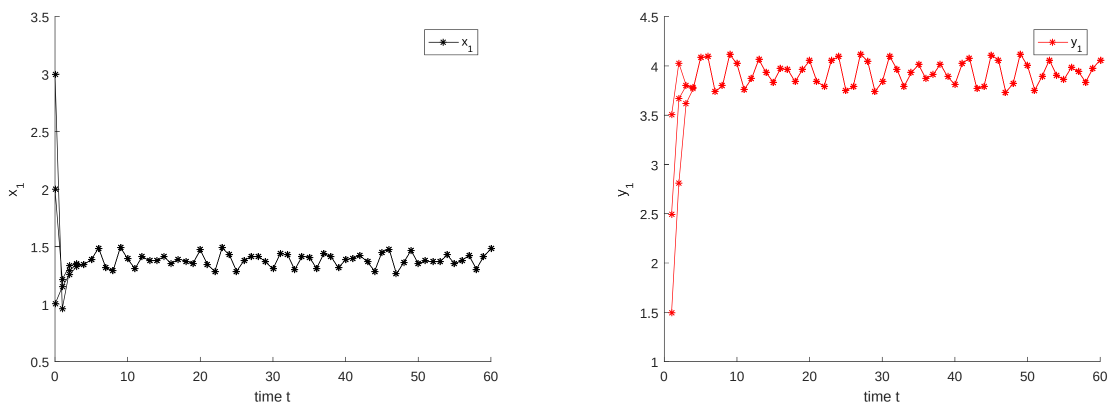

Figure 2: Dynamic behaviors of the solutions $(x_1^*(t), y_1^*(t))$ of system (4.4) with the initial conditions $(x_1^*(0), y_1^*(0)) = (1, 1.5)$, $(2, 2.5)$ and $(3, 3.5)$, respectively.

By calculating, one has

$$\begin{aligned} &d^u - e^l = 0.13 > 0, \quad a^u + c^u \exp\{M_y\} - b^l = 0.1326 > 0, \\ &\frac{E^l(d^l - e^u)}{(e^u - d^l)m^l + q^u E^u} = 2.0833 > \exp\{M_y\} = 1.1573 > \exp\{N_y\} = 1.03 > \frac{b^u - a^l}{c^l} = 0.6667. \end{aligned} \tag{4.5}$$

Obviously, the assumption in (2.5) and (2.13) are satisfied. We obtain from Example 1.2 in [2] that $\mu(t) \equiv 1$, moreover, from (3.9) we have $\mu_1 \equiv 1, \mu_2 \equiv 1$. Thus

$$\begin{aligned} A &= 4b^l \exp\{N_x\} - 2(b^u)^2 \exp\{2M_x\} - 1 = 0.2829 > 0, \\ B &= 2\left[e^l + \frac{q^l E^l}{(E^u + m^u \exp\{M_y\})^2}\right] \exp\{N_y\} \\ &\quad - \left[e^u + \frac{q^u E^u}{(E^l + m^l \exp\{N_y\})^2}\right]^2 \exp\{2M_y\} - 2(c^u)^2 \exp\{2M_y\} = 0.6277 > 0, \end{aligned} \tag{4.6}$$

$\lambda = \min\{A, B\} > 0$ and $-\lambda \in \mathcal{R}^+$. From Figure 2, it is easy to see that for system (4.4) there exists a positive almost periodic solution denoted by $(x_1^*(t), y_1^*(t))$.

In addition, we perform numerical simulation on the system (3.27).

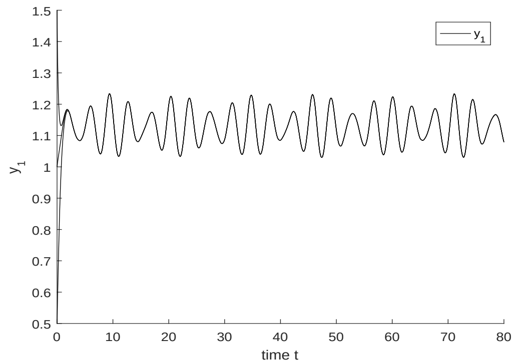

Figure 3: Dynamic behaviors of the solutions $y_1^\star(t)$ of system (4.7) with the initial conditions $y_1^\star(0) = 1.5, 1$ and 0.5, respectively.

Example 4.3. Consider the continuous version:

$$\frac{dy_1}{dt} = y_1(t)\Big(3.2 + 0.1\sin(\sqrt{5}t) - (2.8 + 0.2\cos(\sqrt{3}t))y_1(t)\Big) - \frac{(0.11 + 0.03\sin(\sqrt{6}t))(0.9 + 0.1\sin(\sqrt{3}))y_1^2(t)}{0.9 + 0.1\sin(\sqrt{3}) + (1.2 + 0.2\cos(\sqrt{2}t))y_1(t)} \tag{4.7}$$

We obtain from Example 1.2 in [2] that $\mu(t) \equiv 0$, moreover, from (3.9) we have $\mu_1 \equiv 0, \mu_2 \equiv 0$. Thus, by calculating one has

$$\begin{aligned} &d^u - e^l = 0.7 > 0, \quad C = 5.2554 > 0, \\ &E^l(d^l - e^u) = 0.08 > \Big((e^u - d^l)m^l + q^u E^u\Big)\exp\{M_y\} = 0.0262. \end{aligned} \tag{4.8}$$

From Figure 3, it is easy to see that for system (4.7) there exists a positive almost periodic solution denoted by $y_1^\star(t)$.

Example 4.4. Consider the following discrete system:

$$\begin{aligned} y_1(t+1) = y_1(t)\exp\Big(&1.1 + 0.1\cos(\sqrt{2}t) - (0.89 - 0.01\sin(\sqrt{3}t))y_1(t) \\ &- \frac{(0.02 + 0.01\cos(\sqrt{5}t))(1.4 + 0.1\sin(\sqrt{2}t))y_1(t)}{1.4 + 0.1\sin(\sqrt{2}t) + (0.6 + 0.2\cos(\sqrt{6}t))y_1(t)}\Big). \end{aligned} \tag{4.9}$$

We obtain from Example 1.2 in [2] that $\mu(t) \equiv 0$, moreover, from (3.9) we have $\mu_1 \equiv 0, \mu_2 \equiv 0$. Thus, by calculating one has

$$\begin{aligned} &d^u - e^l = 0.32 > 0, \quad C = 0.1593 > 0, \\ &E^l(d^l - e^u) = 0.13 > \Big((e^u - d^l)m^l + q^u E^u\Big)\exp\{M_y\} = 0.0072. \end{aligned} \tag{4.10}$$

From Figure 4, it is easy to see that for system (4.9) there exists a positive almost periodic solution denoted by $y_1^\star(t)$.

5 Discussion

In this paper, the sufficient conditions of existence and stability of positive almost periodic solutions for system (1.1) on time scale are obtained. Our results shows that the continuous system and discrete system can be unified well on time scales system.

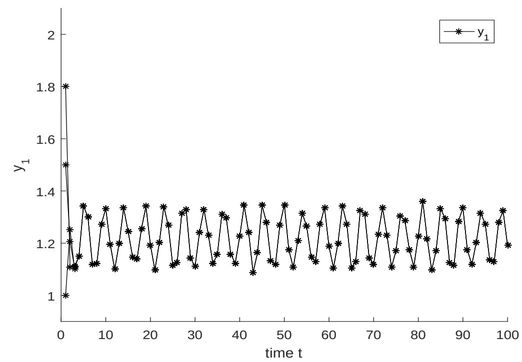

Figure 4: Dynamic behaviors of the solutions $y_1^{\star}(t)$ of system (4.9) with the initial conditions $y_1^{\star}(0) = 1.8, 1.5$ and 1, respectively.

Acknowledgements: The research is supported by the Project for Young and Middle-aged Teachers in Ningde Normal University (2018Q104), Fujian Educational Research Projects (JT180602) and Fujian Science and Technology Project (2019J01841).

References

[1] Hilger S., Ein Maβkettenkalk$\ddot{u}$l mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universit$\ddot{a}$t W$\ddot{u}$rzbury, 1988.
[2] Bohner M., Peterson A., Dynamic Equations on Time Scales: An Introduction with Applications, Birkh$\ddot{a}$user, Boston, 2001.
[3] Atici F.M., Biles D.C., Lebedinsky A., An application of time scales to economics, Math. Comput. Model., 2006, 43(7), 718–726.
[4] Chen A., Du D., Global exponential stability of delayed BAM network on time scale, Neurocomputing, 2008, 71(16-18), 3582–3588.
[5] Du B., Hu X., Ge W.G., Periodic solution of a neutral delay model of single-species population growth on time scales, Commun. Nonlinear Sci. Numer. Simul., 2010, 15(2), 394–400.
[6] Li Y., Wang C., Uniformy almost periodic functions and almost periodic solutions to dynamic equations on time scales, Abstr. Appl. Anal., 2011, Article ID 341520.
[7] Hu M., Wang L., Dynamic inequalities on time scales with applications in permanence of predator-prey system, Discrete Dyn. Nat. Soc., 2012, Article ID 281052.
[8] Zhang H., Li Y., Almost periodic solutions to dynamic equations on time scales, J. Egyptian Math. Soc., 2013, 21(1), 3–10.
[9] Wang Q., Liu Z., Existence and stability of positive almost periodic solutions for a competitive system on time scales, Math. Comput. Simulation, 2017, 138, 65–77.
[10] Gu H., Jiang H., Teng Z., Existence and global exponential stability of equilibrium of competitive neural networks with different time scales and multiple delays, J. Franklin Inst., 2010, 347(2), 719–731.
[11] Liu W., Jiang Y., Bifurcation of a delayed Gause predator-prey model with Michaelis-Menten type harvesting, J. Theoret. Biol., 2018, 438, 116–132.
[12] Hu D., Cao H., Stability and bifuracation analysis in a predator-prey system with Michaelis-Menten type predator harvesting, Nonlinear Anal. Real World Appl., 2017, 33, 58–82.
[13] Gupta R.P., Chandra P., Bifuracation analysis of modified Lesile-Gower predator-prey model with Michaelis-Menten type prey havesting, J. Math. Anal. Appl., 2013, 391(1), 278–295.
[14] Lin Q., Xie X., Chen F., Lin Q., Dynamical analysis of a logistic model with impulsive Holling type-II harvesting, Adv. Difference Equ., 2018, Article number: 112.
[15] Xue Y., Xie X., Chen F., Han R., Almost periodic solution of a discrete commensalism system, Discrete Dyn. Nat. Soc., 2015, Article ID 295483.
[16] Xue Y., Xie X., Lin Q., Almost periodic solution of a discrete competitive system with delays and feedback controls, Open Math., 2019, 17(1), 120–130.