

## ALMOST PERIODIC SOLUTIONS OF POISSON'S EQUATION<sup>1</sup>

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**ABSTRACT.** It is well known that if  $f(t)$  is almost periodic with respect to a real variable  $t$ , and if  $F(t) = \int^t f(s) ds$  is bounded for  $-\infty < t < \infty$ , then  $F(t)$  is also almost periodic in  $t$ . We shall prove a similar result for Poisson's equation.

**1. Introduction.** In this report, we shall prove the following theorem.

**THEOREM.** *Let  $f(P)$  be an almost periodic function of  $P$  in  $R^m$ , and let  $u(P)$  be a bounded continuous function of  $P$  in  $R^m$ . Assume that  $u(P)$  is a solution of*

$$(1.1) \quad \Delta u = f \quad (\Delta = \text{div} \cdot \text{grad, i.e. Laplacian})$$

*in the sense of distribution. Then  $u(P)$  is almost periodic with respect to  $P$  in  $R^m$ .*

A function  $f(P)$  is almost periodic with respect to  $P$  in  $R^m$  when

(i)  $f$  is continuous in  $R^m$ ;

(ii) for every sequence  $\{P_n\}$  in  $R^m$ , the corresponding sequence  $\{f(P+P_n)\}$  contains a subsequence which is convergent uniformly in  $R^m$ .

This definition is equivalent to the definition based on the condition that the set of almost periods of  $f$  is relatively dense in  $R^m$ . If  $f$  is almost periodic with respect to  $P$  in  $R^m$ , then  $f(P)$  is bounded in  $R^m$ .

A function  $u(P)$  is a solution of (1.1) in the sense of distribution, when

$$\int_{R^m} u(P) \Delta \phi(P) dP = \int_{R^m} f(P) \phi(P) dP$$

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for every  $C^\infty$ -function  $\phi$  of  $P$  with compact support. Since  $f(P)$  and  $u(P)$  are bounded in  $R^m$ , there exists a positive constant  $K$  such that

$$(1.2) \quad |u(P) - u(Q)| \leq K |P - Q|$$

for  $P$  and  $Q$  in  $R^m$ , where  $|P - Q|$  denotes the distance between  $P$  and  $Q$  [1, Chapter IV].

2. **Preliminaries.** Let  $f(t)$  be an almost periodic function of one real variable  $t$ . It is known that, if  $\int_0^t f(s) ds$  is bounded for  $t \in R^1$ , then  $\int_0^t f(s) ds$  is almost periodic with respect to  $t$ . We shall prove our theorem in a manner similar to J. Favard's proof of the statement given above [2, pp. 82-85]. To do this, we shall need the estimate (1.2) and the following fact:

Let  $u_1(P)$  and  $u_2(P)$  be bounded continuous functions of  $P$  in  $R^m$ , and assume that  $u_1$  and  $u_2$  are solutions of  $\Delta u = f$  in the sense of distribution. Then  $u_1(P) - u_2(P)$  is identically equal to a constant, since it is a harmonic function which is bounded in  $R^m$ . Hence if

$$(2.1) \quad \inf_P u_1(P) = \inf_P u_2(P),$$

then  $u_1(P)$  is identically equal to  $u_2(P)$ .

3. **Proof of Theorem: Part I.** Since  $f$  is almost periodic and  $u$  satisfies the estimate (1.2), it is sufficient to prove that, if  $\{P_n\}$  is a sequence of points in  $R^m$  such that

$$(3.1) \quad \lim_{n \rightarrow \infty} f(P + P_n) = g(P)$$

uniformly in  $R^m$ , and

$$(3.2) \quad \lim_{n \rightarrow \infty} u(P + P_n) = v(P)$$

uniformly on every compact set in  $R^m$ , then

$$(3.3) \quad \lim_{n \rightarrow \infty} u(P + P_n) = v(P) \quad \text{uniformly in } R^m.$$

First of all, we shall prove that

$$(3.4) \quad \inf v(P) = \inf u(P).$$

It is evident that  $v$  is a solution of  $\Delta v = g$  in the sense of distribution and that  $v$  is a bounded continuous function of  $P$  in  $R^m$ . On the other hand, (3.1) implies  $\lim_{n \rightarrow \infty} g(P - P_n) = f(P)$  uniformly in  $R^m$ . Hence if we choose a subsequence  $\{n_r\}$  of  $\{n\}$  so that  $\lim_{r \rightarrow \infty} v(P - P_{n_r}) = w(P)$  uniformly on each compact set in  $R^m$ , then  $w$  is a bounded

continuous function and a solution of  $\Delta w = f$  in the sense of distribution. Therefore we get  $w(P) = u(P) + C$ , where  $C$  is a constant. Observe that we have  $\inf u \leq \inf v \leq \inf w$  and  $\sup u \geq \sup v \geq \sup w$ . Hence  $C = 0$ , and we get (3.4).

**4. Proof of Theorem: Part II.** Assume that (3.3) is not true. Then there exist a positive constant  $\alpha$  and a sequence  $\{Q_{n_j}\}$  such that

$$(4.1) \quad |u(Q_{n_j} + P_{n_j}) - v(Q_{n_j})| \geq \alpha.$$

Choose a subsequence  $\{n_\nu\}$  of  $\{n_j\}$  so that

$$(4.2) \quad \lim_{\nu \rightarrow \infty} f(P + Q_{n_\nu} + P_{n_\nu}) = h(P)$$

uniformly in  $R^m$  and

$$(4.3) \quad \lim_{\nu \rightarrow \infty} u(P + Q_{n_\nu} + P_{n_\nu}) = w_1(P), \quad \lim_{\nu \rightarrow \infty} v(P + Q_{n_\nu}) = w_2(P)$$

uniformly on each compact set in  $R^m$ . Then (3.1) and (4.2) imply that

$$(4.4) \quad \lim_{\nu \rightarrow \infty} g(P + Q_{n_\nu}) = h(P)$$

uniformly in  $R^m$ . Hence  $w_1$  and  $w_2$  are two bounded continuous functions which are solutions of  $\Delta w = h$  in the sense of distribution. On the other hand,  $\inf w_1 = \inf u = \inf v = \inf w_2$ . Hence  $w_1$  must be identically equal to  $w_2$ . However, this is impossible, since (4.1) and (4.3) imply that

$$|w_1(0) - w_2(0)| \geq \alpha \geq 0.$$

This completes the proof of our theorem.

**5. Remarks.** As we mentioned in §2, we used two properties of Laplacian in the proof of our theorem:

(i) inequality (1.2);

(ii) a harmonic function which is bounded in  $R^m$  is identically equal to a constant.

Therefore, if we want to replace Laplacian by another partial differential operator  $A$  with constant coefficients, it might be helpful to know when an operator  $A$  will enjoy two properties similar to (i) and (ii).

Property (i) is closely related to the translation invariant property of Laplacian and to the existence of Green's function.

Property (ii) may be replaced by the property that every bounded solution of  $Au = 0$  is almost periodic in  $R^m$ . In cases where property

(i) is satisfied and no nontrivial bounded solution of  $Au=0$  approaches zero, it might be possible to show that, if there exist bounded solutions of  $Au=f(P)$ , where  $f(P)$  is almost periodic, then at least one of them is almost periodic in  $R^m$ . There is such a theorem for ordinary differential equations with variable coefficients [2, p. 95].

Finally, it might be interesting to know that

$$\Delta u + u = 0 \quad (\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2)$$

is satisfied by  $u(x, y) = J_0(\sqrt{(x^2+y^2)})$ , where  $J_0$  is the Bessel function of the first kind of order 0. It is evident that  $u(x, y)$  is bounded and  $\lim_{r \rightarrow \infty} J_0(r) = 0$ .

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