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ALMOST QUASI-YAMABE AND GRADIENT ALMOST QUASI-YAMABE SOLITONS ON QUASI-SASAKIAN MANIFOLDS

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Abstract. In this offering exposition, we intend to investigate *almost quasi-Yamabe and gradient almost quasi-Yamabe solitons* within the context of three-dimensional quasi-Sasakian manifolds.

1. Introduction

In [**9**], several years ago, Hamilton publicized the concept of *Yamabe soliton.* According to the author, a Riemannian metric g of a complete Riemannian manifold (M^n, g) is called a *Yamabe soliton* if it obeys

$$\frac{1}{2}\pounds_W\, g = (r-\lambda)\, g, \tag{1.1}$$

where W, λ, r, and $\pounds$ indicate a smooth vector field, a real number, the well-known scalar curvature, and Lie-derivative, respectively. Here, W is termed as the soliton field of the *Yamabe soliton.* A *Yamabe soliton* is called shrinking or expanding in case $\lambda > 0$ or $\lambda < 0$, respectively, whereas *steady* if $\lambda = 0$. Yamabe solitons have been investigated by several geometers in various context (see, [**2**], [**3**], [**8**], [**15**], [**16**], [**17**]). The so called *Yamabe soliton* becomes the *almost Yamabe soliton* if λ is a C^∞ function. In [**1**], Barbosa and Ribeiro introduced the above notion that was completely classified by Seko and Maeta [**14**] on hypersurfaces in Euclidean spaces.

The Yamabe soliton reduces to a *gradient Yamabe soliton* if the soliton field W is gradient of a C^∞ function $\gamma\colon M^n \to \mathbb{R}$. In this occasion, from (1.1), we have

$$\nabla^2\gamma = (r-\lambda)g, \tag{1.2}$$

where $\nabla^2\gamma$ indicates the Hessian of γ. The idea of gradient Yamabe soliton was generalized by Huang and Li [**10**] and named as *quasi-Yamabe gradient soliton.*

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According to Huang and Li, g (Riemannian metric) obeys the equation

$$\nabla^2\gamma = \frac{1}{m}\,\mathrm{d}\gamma \otimes \mathrm{d}\gamma + (r\ - \lambda)\, g, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ and m is a positive constant. If $m = \infty$, the foregoing equation reduces to Yamabe gradient soliton.

A few years ago in [**13**], taking λ as a C^∞ function, Pirhadi and Razavi investigated an *almost quasi-Yamabe gradient soliton.* They got a few fascinating formulas and produced a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient. Recently, Chen [**7**] studied almost quasi-Yamabe solitons within the context of almost Cosymplectic manifolds. According to Chen, a Riemannian metric is said to be an (AQY) metric if there exists a smooth vector field W, a C^∞ function λ, and a positive constant m such that

$$\frac{1}{2}\pounds_W g = \frac{1}{m} W^b \otimes W^b + (r -\ \lambda)\, g \tag{1.4}$$

holds, where the 1-form W^b is associated to W. In this article, the terminology "almost quasi-Yamabe" is written as (AQY) that is used throughout the paper. The (AQY) metric is called closed if the 1-form W^b is closed. The metric becomes trivial if $W \equiv 0$. Furthermore, when $m = \infty$, the previous equation gives the almost Yamabe soliton. If $W = D\gamma$, the previous equation reduces to (AQY) gradient soliton (g, γ, m, λ).

The above discussion motivate us to investigated (AQY) solitons and gradient (AQY) solitons in 3-dimensional quasi-Sasakian manifold.

The present article is structured as follows: At first, we recall a few fundamental facts and formulas of 3-dimensional quasi-Sasakian manifolds, which we need throughout the article. We investigate (AQY) soliton and gradient (AQY) soliton on 3-dimensional quasi-Sasakian manifolds in the next section. Precisely, the following prime Theorems are proved.

Theorem 1.1. *Let the Riemannian metric of a quasi-Sasakian manifold M^3 with the structure-function $\beta =$ constant be a closed* (AQY) *soliton (g, W, m, λ). Then, either the manifold is of constant sectional curvature β^2 or the* (AQY) *soliton's potential vector field is pointwise collinear with the characteristic vector field ξ.*

Theorem 1.2. *Let the Riemannian metric of a quasi-Sasakian manifold M^3 with the structure-function $\beta =$ constant be the gradient* (AQY) *soliton. Then, either the manifold is of constant sectional curvature β^2 or the gradient potential function of the* (AQY) *soliton is pointwise collinear with the Reeb vector field ξ.*

2. Preliminaries

Let M^3 be a connected differentiable manifold equipped with an *almost contact metric structure* (η, ξ, ϕ, g), where η, ξ, ϕ are a 1-form, a vector field, and a $(1,1)$-type tensor field such that [**5**], [**6**]

$$\phi^2 E = -E + \eta(E)\xi, \qquad \eta(\xi) = 1. \tag{2.1}$$

Then also

$$\phi\,\xi = 0, \qquad \eta\circ\phi = 0, \qquad \eta(E) = g(E,\ \xi). \tag{2.2}$$

The fundamental 2-form Φ of M^3 is defined by

$$\Phi(E,F) = g(E,\phi F), \qquad E,F \in \mathfrak{X}(M).$$

Then $\Phi(E,\ \xi) = 0,\ \ E \in \mathfrak{X}(M)$. M^3 is called *quasi-Sasakian* if the above 2-form Φ is closed ($\mathrm{d}\Phi = 0$) and the almost contact structure (η,ξ,ϕ) is normal. This concept was introduced by Blair [**4**]. The normality condition shows that the induced almost complex structure of $M\times\mathbb{R}$ is integrable. Equivalently, M^3 is called normal if the torsion tensor field $N = [\phi,\phi] + 2\xi\otimes \mathrm{d}\eta$ vanishes. It is to be noted that the rank of such *quasi-Sasakian structure* is always odd [**4**].

An almost contact metric manifold M^3 is called a *quasi-Sasakian manifold* if and only if [**11**]

$$\nabla_E \xi = -\beta\phi E, \qquad E \in \mathfrak{X}(M), \tag{2.3}$$

for a certain function β on M^3, called the structure-function of the manifold such that $\xi\beta = 0$. It is clear that such a quasi-Sasakian manifold reduces to a cosymplectic manifold if and only if $\beta = 0$. From (2.3), we have [**11**]

$$(\nabla_E\phi)(F) = \beta(g(E,F)\xi - \eta(F)E) \tag{2.4}$$

and

$$\begin{aligned} R(E,F)\xi = &-(E\beta)\phi F + (F\beta)\phi E \\ &+ \beta^2\{\eta(F)E - \eta(E)F\}, \qquad E,F\in\mathfrak{X}(M). \end{aligned} \tag{2.5}$$

In M^3, the Ricci tensor S is written as [**12**]

$$\begin{aligned} S(F,Z) = &\Big(\frac{r}{2} - \beta^2\Big)g(F,Z) + \Big(3\beta^2 - \frac{r}{2}\Big)\,\eta(F)\eta(Z) \\ &- \eta(F)\mathrm{d}\beta(\phi Z) - \eta(Z)\mathrm{d}\beta(\phi F). \end{aligned} \tag{2.6}$$

From (2.6), it is obvious that if $\beta =$ constant, then M is an η-Einstein manifold.

It is well-known that a Riemannian manifold M^3 assumes the following curvature form

$$\begin{aligned} R(E,F)Z = &\,g(F,Z)QE - g(E,Z)QF + S(F,Z)E - S(E,Z)F \\ &- \frac{r}{2}[g(F,Z)E - g(E,Z)F] \end{aligned} \tag{2.7}$$

for all $E,\ F,\ Z \in \mathfrak{X}(M)$. As a consequence of (2.6), we can easily obtain

$$\begin{aligned} QE = &\Big(\frac{r}{2} - \beta^2\Big)E + \Big(3\beta^2 - \frac{r}{2}\Big)\,\eta(E)\xi \\ &+ \eta(E)(\phi\,\mathrm{grad}\,\eta) - \mathrm{d}\eta(\phi E)\xi, \end{aligned} \tag{2.8}$$

where $\mathrm{d}\gamma(E) = g(\mathrm{grad}\ \gamma, E)$.

Since $\beta=$ constant the equations (2.6) and (2.8) reduce to

$$S(F,Z)=\Big(\frac{r}{2}-\beta^2\Big)g(F,Z)+\Big(3\beta^2-\frac{r}{2}\Big)\eta(F)\eta(Z) \tag{2.9}$$

and

$$QE=\Big(\frac{r}{2}-\beta^2\Big)E+\Big(3\beta^2-\frac{r}{2}\Big)\eta(E)\xi, \tag{2.10}$$

respectively. Now before going to the next section, we state the following result [**7**].

Lemma 2.1. *For any vector fields E, F on M^3, for a gradient (AQY) soliton (M^3,g,γ,m,λ), we have*

$$\begin{aligned} R(E,\ F)D\gamma=&\ \frac{r-\lambda}{m}\{(F\ \ \gamma)E-(E\ \ \gamma)F\}\\ &+(E(r-\lambda))F-(F(r-\lambda))E, \end{aligned} \tag{2.11}$$

where D indicates the gradient operator of g.

3. Proof of the main theorems

Proof of Theorem 1.1. Let us consider a closed (AQY) soliton (g,W,m,λ) on a quasi-Sasakian manifold M^3 with the structure-function $\beta=$ constant. Since W^b is closed, equation (1.4) can be written as

$$\nabla_F W=(r-\lambda)F+\frac{1}{m}g(W,F)W. \tag{3.1}$$

Executing the covariant derivative of (3.1) along the vector field E, we get

$$\begin{aligned} \nabla_E\nabla_F W=&\ (E(r-\lambda))F+(r-\lambda)\nabla_E F+\frac{1}{m}g(\nabla_E W,F)W\\ &+\frac{1}{m}g(W,\nabla_E F)W+\frac{1}{m}g(W,F)\nabla_E W. \end{aligned} \tag{3.2}$$

Exchanging E and F in (3.2), we lead

$$\begin{aligned} \nabla_F\nabla_E W=&\ (F(r-\lambda))E+(r-\lambda)\nabla_F E+\frac{1}{m}g(\nabla_F W,E)W\\ &+\frac{1}{m}g(W,\nabla_F E)W+\frac{1}{m}g(W,E)\nabla_F W \end{aligned} \tag{3.3}$$

and

$$\nabla_{[E,\ F]}W=(r-\lambda)[E,\ F]+\frac{1}{m}g(W,[E,\ F])\,W. \tag{3.4}$$

Utilizing (3.1)–(3.4), and together with $R(E,\ F)W=\nabla_E\nabla_F W-\nabla_F\nabla_E W-\nabla_{[E,F]}W$, we infer

$$\begin{aligned} R(E,\ F)W=&\ (E(r-\lambda))F-(F(r-\lambda))E\\ &+\frac{r-\lambda}{m}g(W,F)E-\frac{r-\lambda}{m}g(W,E)F. \end{aligned} \tag{3.5}$$

Executing the inner product of (3.5) with ξ gives

$$g(R(E,\ F)\,W,\xi)=\{(E(r\ -\ \lambda))-\frac{r\ -\ \lambda}{m}\,g(W,E)\}\eta(F) - \{(F(r\ -\ \lambda))-\frac{r\ -\ \lambda}{m}\,g(W,F)\}\eta(E). \tag{3.6}$$

Again, from relation (2.5), we obtain that

$$g\ (R(E,F)W,\xi)=\beta^2\{g(F,W)\eta(E)-g(E,W)\eta(F)\}. \tag{3.7}$$

Combining equation (3.6) and (3.7), and replacing E and F by ϕE and ξ, respectively, yield that

$$\beta^2 g(\phi E,W)=-\phi E(r-\lambda)+\frac{r-\lambda}{m}g(\phi E,W),$$

which is identical to

$$\Big(\frac{r\ -\ \lambda}{m}-\beta^2\Big)\phi W=\phi D(r-\lambda). \tag{3.8}$$

On the other side, utilizing (2.9) and contracting (3.5) over F, we get

$$\Big(\frac{r}{2}-\beta^2\Big)W-\Big(3\beta^2-\frac{r}{2}\Big)\eta(W)\xi=-2\Big\{D(r-\lambda)-\frac{r-\lambda}{m}W\Big\}. \tag{3.9}$$

Now applying ϕ in the previous equation yields

$$\Big(\frac{r}{2}-\beta^2\Big)\phi W=-2\Big\{\phi D(r-\lambda)-\frac{r-\lambda}{m}\phi W\Big\}, \tag{3.10}$$

which combining with (3.8), gives

$$\Big(\frac{r}{2}-3\beta^2\Big)\phi W=0. \tag{3.11}$$

This implies that either $r=6\beta^2$ or $\phi W=0$.

Case (i): If $r=6\beta^2$, then from (2.9), we get $S=2\beta^2 g$, that is an Einstein manifold. Hence from (2.7), it follows that the manifold is of constant sectional curvature β^2.

Case (ii): If $\phi W=0$, then by operating ϕ, we infer $W=\eta(W)\xi$. This accomplishes the proof. □

Utilizing $\phi W=0$ in (3.8), we infer

$$\phi Dr=\phi D\lambda.$$

By operating ϕ on the foregoing equation and utilizing (2.1), we obtain

$$Dr=D\lambda-(\xi\lambda)\xi. \tag{3.12}$$

If λ is invariant under the characteristic vector field ξ, we conclude from the above

$$r-\lambda=\text{constant}=c\ \ \text{(say)}.$$

Putting this value in (3.5), we get

$$R(E,F)W=\frac{c}{m}\,[g(W,F)E\ -g(W,E)F]. \tag{3.13}$$

This means that the sectional curvature containing the potential vector field is constant.

Again, we know that when λ = constant, the closed (AQY) soliton reduces to the closed quasi-Yamabe soliton. If λ = constant in (3.12), we conclude that the manifold is of constant scalar curvature. Thus, we can write the following corollaries.

Corollary 3.1. *If a non Einstein quasi-Sasakian manifold M^3 with the structure-function β = constant admits a closed* (AQY) *soliton (g, γ, m, λ), then the sectional curvature containing the potential vector field is constant, provided λ is invariant under the characteristic vector field ξ.*

Corollary 3.2. *If a quasi-Sasakian manifold M^3 with the structure-function β = constant admits a closed quasi-Yamabe soliton (g, W, m, λ), then either the manifold is of constant sectional curvature β^2 or the manifold is of constant scalar curvature.*

Proof of Theorem 1.2. Let us presume that the Riemannian metric of a quasi-Sasakian manifold M^3 with the structure-function β = constant be the gradient (AQY) soliton. Then the inner product of (2.11) with ξ gives

$$g(R(E,F)\,D\gamma,\xi) = \frac{r-\lambda}{m}\{(F\,\gamma)\eta(E) - (E\,\gamma)\eta(F)\} + (E(r-\lambda))\eta(F) - (F(r-\lambda))\eta(E). \tag{3.14}$$

On the other side, from equation (2.5), we obtain that

$$g\,(R(E,F)\,D\,\gamma,\,\xi) = \beta^2\{(F\,\gamma)\,\eta(E) - (E\,\gamma)\eta(F)\}. \tag{3.15}$$

Combining equation (3.14) and (3.15), and superseding E and F by ϕE and ξ, respectively, reveal that

$$\Big(\beta^2 - \frac{r-\lambda}{m}\Big)(\phi EW) + \phi E(r-\lambda) = 0. \tag{3.16}$$

On the other side, contracting (2.11) over E and using (2.10), we get

$$\Big(\frac{r}{2} - \beta^2\Big)(F\gamma) + \Big(3\beta^2 - \frac{r}{2}\Big)\eta(F)\eta(D\gamma) = 2\frac{r-\lambda}{m}(F\gamma) - 2(F(r-\lambda)). \tag{3.17}$$

Replacing F by ϕF in the foregoing equation and comparing with (3.16) give

$$\Big(\frac{r}{2} - 3\beta^2\Big)(\phi E\gamma) = 0. \tag{3.18}$$

This implies that either $r = 6\beta^2$ or $(\phi E\gamma) = 0$.

Case (i): If $r = 6\beta^2$, then by similar argument like the Theorem 1.1, we conclude that the manifold is of constant sectional curvature β^2.

Case (ii): If $(\phi E\gamma) = 0$, then by operating ϕ, we infer

$$D\gamma = (\xi\gamma)\xi. \tag{3.19}$$

This finishes the proof. □

By the the execution of covariant derivative of (3.19) along E together with (2.3) and (3.1), we find that

$$\frac{1}{m}\, g(E, D\gamma)\; D\gamma + (r-\lambda)E = (E(\xi\gamma))\xi - \beta(\xi\gamma)\phi E,$$

which becomes

$$\frac{1}{m}(\xi\gamma)^2\; \eta(E)\xi + (r-\lambda)E = \; (E(\xi\gamma))\xi - \beta(\xi\gamma)\phi E, \tag{3.20}$$

where equation (3.19) is used. Replacing $E = \xi$ in (3.20), and then taking inner product operation with ξ, we get

$$\frac{1}{m}(\xi\gamma)^2 + (r-\lambda) = \xi(\xi\gamma). \tag{3.21}$$

Again, by contracting equation (3.20) along E, we infer

$$\frac{1}{m}\; (\xi\gamma)^2 + 3(r-\lambda) = \xi\; (\xi\gamma). \tag{3.22}$$

From the last two equation, we obtain

$$(r-\lambda) = 0. \tag{3.23}$$

Using (3.23), from (3.22), we get that

$$\frac{1}{m}(\xi\; \gamma)^2 = \xi(\xi\; \gamma).$$

Let $\xi = \frac{\partial}{\partial u}$, then the previous equation becomes

$$\frac{\partial^2\gamma}{\partial u^2} = \frac{1}{m}\Big(\frac{\partial\gamma}{\partial u}\Big)^2. \tag{3.24}$$

From the foregoing partial differential equation, we can obtain $\gamma = -m \ln u$, $u > 0$, and m is a non-zero positive integer satisfying (3.24). By considering the above arguments, we can write the following corollaries.

Corollary 3.3. *If a non Einstein quasi-Sasakian manifold M^3 with the structure-function β = constant admits a gradient* (AQY) *soliton (g, γ, m, λ), then $r = \lambda$.*

Corollary 3.4. *If a non Einstein quasi-Sasakian manifold M^3 equipped with a gradient* (AQY) *soliton (g, γ, m, λ) satisfies the partial differential equation* (3.24)*, provided the structure-function β = constant, then the potential function is given by $\gamma = -m \ln u$, $u > 0$.*

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