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Almost \ast -Ricci soliton on paraKenmotsu manifolds

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Abstract We consider almost \ast -Ricci solitons in the context of paracontact geometry, precisely, on a paraKenmotsu manifold. First, we prove that if the metric g of η -Einstein paraKenmotsu manifold is \ast -Ricci soliton, then M is Einstein. Next, we show that if η -Einstein paraKenmotsu manifold admits a gradient almost \ast -Ricci soliton, then either M is Einstein or the potential vector field collinear with Reeb vector field ξ . Finally, for three-dimensional case we show that paraKenmotsu manifold is of constant curvature -1 . An illustrative example is given to support the obtained results.

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1 Introduction

On the analogy of almost contact manifolds, Sato [27] introduced the notion of almost paracontact manifolds. An almost contact manifold is always odd dimensional, but an almost paracontact manifold could be of even dimension as well. Takahashi [31] defined almost contact manifolds, in particular, Sasakian manifolds equipped with an associated pseudo-Riemannian metric. Later, Kaneyuki and Williams [17] introduced the notion of an almost paracontact pseudo-Riemannian structure, as a natural odd dimensional counterpart to paraHermitian structure. In [37], Zamkovoy showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n + 1, n)$. In recent years, almost paracontact structure has been studied by many authors, particularly since the appearance of [37]. The curvature identity for different classes of almost paracontact geometry was obtained in [9, 35, 37]. The notion of paraKenmotsu manifold was introduced by Welyczko [34]. This structure is an analogy of Kenmotsu manifold [18] in paracontact geometry. ParaKenmotsu (briefly p-Kenmotsu) and special paraKenmotsu (briefly sp-Kenmotsu) manifolds were studied by Sinha and Prasad [29], Blaga [2], Sai Prasad and Satyanarayana [25], Prakasha and Vikas [22], and many others.

A Ricci soliton is a generalization of an Einstein metric. We reminisce the notion of Ricci soliton according to [15]. On the manifold M , a Ricci soliton is a triple (g, V, λ) with g , a pseudo-Riemannian metric, V , a vector field called potential vector field and λ , a real scalar, such that

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$$(L_V g)(X, Y) + 2\text{Ric}(X, Y) = 2\lambda g(X, Y), \quad (1)$$

where L denotes Lie derivative along V and Ric denotes the Ricci tensor. The Ricci soliton is a special self similar solution of the Hamilton's Ricci flow: $\frac{\partial}{\partial t} g(t) = -\text{Ric}(t)$ with initial condition $g(0) = g$; and is said to be shrinking, steady, and expanding accordingly, as λ is positive, zero, and negative, respectively. If the vector field V is the gradient of a smooth function f on M , that is, $V = \nabla f$, then we say that Ricci soliton is gradient and f is potential function. For a gradient Ricci soliton, Eq. (1) takes the form:

$$\text{Hess } f + \text{Ric} = \lambda g,$$

where Hess denotes the Hessian operator ∇^2 (∇ denotes the Riemannian connection of g). We recommend the reference [8] for more details about the Ricci flow and Ricci soliton. In the context of paracontact geometry, Ricci solitons were first initiated by Calvaruso and Perrone in [6]. Then, these are extensively studied by [1, 2, 5, 10, 26] and many others. In this junction, it is suitable to mention that η -Ricci solitons on paraSasakian manifolds were studied in the paper [19, 23]

In 2014, Kaimakamis and Panagiotidou [16] introduced the concept of $*$ -Ricci solitons within the framework of real hypersurfaces of a complex space form, where they essentially modified the definition of Ricci soliton by replacing the Ricci tensor Ric in (1) with the $*$ -Ricci tensor Ric^* . A pseudo-Riemannian metric g on a manifold M is called a $*$ -Ricci soliton if there exist a constant λ and a vector field V , such that

$$(L_V g)(X, Y) + 2\text{Ric}^*(X, Y) = 2\lambda g(X, Y), \quad (2)$$

for all vector fields X, Y on M . Moreover, if the vector field V is a gradient of a smooth function f , then we say that $*$ -Ricci soliton is gradient and equation (2) takes the form

$$\text{Hess } f + \text{Ric}^* = \lambda g. \quad (3)$$

Note that a $*$ -Ricci soliton is trivial if the vector field V is Killing, and in this case, the manifold becomes $*$ -Einstein. Here, it is suitable to mention that the notion of $*$ -Ricci tensor was first introduced by Tachibana [30] on almost Hermitian manifolds and further studied by Hamada [13] on real hypersurfaces of non-flat complex space forms. If λ appearing in (2) and (3) is a variable smooth function on M , then g is called almost $*$ -Ricci soliton and gradient almost $*$ -Ricci soliton, respectively.

Very recently in 2018, Ghosh and Patra [12] first undertook the study of $*$ -Ricci solitons on almost contact metric manifolds. The case of $*$ -Ricci soliton in paraSasakian manifold was treated by Prakasha and Veerasha in [24]. Here, they proved that if the metric of paraSasakian manifold is a $*$ -Ricci Soliton, then it is η -Einstein ([24], Lemma 5). In this connection, it is suitable to mention that the present authors [33] studied $*$ -Ricci soliton on η -Einstein Kenmotsu and three-dimensional Kenmotsu manifolds, and proved that if metric of a η -Einstein Kenmotsu manifold is $*$ -Ricci soliton, then it is Einstein (see [33], Theorem 3.2). For three-dimensional case, it is proved that if M admits a $*$ -Ricci soliton, then it is of constant sectional curvature -1 (see [33], Theorem 3.3). It is mentioned that any three-dimensional paraKenmotsu manifold is η -Einstein (i.e., the Ricci tensor Ric is of the form $\text{Ric} = ag + b\eta \otimes \eta$, where a, b are known as associated functions). However, in higher dimensions this is not true. We also know (see [38], Proposition 4.1) that for dimension > 3 , the associated functions of an η -Einstein paraKenmotsu manifold are not constant, like paraSasakian manifolds [37].

Inspired by above-mentioned works, here, we consider $*$ -Ricci soliton in the framework of paraKenmotsu manifold. The present paper is organized as follows: In Sect. 2, we reminisce some fundamental formulas and properties of paraKenmotsu manifolds. In Sect. 3, we prove that if η -Einstein paraKenmotsu manifold admits $*$ -Ricci soliton, then M is Einstein. Next, we consider a gradient almost $*$ -Ricci soliton and show that either M is Einstein or potential vector field collinear with Reeb vector field. Also for three-dimensional case, we prove that if three-dimensional paraKenmotsu manifold admits $*$ -Ricci soliton, then it is of constant negative curvature -1 . In Sect. 4, we given an example to verify our main results.

2 Preliminaries

In this section, we reminisce some basic notions of almost paracontact metric manifold and refer to [4, 17, 21, 32, 37] for more information and details.

A $2n+1$ -dimensional smooth manifold M is said to have an almost paracontact structure if it admits a $(1,1)$ -tensor field φ , a vector field ξ , and a 1-form η satisfying the following conditions:



- (i) $\varphi^2 = I - \eta \otimes \xi, \eta(\xi) = 1.$
- (ii) The tensor field φ induces an almost paracomplex structure on each fiber of $\mathcal{D} = \ker(\eta)$, i.e., the ± 1 -eigen distributions $\mathcal{D}^\pm := \mathcal{D}\varphi(\pm)$ of φ have equal dimension n .

From the definition, it pursues that $\varphi\xi = 0, \eta \circ \varphi = 0$ and $\text{rank}(\varphi) = 2n$. An almost paracontact structure is said to be normal [17] if and only if the (1,2) type torsion tensor $N_\varphi := [\varphi, \varphi] - 2d\eta \otimes \xi$ vanishes identically, where $[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, \varphi Y] - \varphi[X, \varphi Y]$. If an almost paracontact manifold is endowed with a pseudo-Riemannian metric g , such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{4}$$

where signature of g is necessarily $(n + 1, n)$ for all $X, Y \in TM$, then $(M, \varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold. By Q and r , we will indicate the Ricci operator determined by $S(X, Y) = g(QX, Y)$ and the scalar curvature of the metric g , respectively. The fundamental 2-form Φ of an almost paracontact metric structure (φ, ξ, η, g) is defined by $\Phi(X, Y) = g(X, \varphi Y)$. If $\Phi = d\eta$, then the manifold $(M, \varphi, \xi, \eta, g)$ is called a paracontact metric manifold and g is an associated metric.

An almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is called paraKenmotsu manifold if it satisfies (see [28,38]):

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{5}$$

for any vector fields $X, Y \in TM$. In [38], Zamkovoy proved that $(M, \varphi, \xi, \eta, g)$ is normal but not quasi-parasasakian and hence not parasasakian. Also in a paraKenmotsu manifold, we have the following formulas (see [38]):

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{8}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{9}$$

$$\text{Ric}(X, \xi) = -2n\eta(X), \tag{10}$$

$$(L_\xi g)(X, Y) = -2\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{11}$$

Note that (11) implies that ξ is not Killing in paraKenmotsu manifold. An almost paracontact metric manifold M is said to be η -Einstein if there exist smooth functions a and b , such that

$$\text{Ric}(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{12}$$

for all $X, Y \in TM$. If $b = 0$, then M becomes an Einstein manifold. Following Zamkovoy ([38], Proposition 4.1), it is showed that if M is an η -Einstein paraKenmotsu manifold of dimension > 3 , then we have:

$$Z(b) - 2b\eta(Z) = 0,$$

for any $Z \in TM$.

Now, we recall the notion of $*$ -Ricci tensor. This is defined by (see [3,11]):

$$\text{Ric}^*(X, Y) = \sum_i g(R(X, e_i)\varphi e_i, \varphi Y) = \frac{1}{2} \sum_i g(\varphi R(X, \varphi Y)e_i, e_i), \tag{13}$$

where e_i is a local orthonormal frame and the last equality follows from the first Bianchi identity. It should be remarked that Ric^* is not symmetric, in general. Thus, the condition $*$ -Einstein (that is, Ric^* is a constant multiple of the metric g) automatically requires a symmetric property of the $*$ -Ricci tensor [14].

3 Main results

Before entering our main results, first, we find the expression of $*$ -Ricci tensor in paraKenmotsu manifolds

Lemma 3.1 *A $*$ -Ricci tensor on a $(2n + 1)$ -dimensional paraKenmotsu manifold $(M, \varphi, \xi, \eta, g)$ is given by:*

$$\text{Ric}^*(X, Y) = -\text{Ric}(X, Y) - (2n - 1)g(X, Y) - \eta(X)\eta(Y), \quad (14)$$

for any vector fields X, Y on M .

Proof In a paraKenmotsu manifold, the following formula is known (see [38])

$$\begin{aligned} R(X, Y)\varphi Z - \varphi R(X, Y)Z &= g(Y, Z)\varphi X - g(X, Z)\varphi Y \\ &\quad - g(Y, \varphi Z)X + g(X, \varphi Z)Y. \end{aligned} \quad (15)$$

From (15), it follows that:

$$\begin{aligned} R(X, Y, \varphi Z, \varphi W) - g(\varphi R(X, Y)Z, \varphi W) &= g(Y, Z)g(\varphi X, \varphi W) - g(X, Z)g(\varphi Y, \varphi W) \\ &\quad - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W). \end{aligned}$$

Making use of (4) with the above equation takes the form:

$$\begin{aligned} R(X, Y, \varphi Z, \varphi W) &= -R(X, Y, Z, W) - g(Y, Z)g(X, W) + g(X, Z)g(Y, W) \\ &\quad - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W). \end{aligned} \quad (16)$$

Contracting (16) over Y and Z and by definition of $*$ -Ricci tensor, we obtain (14). This completes the proof. \square

In view of recent results on paraSasakian manifold [24] and η -Einstein Kenmotsu manifold [33], a natural question arises whether there exists paraKenmotsu manifold admits a $*$ -Ricci soliton. For this, we consider an η -Einstein paraKenmotsu manifold; such a manifold is in general not paraSasakian. Now, we prove the following.

Theorem 3.2 *If the metric of η -Einstein paraKenmotsu manifold of dimension > 3 is a $*$ -Ricci soliton, then it is Einstein manifold.*

Proof Since M is η -Einstein, taking $Y = \xi$ in (12) and making use of (10), we have:

$$a + b = -2n. \quad (17)$$

Contracting (12) gives the scalar curvature $r = (2n + 1)a + b$. Combining this with (17) yields $a = (1 + \frac{r}{2n})$ and $b = -\{(2n + 1) + \frac{r}{2n}\}$. Thus, Eq. (12) takes the form:

$$\text{Ric}(X, Y) = \left(1 + \frac{r}{2n}\right)g(X, Y) - \left\{(2n + 1) + \frac{r}{2n}\right\}\eta(X)\eta(Y). \quad (18)$$

In view of (14) and (18), Eq. (2) can be written as:

$$(L_V g)(Y, Z) = \left\{2(2n + \lambda) + \frac{r}{n}\right\}g(Y, Z) - \left\{4n + \frac{r}{n}\right\}\eta(Y)\eta(Z). \quad (19)$$

Differentiating (19) along an arbitrary vector field X and using (7), we obtain:

$$\begin{aligned} (\nabla_X L_V g)(Y, Z) &= \frac{Xr}{n}g(Y, Z) - \frac{Xr}{n}\eta(Y)\eta(Z) - \left(4n + \frac{r}{n}\right)\{g(X, Y)\eta(Z) \\ &\quad + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\}. \end{aligned} \quad (20)$$

We know the following commutation formula (see [36]):

$$\begin{aligned} (L_V \nabla_X g - \nabla_X L_V g - \nabla_{[V, X]}g)(Y, Z) \\ = -g((L_V \nabla)(X, Y), Z) - g((L_V \nabla)(X, Z), Y), \end{aligned}$$



for all vector fields X, Y, Z on M . Since g is parallel with respect to Levi-Civita connection ∇ , the above relation becomes:

$$(\nabla_X L_V g)(Y, Z) = g((L_V \nabla)(X, Y), Z) + g((L_V \nabla)(X, Z), Y). \tag{21}$$

We know that $L_V \nabla$ is a symmetric tensor of type $(1, 2)$ and so it follows from (21) that

$$g((L_V \nabla)(X, Y), Z) = \frac{1}{2} \{(\nabla_X L_V g)(Y, Z) + (\nabla_Y L_V g)(Z, X) - (\nabla_Z L_V g)(X, Y)\}. \tag{22}$$

By a straightforward combinatorial computation, and keeping in mind that $L_V \nabla$ is a symmetric operator, the foregoing equation gives:

$$\begin{aligned} 2n(L_V \nabla)(X, Y) &= (Xr)Y - (Xr)\eta(Y)\xi + (Yr)X \\ &\quad - (Yr)\eta(X)\xi - g(X, Y)Dr + \eta(X)\eta(Y)Dr \\ &\quad - 2(4n^2 + r) \{g(X, Y)\xi - \eta(X)\eta(Y)\xi\}, \end{aligned} \tag{23}$$

for all vector fields Z and D is the gradient operator of g . Setting $X = Y = e_i$ (where $\{e_i : i = 1, 2, \dots, 2n+1\}$ is an orthonormal frame) in (23) and summing over i , we find:

$$n \sum_{i=1}^{2n+1} \varepsilon_i(L_V \nabla)(e_i, e_i) = (1 - n)Dr - (\xi r)\xi - 2n(4n^2 + r)\xi, \tag{24}$$

where $\varepsilon_i = g(e_i, e_i)$. Now, taking covariant differentiation of $*$ -Ricci soliton Eq. (2) along a vector field X , we obtain $(\nabla_X L_V g)(Y, Z) = -2(\nabla_X Ric^*)(Y, Z)$. Substituting this in (22), we have:

$$g((L_V \nabla)(X, Y), Z) = (\nabla_Z Ric^*)(X, Y) - (\nabla_X Ric^*)(Y, Z) - (\nabla_Y Ric^*)(X, Z). \tag{25}$$

Again, taking covariant differentiation of (14) with respect to Z and then using (7), we get:

$$\begin{aligned} (\nabla_Z Ric^*)(X, Y) &= -(\nabla_Z Ric)(X, Y) - \{g(Z, X)\eta(Y) \\ &\quad + g(Z, Y)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}. \end{aligned} \tag{26}$$

Combining (26) with (25) yields:

$$\begin{aligned} g((L_V \nabla)(X, Y), Z) &= -(\nabla_Z Ric)(X, Y) + (\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) \\ &\quad + 2\{g(X, Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\}. \end{aligned} \tag{27}$$

Replacing X and Y by e_i in (27) and summing over i , we find:

$$\sum_{i=1}^{2n+1} \varepsilon_i(L_V \nabla)(e_i, e_i) = 4n\xi. \tag{28}$$

In view of (28) and (24), we at once obtain:

$$(n - 1)Dr + (\xi r)\xi + 2n\{2n(2n + 1) + r\}\xi = 0. \tag{29}$$

Taking inner product of (29) with ξ yields $(\xi r) + 2(2n(2n + 1) + r) = 0$. Making use of this in (29) provides $Dr = (\xi r)\xi$, as $n > 1$. Next, substituting $Y = \xi$ in (23), it follows that

$$2n(L_V \nabla)(X, \xi) = (\xi r)\varphi^2 X. \tag{30}$$

Differentiating (30) along an arbitrary vector field Y and using (6) and (30), we find:

$$\begin{aligned} 2n(\nabla_Y L_V \nabla)(X, \xi) + 2n(L_V \nabla)(X, Y) &= (Y(\xi r))\varphi^2 X \\ &\quad - (\xi r)\{g(X, Y)\xi + \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi\}. \end{aligned} \tag{31}$$

According to Yano [36], we have the following well-known commutation formula:

$$(L_V R)(X, Y)Z = (\nabla_X L_V \nabla)(Y, Z) - (\nabla_Y L_V \nabla)(X, Z). \tag{32}$$

Replacing Z by ξ in (32) and taking into account of (31), we obtain:

$$2n(L_V R)(X, Y)\xi = (X(\xi r))\varphi^2 Y - (Y(\xi r))\varphi^2 X + 2(\xi r)\{\eta(X)Y - \eta(Y)X\}. \quad (33)$$

Contracting this over X and noting that $Dr = (\xi r)\xi$, we have $(L_V \text{Ric})(Y, \xi) = 0$. Next, taking Lie derivative of (10) along V , making use of last equation and (18), we have:

$$\begin{aligned} & \left(1 + \frac{r}{2n}\right)g(Y, L_V \xi) - \left\{(2n + 1) + \frac{r}{2n}\right\}\eta(Y)\eta(L_V \xi) \\ & = -4n\lambda\eta(Y) - 2ng(Y, L_V \xi). \end{aligned} \quad (34)$$

Taking $Y = \xi$ in (34), we have $\lambda = 0$. Furthermore, substituting ξ for Y and Z in (19) gives $\eta(L_V \xi) = 0$. Thus, making use of $\lambda = 0$ and $\eta(L_V \xi) = 0$, Eq. (34) becomes:

$$\{2n(2n + 1) + r\}L_V \xi = 0. \quad (35)$$

Now, if $r = -2n(2n + 1)$, then it follows from (18) that M is Einstein.

Suppose that we assume that $r \neq -2n(2n + 1)$ in some open set \mathcal{O} of M . Then, on \mathcal{O} , $L_V \xi = 0$. This together with (6) yields:

$$\nabla_\xi V = V - \eta(V)\xi.$$

Taking $Y = \xi$ in (19) and using $\lambda = 0$, we have $(L_V g)(X, \xi) = 0$. From this, we have $L_V \eta = 0$. Replacing Y by ξ in the well-known formula (see [36]):

$$(L_V \nabla)(X, Y) = L_V \nabla_X Y - \nabla_X L_V Y - \nabla_{[V, X]} Y,$$

and by virtue of (6), (30), $L_V \xi = 0$ and $L_V \eta = 0$, we obtain $(\xi r) = 0$. Since $Dr = (\xi r)\xi$, we see that r is constant. Thus, (29) implies that $r = -2n(2n + 1)$ on \mathcal{O} . This contradicts our assumption. This establish the proof. \square

Now, we consider gradient almost $*$ -Ricci soliton in η -Einstein paraKenmotsu manifolds and prove the following:

Theorem 3.3 *Let M be a $(2n + 1)$ -dimensional η -Einstein paraKenmotsu manifold. If g represents a gradient almost $*$ -Ricci soliton, then either M is Einstein or the potential vector field is pointwise colinear with the Reeb vector field ξ .*

Proof If the metric g of a η -Einstein paraKenmotsu manifold is gradient almost $*$ -Ricci soliton, then from (14) and (3), we obtain:

$$\nabla_X Df = QX + (2n - 1 + \lambda)X + \eta(X)\xi, \quad (36)$$

for any vector field X on M . Taking covariant differentiation of (36) in the direction of an arbitrary vector field Y on M yields:

$$\begin{aligned} \nabla_Y \nabla_X Df &= (\nabla_Y Q)X + Q\nabla_Y X + (2n - 1 + \lambda)\nabla_Y X + (Y\lambda)X \\ &+ (\nabla_Y \eta)(X)\xi + \eta(\nabla_Y X)\xi + \eta(X)\nabla_Y \xi. \end{aligned} \quad (37)$$

Making use of (36) and (37) in the well-known expression of curvature tensor $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$, we deduce:

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X + (X\lambda)Y - (Y\lambda)X + \eta(Y)X - \eta(X)Y. \quad (38)$$

In view of (18), we have:

$$QX = \left(1 + \frac{r}{2n}\right)X - \left\{(2n + 1) + \frac{r}{2n}\right\}\eta(X)\xi. \quad (39)$$

Differentiating the foregoing equation along an arbitrary vector field Y and using (7), we obtain:

$$(\nabla_Y Q)X = \frac{Yr}{2n}\{X - \eta(X)\xi\} - \left\{(2n + 1) + \frac{r}{2n}\right\}\{g(X, Y)\xi - \eta(X)Y\}. \quad (40)$$



In view of (40), we get from (38) that

$$R(X, Y)Df = \frac{Xr}{2n}(Y - \eta(Y)\xi) - \frac{Yr}{2n}(X - \eta(X)\xi) + \left\{ (2n + 2) + \frac{r}{2n} \right\} (\eta(Y)X - \eta(X)Y) + (X\lambda)Y - (Y\lambda)X. \tag{41}$$

By virtue of above equation, we can easily see that

$$g(R(X, Y)Df, \xi) = (X\lambda)\eta(Y) - (Y\lambda)\eta(X). \tag{42}$$

Also, we have from (8) that

$$g(R(X, Y)\xi, Df) = (Yf)\eta(X) - (Xf)\eta(Y). \tag{43}$$

Comparing (42) with (43) and substituting Y by ξ in the resulting equation, we obtain:

$$d(\lambda - f) = \xi(\lambda - f)\eta, \tag{44}$$

where d is the exterior derivative. This means that $\lambda - f$ is invariant along the distribution \mathcal{D} (where \mathcal{D} is $\text{Ker}\eta$); that is, $\lambda - f$ is constant for all vector field $X \in \mathcal{D}$.

Contracting (38) over Y , we obtain:

$$S(X, Df) = -\frac{1}{2}Xr - 2n(X\lambda) + 2n\eta(Y). \tag{45}$$

In view of (39), the foregoing equation gives:

$$\left(1 + \frac{r}{2n}\right)(Xf) - \left\{ (2n + 1) + \frac{r}{2n} \right\} \eta(X)(\xi f) + \frac{1}{2}(Xr) + 2n(X\lambda) - 2n\eta(X) = 0, \tag{46}$$

for all $X \in TM$. Replacing X by ξ in the above equation, we have:

$$2n\xi(\lambda - f) + \frac{1}{2}(\xi r) - 2n = 0. \tag{47}$$

Now, plugging Y by ξ in (41) and taking inner product with Y yield:

$$g(R(X, \xi)Df, Y) = -\frac{(\xi r)}{2n}(g(X, Y) - \eta(X)\eta(Y)) + \left\{ (2n + 2) + \frac{r}{2n} \right\} (g(X, Y) - \eta(X)\eta(Y)) + (X\lambda)\eta(Y) - (\xi\lambda)g(X, Y). \tag{48}$$

By virtue of (9), we have:

$$g(R(X, \xi)Y, Df) = (\xi f)g(X, Y) - (Xf)\eta(Y). \tag{49}$$

Comparing (48) with (49), one can get:

$$-\frac{(\xi r)}{2n}(g(X, Y) - \eta(X)\eta(Y)) + \left\{ (2n + 2) + \frac{r}{2n} \right\} (g(X, Y) - \eta(X)\eta(Y)) + X(\lambda - f)\eta(Y) - \xi(\lambda - f)g(X, Y) = 0. \tag{50}$$

Contracting the above equation, we get:

$$-(\xi r) + 2n\left(2n + 2 + \frac{r}{2n}\right) - 2n\xi(\lambda - f) = 0. \tag{51}$$

Making use of (51) in (47), we easily obtain:

$$(\xi r) = 2\{r + 2n(2n + 1)\}. \tag{52}$$

By virtue of (52), we can easily find that

$$\xi(\lambda - f) = -\left(\frac{r}{2n} + 2n\right). \quad (53)$$

Making use of (53) in (44), we obtain that

$$d(\lambda - f) = -\left(\frac{r}{2n} + 2n\right)\eta. \quad (54)$$

Applying the well-known Poincare lemma and using the fact $d\eta = 0$ on the above equation, we obtain $-dr \wedge \eta = 0$, and making use of (52), we have:

$$Dr = 2\{r + 2n(2n + 1)\}\xi. \quad (55)$$

Suppose that X in (46) is orthogonal to ξ . Keeping in mind that $\lambda - f$ is constant along \mathcal{D} and making use of (54) and (55), one gets $\{r + 2n(2n + 1)\}(Xf) = 0$, for all $X \in \mathcal{D}$. This implies that

$$\{r + 2n(2n + 1)\}(Df - (\xi f)\xi) = 0. \quad (56)$$

Suppose $r = -2n(2n + 1)$, then using this relation in (39), we see that $QX = -2nX$, and hence M is Einstein. If $r \neq -2n(2n + 1)$, then we have $Df = (\xi f)\xi$. This shows that potential vector field is collinear with ξ , and this completes the proof. \square

Now, we study $*$ -Ricci soliton in three-dimensional paraKenmotsu manifold and prove the following:

Theorem 3.4 *If the metric g of three-dimensional paraKenmotsu manifold is a $*$ -Ricci soliton, then it is of constant curvature -1 .*

Proof It is known that for any three-dimensional pseudo-Riemannian manifold, we have the following well-known expression:

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y \\ &\quad - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \quad (57)$$

Setting $Y = Z = \xi$ in the above relation and making use of (8) and (10) give:

$$QX = \left(1 + \frac{r}{2}\right)X - \left(3 + \frac{r}{2}\right)\eta(X)\xi,$$

which is equivalent to

$$\text{Ric}(X, Y) = \left(1 + \frac{r}{2}\right)g(X, Y) - \left(3 + \frac{r}{2}\right)\eta(X)\eta(Y). \quad (58)$$

Proceeding in the similar manner as in proof of Theorem 3.2. In dimension 3, that is, for $n = 1$ all Eqs. (19)–(33) holds true. Thus, (33) becomes:

$$\begin{aligned} 2(L_V R)(X, Y)\xi &= (X(\xi r))\varphi^2 Y - (Y(\xi r))\varphi^2 X \\ &\quad + 2(\xi r)\{\eta(X)Y - \eta(Y)X\}. \end{aligned} \quad (59)$$

Lie differentiating (8) along V and making use of (19) give:

$$\begin{aligned} (L_V R)(X, Y)\xi + R(X, Y)L_V \xi &= 2\lambda\{\eta(X)Y - \eta(Y)X\} \\ &\quad + g(X, L_V \xi)Y - g(Y, L_V \xi)X. \end{aligned} \quad (60)$$

In view of (59) and (60), we have:

$$\begin{aligned} &(X(\xi r))\varphi^2 Y - (Y(\xi r))\varphi^2 X + 2(\xi r)\{\eta(X)Y - \eta(Y)X\} + 2R(X, Y)L_V \xi \\ &= 4\lambda\{\eta(X)Y - \eta(Y)X\} + 2\{g(X, L_V \xi)Y - g(Y, L_V \xi)X\}. \end{aligned}$$



Contracting the foregoing equation over X and using (58), we have:

$$\begin{aligned} (r + 6)g(Y, L_V\xi) - (r + 6)\eta(Y)\eta(L_V\xi) \\ = Y(\xi r) + \{\xi(\xi r) + 4(\xi r) - 8\lambda\}\eta(Y). \end{aligned} \tag{61}$$

Setting $Y = \xi$ in (61) and using (29), we have $\lambda = 0$. In view of (19), we obtain $(L_Vg)(Y, \xi) = 0$, which implies $\eta(L_V\xi) = 0$. Thus, using $\lambda = 0$, $\eta(L_V\xi) = 0$ and (29), Eq. (61) reduces: to

$$(r + 6)g(Y, L_V\xi) = -2\{Yr - (\xi r)\eta(Y)\}. \tag{62}$$

Suppose that $r = -6$, then from (58), we can see that it is Einstein and $QX = -2X$. This together with (57) gives:

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X, \tag{63}$$

showing that M is of constant curvature -1 .

On the other hand, suppose that $r \neq -6$ in some open set \mathcal{O} of M . Then, Eq. (62) can be written as:

$$L_V\xi = f\{Dr - (\xi r)\xi\}, \tag{64}$$

where $f = -\frac{2}{r+6}$. Replacing Y by ξ in the well-known commutation formula [36]:

$$(L_V\nabla)(X, Y) = L_V\nabla_X Y - \nabla_X L_V Y - \nabla_{[V, X]} Y,$$

and using (6), (30), and (64), one can easily get:

$$\begin{aligned} f\{(Xr)\eta(Y) + (Yr)\eta(X) - 2(\xi r)\eta(X)\eta(Y) + g(\nabla_X Dr, Y) - (X(\xi r))\eta(Y)\} \\ + (Xf)\{Yr - (\xi r)\eta(Y)\} + \left\{f(\xi r) - \frac{1}{2}(\xi r)\right\}g(\varphi X, \varphi Y) = 0. \end{aligned}$$

Interchanging X, Y in the foregoing equation and recalling the Poincare lemma: $g(\nabla_X Dr, Y) = g(X, \nabla_Y Dr)$, we find:

$$\begin{aligned} f\{(Y(\xi r))\eta(X) - (X(\xi r))\eta(Y)\} + (Xf)\{Yr - (\xi r)\eta(Y)\} \\ - (Yf)\{Xr - (\xi r)\eta(X)\} = 0. \end{aligned} \tag{65}$$

Substituting Y by ξ in (65) and making use of (29), we obtain:

$$(2f - \xi f)(Xr - (\xi r)\eta(X)) = 0,$$

which implies that

$$(2f - \xi f)(Dr - (\xi r)\xi) = 0. \tag{66}$$

From (66), we have either $Dr = (\xi r)\xi$ or $\xi f = 2f$.

Case 1: First, we assume that $Dr = (\xi r)\xi$. By virtue of this, Eq. (64) can be written as $L_V\xi = 0$ on \mathcal{O} . This together with (6) gives:

$$\nabla_\xi V = V - \eta(V)\xi. \tag{67}$$

By virtue of $(L_Vg)(X, \xi) = 0$ and (67), one gets:

$$g(\nabla_X, V) = -g(X, \nabla_\xi V) = -g(X, V) + \eta(X)\eta(V). \tag{68}$$

Replacing Y by ξ in the well-known formula [36]

$$(L_V\nabla)(X, Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y} V + R(V, X)Y, \tag{69}$$

and making use of (6), (8), (30), (67), and (68), we find $\xi r = 0$. Hence from (29), it follows that $r = -6$ on \mathcal{O} , which yields a contradiction.

Case 2: Now, assume that $\xi f = 2f$. This together with $f = -\frac{2}{r+6}$, we have $\xi r = -2(r+6)$, and consequently, we get $g(Dr, \xi) = -2(r+6)$. Since $r \neq -6$ in some open set \mathcal{O} , so the last equation implies that $Dr = f\xi$, for some smooth function f . In fact, we have $Dr = (\xi r)\xi$, and so, by Case 1, we get a contradiction. This establishes the proof of the theorem. \square

As we know, in differential geometry, symmetric spaces play an important role. In the late 20s, Cartan [7] initiated Riemannian symmetric spaces and obtained a classification of those spaces. If the Riemannian curvature tensor of a Riemannian manifold satisfies the condition $\nabla R = 0$, then this manifold is called locally symmetric [7]. For every point of this manifold, this symmetry condition is equivalent to the fact that the local geodesic symmetry is an isometry [20]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

Definition 3.5 An almost paracontact metric manifold is said to be locally φ -symmetric if

$$\varphi^2(\nabla_W R)(X, Y)Z = 0,$$

for all vector fields W, X, Y, Z orthogonal to ξ .

It is known that a three-dimensional paraKenmotsu manifold is locally φ -symmetric if and only if the scalar curvature is constant [38]. Therefore, by Theorem 3.4, we state the following.

Corollary 3.6 A three-dimensional paraKenmotsu manifold admitting $*$ -Ricci soliton is locally φ -symmetric.

Remark 3.7 Corollary 3.6 builds the connection between $*$ -Ricci soliton and symmetry of the manifold. The symmetry of a manifold is vital, because it is connected with the curvature of the manifold. The curvature has important physical significance in the theory of gravitation.

4 Example

In this section, we give an example of $*$ -Ricci solitons in three-dimensional paraKenmotsu manifold which verifies Theorem 3.4 and Corollary 3.6.

Example 4.1 We consider three-dimensional manifold $M = \{(x, y, z) \in \mathcal{R}^3, z \neq 0\}$ with the Cartesian coordinates (x, y, z) and the vector fields:

$$\partial_1 = \varphi\partial_2, \quad \partial_2 = \varphi\partial_1, \quad \varphi\partial_3 = 0,$$

where

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form $\eta = dz$ defines an almost paracontact structure on M with characteristic vector field $\xi = \partial_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Let g be a pseudo-Riemannian metric defined by:

$$[g(\partial_i, \partial_j)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

Using Koszul formula, we have:

$$\begin{aligned} \nabla_{\partial_1}\partial_1 &= -\partial_3, & \nabla_{\partial_1}\partial_2 &= 0, & \nabla_{\partial_1}\partial_3 &= \partial_1, \\ \nabla_{\partial_2}\partial_1 &= 0, & \nabla_{\partial_2}\partial_2 &= \partial_3, & \nabla_{\partial_2}\partial_3 &= \partial_2, \\ \nabla_{\partial_3}\partial_1 &= 0, & \nabla_{\partial_3}\partial_2 &= 0, & \nabla_{\partial_3}\partial_3 &= 0. \end{aligned}$$

It is not hard to verify that the conditions (5) and (6) for paraKenmotsu manifold are satisfied. Hence, the manifold under consideration is a paraKenmotsu manifold. The components of the curvature tensor are:

$$\begin{aligned} R(\partial_1, \partial_2)\partial_1 &= \partial_2, & R(\partial_1, \partial_2)\partial_2 &= \partial_1, & R(\partial_1, \partial_2)\partial_3 &= 0, \\ R(\partial_1, \partial_3)\partial_1 &= \partial_3, & R(\partial_1, \partial_3)\partial_2 &= 0, & R(\partial_1, \partial_3)\partial_3 &= -\partial_1, \\ R(\partial_2, \partial_3)\partial_1 &= 0, & R(\partial_2, \partial_3)\partial_2 &= -\partial_3, & R(\partial_2, \partial_3)\partial_3 &= -\partial_2. \end{aligned}$$



The components of Ricci tensor and $*$ -Ricci tensor are:

$$\begin{aligned} \text{Ric}(\partial_1, \partial_1) &= -2, & \text{Ric}(\partial_2, \partial_2) &= 2, & \text{Ric}(\partial_3, \partial_3) &= -2, \\ \text{Ric}^*(\partial_1, \partial_1) &= 1, & \text{Ric}^*(\partial_2, \partial_2) &= -1, & \text{Ric}^*(\partial_3, \partial_3) &= 0. \end{aligned} \quad (70)$$

If we choose $V = \partial_1 - \partial_3$, then we see that

$$\begin{aligned} (L_V g)(\partial_1, \partial_1) &= Vg(\partial_1, \partial_1) - 2g(L_V \partial_1, \partial_1) \\ &= -2g([\partial_1 - \partial_3, \partial_1], \partial_1) = -2. \end{aligned} \quad (71)$$

Thus, V is not a Killing vector field. Now, the $*$ -Ricci soliton equation

$$(L_V g)(\partial_1, \partial_1) + 2\text{Ric}^*(\partial_1, \partial_1) = 2\lambda g(\partial_1, \partial_1),$$

gives $\lambda = 0$ (we know that if paraKenmotsu manifold admits $*$ -Ricci soliton, then $\lambda = 0$). Similarly, we can check the other components and verify that M satisfies:

$$(L_V g)(X, Y) + 2\text{Ric}^*(X, Y) = 2\lambda g(X, Y).$$

Hence, the metric g is a $*$ -Ricci soliton. Using (70), we have constant scalar curvature as follows:

$$r = \text{Ric}(\partial_1, \partial_1) - \text{Ric}(\partial_2, \partial_2) + \text{Ric}(\partial_3, \partial_3) = -6. \quad (72)$$

Because of scalar curvature $r = -6$, from Theorem 3.4, we can conclude that M is an Einstein manifold. It is easy to verify that the manifold is locally φ -symmetric. Hence, the results of Theorem 3.4 and Corollary 3.6 are verified.

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