2014

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Publisher Citation
Almost Ricci solitons and K-contact geometry, Monatshefte fur Mathematik 175(4) (Dec. 2014),621-628. First published online 04 July 2014

Comments
This is the author's accepted version of the article published in Monatshefte fur Mathematik The final published article is available at Springer: http://dx.doi.org/10.1007/s00605-014-0657-8

Monatshefte fur Mathematik, Vol. 174 (2014), 1. 621-628.

ALMOST RICCI SOLITONS AND K-CONTACT GEOMETRY

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Abstract: We give a short Lie-derivative theoretic proof of the following recent result of Barros et al. "A compact non-trivial almost Ricci soliton with constant scalar curvature is gradient, and isometric to a Euclidean sphere". Next, we obtain the result: A complete almost Ricci soliton whose metric g is K-contact and flow vector field X is contact becomes Ricci soliton with constant scalar curvature. In particular, for X strict, g becomes compact Sasakian Einstein. Finally, we show that the Lie-bracket of two distinct Ricci soliton vector fields with the same metric generates a steady Ricci soliton.

2010 MSC:53C25,53C44,53C21

Keywords: Almost Ricci soliton, Conformal vector field, Constant scalar curvature, K-contact metric, Einstein Sasakian metric.

1 Introduction

Modifying the Ricci soliton equation by allowing the dilation constant λ to become a variable function, Pigola et al. [8] defined an almost Ricci soliton as a Riemannian manifold (M, g) satisfying the condition:

$$\pounds_X g_{ij} + 2R_{ij} = 2\lambda g_{ij}. \tag{1}$$

where X is a vector field on M, g_{ij} and R_{ij} are the components of the metric tensor g and its Ricci tensor in local coordinates (x^i), $\pounds_X$ is the Lie-derivative operator along X, and λ is a smooth function on M. A simple example is the canonical metric g on a Euclidean sphere with X a non-homothetic conformal vector field. For λ constant, (1) becomes the Ricci soliton. The almost Ricci soliton is said to be shrinking, steady, and expanding according as λ is positive, zero, and negative respectively; otherwise is indefinite. If the vector field X is the gradient of a smooth function f, upto the addition of a Killing vector field, (M, g, X, λ) is called a gradient almost Ricci soliton, in which case the equation (1) assumes the form:

$$\nabla_i \nabla_j f + R_{ij} = \lambda g_{ij}. \tag{2}$$

For an almost Ricci soliton with X homothetic, g is Einstein and hence λ becomes constant and it becomes the trivial Ricci soliton. For X non-homothetic, g is a non-trivial almost Ricci soliton. We also note for an almost Ricci soliton that X is conformal if and only if g is Einstein.

Ricci solitons are special solutions of the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t), \tag{3}$$

of the form $g_{ij}(t) = \sigma(t)\psi_t^* g_{ij}$ with initial condition $g_{ij}(0) = g_{ij}$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function. In the same vein, we can view almost Ricci soliton as a special solution of Ricci flow, by considering the ansatz:

$$g_{ij}(t) = \sigma(t, x^k)\psi_t^* g_{ij}, \tag{4}$$

where ψ_t are diffeomorphisms of M generated by the family of vector fields $Y(t)$, and $\sigma(t, x^k)$ can be viewed as a pointwise scaling function that depends not only on time t, but also on the coordinates x^k of points. The initial conditions: $g_{ij}(0) = g_{ij}$, $\psi_0 =$ identity, imply $\sigma(0, x^k) = 1$. Differentiating (4) with respect to t, using the Ricci flow equation (3), and substituting $t = 0$ shows

$$-2R_{ij} = (\frac{\partial}{\partial t}\sigma(t, x^k))|_{t=0} g_{ij} + \pounds_{Y(0)} g_{ij},$$

Labelling $Y(0)$ as X and the time-independent function $(\frac{\partial}{\partial t}\sigma(t, x^k))|_{t=0}$ as -2λ, we obtain the almost Ricci soliton equation (1).

2 Compact Almost Ricci Soliton

It is well known that a compact Ricci soliton is gradient. This need not be true for almost Ricci soliton. In [3], Barros and Ribeiro Jr. showed that a compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. Intrigued by the fact that a compact Ricci soliton with constant scalar curvature is trivial (i.e. X is Killing and g is Einstein), Barros, Batista and Ribeiro Jr. [2] proved the following result.

Theorem 1 (B-B-R) *Let (M^n, g, X, λ) be a compact oriented almost Ricci soliton. If Ric, S and dv_g denote respectively the Ricci tensor, scalar curvature and the volume form with respect to g, then*

$$\int_M |Ric - \frac{S}{n}g|^2 dv_g = \frac{n-2}{2n}\int_M g(\nabla S, X) dv_g. \tag{5}$$

If, in addition; $n > 2$, the almost Ricci soliton is non-trivial and the scalar curvature is constant, then (M, g) is isometric to a Euclidean sphere and the almost Ricci soliton is gradient.

In this paper we provide a short Lie-derivative theoretic proof of this result, based on equations of evolution of Christoffel symbols and curvature quantities along the flow vector field X. We denote the Levi-Civita connection, connection coefficients, and components of curvature tensor of g by ∇, Γ^i_{jk}, and R^h_{kji} respectively.

Another Proof Of Theorem 1 (B-B-R). Let us denote the inverse of g_{ij} by g^{ij}. Taking the Lie-derivative of the relation $g_{ij}g^{jk} = \delta^k_i$ along X, using equation (1) and subsequently operating the resulting equation by g^{il} we immediately get

$$\pounds_X g^{kl} = 2R^{kl} - 2\lambda g^{kl}. \tag{6}$$

Next, the use of equation (1) in the formula (page 23, Yano [9]):

$$\pounds_X \Gamma^h_{ij} = \frac{1}{2} g^{ht}[\nabla_j(\pounds_X g_{it}) + \nabla_i(\pounds_X g_{jt}) - \nabla_t(\pounds_X g_{ij})],$$

yields the evolution equation

$$\begin{aligned} \pounds_X \Gamma^h_{ij} &= \nabla^h R_{ij} - \nabla_j R^h_i - \nabla_i R^h_j - (\nabla^h \lambda) g_{ij} \\ &+ (\nabla_j \lambda)\delta^h_i + (\nabla_i \lambda)\delta^h_j. \end{aligned} \tag{7}$$

Let us follow the notational convention: $\nabla_k \nabla_j Z^h - \nabla_j \nabla_k Z^h = R^h_{kji} Z^i$, where Z^i are components of an arbitrary vector field, and $R^k_{kji} = R_{ji}$. Using equation (7) in the following commutation formula (page 23, [9]):

$$\nabla_k(\pounds_X \Gamma^h_{ij}) - \nabla_j(\pounds_X \Gamma^h_{ik}) = \pounds_X R^h_{kji},$$

we obtain the evolution equation:

$$\begin{aligned}\pounds_X R^h_{kji} &= \nabla_j\nabla_k R^h_i - \nabla_k\nabla_j R^h_i + \nabla_j\nabla_i R^h_k - \nabla_k\nabla_i R^h_j \\ &+ \nabla_k\nabla^h R_{ij} - \nabla_j\nabla^h R_{ik} + (\nabla_k\nabla_i\lambda)\delta^h_j \\ &- (\nabla_k\nabla^h\lambda)g_{ij} - (\nabla_i\nabla_j\lambda)\delta^h_k + (\nabla_j\nabla^h\lambda)g_{ik}.\end{aligned}$$

Contracting this equation with g^{hk} and using the twice contracted Bianchi identity: $\nabla_i R^i_j = \frac{1}{2}\nabla_j S$, we have

$$\begin{aligned}\pounds_X R_{ji} &= \nabla_j\nabla_i S - \nabla_h\nabla_j R^h_i - \nabla_h\nabla_i R^h_j \\ &+ \Delta R_{ij} - (\Delta\lambda)g_{ij} - (n-2)\nabla_i\nabla_j\lambda.\end{aligned}$$

Lie-differentiating $S = R_{ij}g^{ij}$ along X, and using the above equation and equation (6) provides the evolution equation for the scalar curvature:

$$\pounds_X S = 2R_{ij}R^{ij} + \Delta S - 2\lambda S - 2(n-1)\Delta\lambda. \tag{8}$$

Writing $\pounds_X S$ as $g(\nabla S, X)$, integrating the above equation over the compact M and using the Gauss divergence theorem we get

$$\int_M [R_{ij}R^{ij} - \lambda S - \frac{1}{2}g(\nabla S, X)]dv_g = 0. \tag{9}$$

At this point, we note

$$div(SX) = \nabla_i(SX^i) = g(\nabla S, X) + S div X,$$

and integrate it over M in order to get

$$\int_M [g(\nabla S, X) + S div X]dv_g = 0. \tag{10}$$

Now we contract equation (1) with g^{ij} in order to get $div X = n\lambda - S$, and use it in (10) to obtain

$$\int_M (n\lambda S - S^2 + g(\nabla S, X))dv_g = 0.$$

Eliminating $\int_M (\lambda S)dv_g$ between the above equation and (9) and noting $|Ric - \frac{S}{n}g|^2 = R_{ij}R^{ij} - \frac{S^2}{n}$ we obtain equation (5), proving the first part of the theorem.

For the second part, we use the hypothesis that S is constant in equation (5) to conclude that g is Einstein. Thus, equation (1) reduces to $\pounds_X g_{ij} = 2(\lambda - \frac{S}{n})g_{ij}$, i.e. X is a non-homothetic conformal vector field on M. With the setting $\lambda - \frac{S}{n} = \rho$, the foregoing conformal equation assumes the form

$$\pounds_X g_{ij} = 2\rho g_{ij}. \tag{11}$$

Using the conformal integrability condition (p. 26, [9])

$$\pounds_X R_{ij} = (2-n)\nabla_i \nabla_j \rho - (\Delta\rho) g_{ij}$$

and the Einstein condition $R_{ij} = \frac{S}{n} g_{ij}$ we get

$$(\Delta\rho + \frac{2S}{n}\rho) g_{ij} = (2-n)\nabla_i \nabla_j \rho. \tag{12}$$

Contracting it with g^{ij} gives $\Delta\rho = -\frac{S}{n-1}\rho$. Using this in the identity: $\Delta\rho^2 = \nabla^i \nabla_i(\rho^2) = 2[|\nabla\rho|^2 + \rho\Delta\rho]$, and integrating over M gives $\int_M |\nabla\rho|^2 = \frac{S}{n-1}\int_M \rho^2$. This shows that $S > 0$. Consequently, equation (12) becomes

$$\nabla_i \nabla_j \rho = -\frac{S}{n(n-1)}\rho g_{ij}. \tag{13}$$

This implies, by virtue of Obata's theorem [7]: "A complete Riemannian manifold (M, g) of dimension $n \geq 2$ admits a non-trivial solution ρ of the system of partial differential equations $\nabla_i \nabla_j \rho = -c^2 \rho g_{ij}$ (c a positive constant) if and only if M is isometric to a Euclidean sphere of radius $1/c$" that (M, g) is isometric to a Euclidean sphere of radius $\sqrt{\frac{n(n-1)}{S}}$.

Equation (13) can also be expressed as $\pounds_{\nabla\rho} g_{ij} = \frac{2S}{n(1-n)}\rho g_{ij}$. Combining this with (11) we obtain

$$\pounds_{X - \frac{n(n-1)}{S}\nabla\rho} g_{ij} = 0.$$

Hence $X = \nabla(\frac{n(1-n)}{S}\rho)$ + a Killing vector field, i.e. the almost Ricci soliton is gradient, completing the proof.

3 K-Contact Metric As Almost Ricci Soliton

A $(2m+1)$-dimensional smooth manifold M is called a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M. For a given contact 1-form η there exists a unique vector field ξ (Reeb vector field) such that $(d\eta)(\xi, .) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric g and a (1,1)-tensor field φ such that

$$(d\eta)(Y, Z) = g(Y, \varphi Z), \eta(Z) = g(\xi, Z), \varphi^2 = -I + \eta \otimes \xi, \tag{14}$$

for arbitrary vector fields Y, Z on M. We call g an associated metric of η and (φ, η, ξ, g) a contact metric structure. A K-contact metric is a contact metric for which ξ is Killing, equivalently:

$$Ric(\xi, Y) = 2mg(\xi, Y), \tag{15}$$

for an arbitrary vector field Y on M. This condition is also equivalent to:

$$Ric(\xi, \xi) = 2m. \tag{16}$$

For details we refer to [4]. A contact metric g on M^{2m+1} is called Sasakian if the almost Kaehler structure induced on the cone $(\mathcal{R}^+ \times M)$ with metric $dr^2 + r^2 g$, is Kaehler (see Boyer and Galicki [5]). A Sasakian metric is K-contact, but the converse need not be true, except in dimension 3.

We would like to consider an almost Ricci soliton (M, g, X, λ) such that g is a K-contact metric and X is a contact vector field. Let us recall that a vector field X on a contact manifold is said to be a contact vector field if

$$\pounds_X \eta = f\eta, \tag{17}$$

for a smooth function f on M. The contact vector field X is called strict when $f = 0$.

Using Cartan's magic formula, we find that $\pounds_\xi \eta = di_\xi \eta + i_\xi d\eta = d(1) + d\eta(\xi, .) = 0$, i.e. ξ is a strict contact vector field. We note from equation (1) that, if we take g as a K-contact metric and X as ξ, then (as ξ is Killing), the K-contact metric g reduces to an Einstein metric and λ becomes constant, equal to the Einstein constant $2m$, as seen from equation (16). We generalize this special situation in the form of the following result.

Theorem 2 *Let (M, g, X, λ) be a complete almost Ricci soliton with g a K-contact metric and X a contact vector field. Then it becomes Ricci soliton and g has constant scalar curvature. In particular, if X is strict, then g is Sasakian Einstein.*

Proof. First of all, we have, by definition of the contact structure, $\omega = \eta \wedge (d\eta)^m \neq 0$ and thus is a volume element. Denote it by ω. Using the hypothesis (17) we compute $\pounds_X d\eta = d\pounds_X \eta = (df) \wedge \eta + f(d\eta)$. Consequently, the formula: $\pounds_X \omega = (divX)\omega$ yields the relation $divX = (m+1)f$. On the other hand, the trace of equation (1) is $divX = (2m+1)\lambda - S$. Comparing the two values of $divX$ we have

$$S = (2m+1)\lambda - (m+1)f. \tag{18}$$

Next, we Lie-differentiate the second equation in (14) along X, and then use equations (1), (15) and (17) in order to get

$$\pounds_X \xi = (f - 2\lambda + 4m)\xi. \tag{19}$$

The Lie-derivative of $g(\xi, \xi) = 1$ (as ξ is unit) along X, and the use of equations (1) and (16) provides $g(\pounds_X \xi, \xi) = 2m - \lambda$. The inner product of (19) with ξ and the foregoing equation lead us to the relation: $f = \lambda - 2m$. Consequently, we have

$$\pounds_X \eta = (\lambda - 2m)\eta, \quad \pounds_X \xi = (2m - \lambda)\xi. \tag{20}$$

At this point, we take the Lie-derivative of the first equation in (14), along X and use equations (1) and (17) in order to obtain

$$\eta(Z)\nabla f - (Zf)\xi + 2(f - 2\lambda)\varphi Z = -4Q\varphi Z + 2(\pounds_X \varphi)Z, \tag{21}$$

where Z is an arbitrary vector field on M, and Q is the Ricci tensor of type (1,1), defined by $g(Q., .) = Ric(., .)$. Substituting ξ for Z in equation (21) and using the property $\varphi\xi = 0$ and equation (20) we find $\nabla f = (\xi f)\xi$, i.e. $df = (\xi f)\eta$. Taking its exterior derivative, using Poincare lemma: $d^2 = 0$, and then wedge product with η we have $(\xi f)\eta \wedge d\eta = 0$. As $\eta \wedge d\eta$ cannot vanish anywhere, otherwise the definition of the contact structure would be violated, we conclude that $\xi f = 0$, and hence $df = 0$, i.e. f is constant on M. Consequently, equation (21) reduces to the following evolution equation for φ:

$$\pounds_X \varphi = 2Q\varphi - (2m + \lambda)\varphi. \tag{22}$$

As shown earlier, $f = \lambda - 2m$, and f is constant, we conclude that λ is constant and hence the almost Ricci soliton becomes Ricci soliton. Appealing to equation (18), we find that S is constant. This proves first part. For the second part, the hypothesis $f = 0$ immediately implies $\lambda = 2m$ and thus we get from (18) that $S = 2m(2m+1)$. Plugging these findings in equation (8) and carrying out a straightforward computation shows $|Ric - 2mg|^2 = 0$. Hence $Ric = 2mg$, i.e. g is Einstein with Einstein constant $2m$.

As (M, g) is complete, thanks to Myers' theorem, (M, g) becomes compact. In order to turn g into Sasakian, we recall the following result of Morimoto [6]: "Let (M, η, g) be a compact K-contact manifold such that g is η-Einstein, i.e. its Ricci tensor satisfies $Ric = ag + b\eta \otimes \eta$ for real constants a, b. If $a > -2$, then g is Sasakian". This result was also proved independently by Boyer and Galicki [5], and Apostolov et al. [1]. In our case, $a = 2m$ and $b = 0$, and hence the aforementioned result holds. Thus, we conclude that g is Sasakian, and complete the proof.

4 Commutation Of Ricci Soliton Vector Fields

We consider two distinct Ricci solitons with the same Riemannian metric and show that the Lie-bracket of their flow vector fields give rise to a steady Ricci soliton. More precisely, we prove

Proposition 1 *Let (M, g, X_1, λ_1) and (M, g, X_2, λ_2) be two distinct non-trivial Ricci solitons. Then, $[X_1, X_2]$ determines a steady Ricci soliton on M with a metric homothetic to g.*

Proof By hypothesis, we have

$$\pounds_{X_1} g + 2Ric = 2\lambda_1 g, \quad \pounds_{X_2} g + 2Ric = 2\lambda_2 g, \tag{23}$$

where λ_1 and λ_2 are constants. As these are two distinct Ricci solitons, we may assume without any loss of generality, that $\lambda_1 < \lambda_2$. The two equations in (23) show that $X_1 = X_2 + H$ where H is a homothetic vector field satisfying $\pounds_H g = 2(\lambda_1 - \lambda_2)g$. The following computation:

$$\begin{aligned} \pounds_{[X_1,X_2]} g &= \pounds_{[X_2+H,X_2]} g = \pounds_{[H,X_2]} g = \pounds_H \pounds_{X_2} g - \pounds_{X_2} \pounds_H g \\ &= \pounds_H(-2Ric + 2\lambda_2 g) - \pounds_{X_2}(2(\lambda_1 - \lambda_2)g) = 4(\lambda_1 - \lambda_2)Ric, \end{aligned}$$

shows that

$$\pounds_{\frac{1}{2(\lambda_2-\lambda_1)}[X_1,X_2]}g + 2Ric = 0.$$

Taking into account the fact that the Ricci tensor is invariant under a homothetic transformation and noticing that $[X_1, X_2]$ cannot be conformal (otherwise g would become Einstein) we conclude that $(M, \frac{1}{2(\lambda_2-\lambda_1)}g, [X_1, X_2], 0)$ is a Ricci soliton which is steady. This completes the proof.

5 Concluding Remarks

1. In the proof of Theorem 1, Barros, Batista and Ribiero Jr. used a result of Yano and Nagano and the Hodge-de Rham decomposition. Our proof uses a theorem of Obata and does not need Hodge-de Rham decomposition.

2. The hypotheses of Theorem 2 can be interpreted in terms of contact Hamiltonians as follows. The contact Hamiltonian associated to a contact vector field X defined by equation (17) is a function $\mathcal{H}$ defined as $\eta(X)$, and the function f turns out to be equal to $\xi\mathcal{H}$. The vector field X is the Hamiltonian vector field associated to $\mathcal{H}$. Computing $\pounds_X\mathcal{H} = \pounds_X(\eta(X)) = (\pounds_X\eta)X = f\eta(X) = f\mathcal{H} = (\xi\mathcal{H})\mathcal{H}$ shows that the contact vector field X is strict, i.e. $f = \xi\mathcal{H} = 0$ if and only if the associated Hamiltonian $\mathcal{H}$ is a first integral of X, i.e. is preserved along the flow of the Hamiltonian vector field X.

3. For the second part of Theorem 2, we found that $\lambda = 2m$, $Ric = 2mg$ and hence $Q = 2mI$. Using these and the hypothesis $f = 0$ in equations (20) and (22) we infer that X preserves all structure tensors η, ξ, g, φ, and hence is an infinitesimal automorphism of the Sasakian structure on M.

Acknowledgments: The author thanks Dr. Amalendu Ghosh for help on a couple of points. This work has been supported by University Research Scholarship of the University of New Haven.

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